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Efficient Derivative-Free Class of Seventh Order Method for Non-differentiable Equations

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Abstract. Many applications from a wide variety of disciplines in the natural sciences and also in engineering are reduced to solving of an equation or a system of equations in a correspondingly chosen abstract area. For most of these problems, the solutions are found iterative, because their analytic versions are difficult to find or impossible. This article encompasses efficient, derivatives-free, high-convergence iterative methods. Convergence of two types: Local and Semi-local areas will be investigated under the conditions of the φ, ψ-continuity utilizing operators on the method. The new method can also be applied to other methods, using inverses of the linear operator or the matrix.

1. Introduction

In the area of Applied Science and Technology, a great number of problems can be resolved by converting them into nonlinear form equation

$$G(x) = 0 \tag{1}$$

where $G : \mathcal{B} \subset \mathcal{U} \to \mathcal{U}$ is differentiable as per Fréchet, $\mathcal{U}$ denotes complete normed linear space and $\mathcal{B}$ is a non-empty, open and convex set.

Normally, the solutions to these non-linear equations can not be obtained in a closed-form. Therefore, the most frequently used solving techniques are of iterative nature. Newton's Method is a well-known iterative method for handling non-linear equations. Recently, with advances in Science and Mathematics many new iterative methods of higher order have been discovered for the handling of non-linear equations and are currently being used [1, 2, 4–8, 10–22]. The computation of derivatives of second and higher order is a great disadvantage for the iterative systems of higher order and is not suitable for the practical application. Because of the computation of G'', the

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cubically converging classical schemas are not appropriate with respect to the cost of calculations. We found that many such methods rely on Taylor series extensions to prove convergence results and require the existence of derivative with at least an order of magnitude greater than that of the methodology [1, 2, 4, 10–19, 21, 22]. Here we consider, for example, a three-step two-parameter family of derivative free methods with seventh-order of convergence for solving systems of nonlinear equations proposed in [18] and which may be expressed in the following formulation:

For $x_0 \in \mathcal{B}$ and each $n = 0, 1, 2, \ldots$

$$\begin{aligned} w_n &= x_n + aG(x_n), \quad s_n = x_n - aG(x_n), \quad A_n = [w_n, s_n; G], \\ y_n &= x_n - A_n^{-1}G(x_n), \\ z_n &= y_n - A_n^{-1}G(y_n), \\ u_n &= z_n + bG(z_n), \quad v_n = z_n - bG(z_n), \quad Q_n = [u_n, v_n; G], \\ x_{n+1} &= z_n - (pI + A_n^{-1}Q_n(qI + A_n^{-1}Q_n(rI + dA_n^{-1}Q_n)))A_n^{-1}G(z_n), \end{aligned} \tag{2}$$

where $a, b, p, q, r, d \in \mathbb{R}$, $[\cdot, \cdot; G] : \mathcal{B} \times \mathcal{B} \to \mathscr{W}(\mathcal{U})$, the space of bounded linear operators from $\mathcal{U}$ into $\mathcal{U}$. The local convergence analysis of the method (2) is provided in [18] using the Taylor series expansion approach and conditions reaching the eighth derivative of the operator G. These derivatives do not appear on the method (2). The convergence order is shown to be seven provided that $p = \frac{17}{4}, q = -\frac{27}{4}, r = \frac{19}{4}$ and $d = -\frac{5}{4}$. The conditions on high order derivatives restrict the applicability of the method (2) for solving equations where at least $G^{(8)}$ should exist. Although, the method may converge. Let us consider the toy example for $\mathcal{B} = [-1, 2]$ and G defined by

$$G(t) = \begin{cases} t^4 \log t + 5t^7 - 5t^6, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$$

It follows by this definition that $G(\xi) = G(1) = 0$ but $G^{(4)}$ is not bounded on $\mathcal{B}$. Thus, the results in [18] cannot assure that $\lim_{n\to\infty} x_n = \xi = 1$. But, the method converges to 1.

Therefore, there is a need to weaken the conditions. In this article, we use only conditions on the operators on the method (2). Therefore, the method can be utilized to solve non-differentiable equations. Furthermore, the results should also demonstrate the isolation of the solution and the bounds of error in advance. This is what is new and what motivates our article. This means extending its applicability, taking advantage of weaker conditions for such methods. In addition, we are also discussing a more interesting case of semi-local convergence. It is obvious that the aforementioned goals can be easily achieved in a similar way for other iterative methods [1, 2, 4, 10–17, 19, 21, 22]. Furthermore, our bounds of error is more precise and our criteria for convergence apply even if the assumptions referred to in the references above are infringed.

The remainder of the article is organized as follows: Analysis of local convergence is provided in Section 2. Majorizing sequences will be introduced and analyzed for the semi-local convergence analysis of 2 in Section 3. Results demonstrating isolation of the solution is discussed in Section

4. Numeric experiments that use convergence results from the previous sections are described in Section 5. The concluding remarks of Section 6 bring this article to an end.

2. Convergence 1: Local

Let $M = [0, +\infty)$. The following conditions are used:

(C_1) There exist continuous and non-decreasing functions (CNF) $\varphi_0 : M \times M \to M$, $\delta_1 : M \to M$, $\delta_2 : M \to M$, a solution $\xi \in \mathcal{B}$ of the equation $G(x) = 0$ and a linear operator $\mathscr{P}$ such that for each $w = x + aG(x)$, $s = x - aG(x)$ and $\mathscr{P}^{-1} \in \mathscr{W}(\mathcal{U})$

$$\|\mathscr{P}^{-1}([w, s; G] - \mathscr{P})\| \leq \varphi_0(\|w - \xi\|, \|s - \xi\|),$$

$$\|w - \xi\| \leq \delta_1(\|x - \xi\|)$$

and

$$\|s - \xi\| \leq \delta_2(\|x - \xi\|).$$

(C_2) The equation $\varphi_0(\delta_1(t), \delta_2(t)) - 1 = 0$ has a smallest positive solution denoted by ρ_0. Let $M_0 = [0, \rho_0)$ and $\mathcal{B}_0 = \mathcal{B} \cap S(\xi, \rho_0)$.

(C_3) There exist CNF $\varphi : M_0 \times M_0 \times M_0 \to M$, $\delta_3 : M_0 \to M$, $\delta_4 : M_0 \to M$, $\varphi_1 : M_0 \times M_0 \times M_0 \times M_0 \to M$, $\varphi_2 : M_0 \to M$ such that for each $x, z \in \mathcal{B}_0$, $u = z + bG(z)$, $v = z - bG(z)$,

$$\|u - \xi\| \leq \delta_3(\|z - \xi\|), \quad \|v - \xi\| \leq \delta_4(\|z - \xi\|),$$

$$\|\mathscr{P}^{-1}([w, s; G] - [x, \xi; G])\| \leq \varphi(\|x - \xi\|, \|w - \xi\|, \|s - \xi\|),$$

$$\|\mathscr{P}^{-1}([w, s; G] - [u, v; G])\| \leq \varphi_1(\|w - \xi\|, \|s - \xi\|, \|u - \xi\|, \|v - \xi\|)$$

and

$$\|\mathscr{P}^{-1}([z, \xi; G] - \mathscr{P})\| \leq \varphi_2(\|z - \xi\|).$$

(C_4) The equations $h_i(t) - 1 = 0$, $i = 1, 2, 3$ have smallest solutions $r_i \in M_0 - \{0\}$, respectively where the functions $h_i : M_0 \to M$ are defined by

$$h_1(t) = \frac{\varphi(t, \delta_1(t), \delta_2(t))}{1 - \varphi_0(\delta_1(t), \delta_2(t))},$$

$$h_2(t) = \frac{\varphi(h_1(t)t, \delta_1(t), \delta_2(t))h_1(t)}{1 - \varphi_0(\delta_1(t), \delta_2(t))}$$

$$\epsilon(t) = \frac{\varphi_1(\delta_1(t), \delta_2(t), \delta_3(h_2(t)t), \delta_4(h_2(t)t)}{1 - \varphi_0(\delta_1(t), \delta_2(t))},$$

$$\lambda(t) = |p + q + r + d - 1| + |p + 2r + 3d|\epsilon(t) + |r + 3d|\epsilon(t)^2 + |d|\epsilon(t)^3,$$

$$h_3(t) = \left[\frac{\varphi(h_2(t)t, \delta_1(t), \delta_2(t))}{1 - \varphi_0(\delta_1(t), \delta_2(t))} + \frac{\lambda(t)(1 + \varphi_2(h_2(t)t))}{1 - \varphi_0(\delta_1(t), \delta_2(t))}\right]h_2(t).$$

Set $r = min\{r_i\}$. Let $M_1 = [0, r)$. It follows by these definitions that for each $t \in M_1$

$$0 \leq \varphi_0(\delta_1(t), \delta_2(t)) < 1,$$

$$0 \leq \epsilon(t),$$

$$0 \leq \lambda(t)$$

and

$$0 \leq h_i(t) < 1.$$

Notice that for $x_0 \in S(\xi, r) - \{\xi\}$ the conditions (C_1)-(C_2) and (C_4) imply

$$\begin{aligned}\|\mathscr{P}^{-1}([w_0, s_0; G] - \mathscr{P})\| &\varphi_0(\|w_0 - \xi\|, \|s_0 - \xi\|) \\ &\leq \varphi_0(\delta_1(r), \delta_2(r)) < 1.\end{aligned}$$

Thus $A_0^{-1} \in \mathscr{W}(\mathcal{U})$ by the Banach lemma on invertible operators [3, 9, 10] and the first iterate y_0 is well-defined by the first sub-step of the method (2).

(C_5) $S[\xi, r] \subset \mathcal{B}$.

The motivation for the development of the functions h_i follows in turn by the estimates

$$\begin{aligned}\|A_n^{-1}\mathscr{P}\| &\leq \frac{1}{1 - \varphi_0(\|w_n - \xi\|, \|s_n - \xi\|)} \leq \frac{1}{1 - \varphi_0(\delta_1(\|x_n - \xi\|), \delta_2(\|x_n - \xi\|))}, \\ y_n - \xi &= A_n^{-1}(A_n - [x_n, \xi; G])(x_n - \xi), \\ \|y_n - \xi\| &\leq \frac{\varphi(\|x_n - \xi\|, \|w_n - \xi\|, \|s_n - \xi\|)\|x_n - \xi\|}{1 - \varphi_0(\delta_1(\|x_n - \xi\|), \delta_2(\|x_n - \xi\|))} \\ &\leq h_1(\|x_n - \xi\|)\|x_n - \xi\| \leq \|x_n - \xi\| < r.\end{aligned}$$

Similarly,

$$\begin{aligned}\|z_n - \xi\| &\leq \frac{\varphi(\|y_n - \xi\|, \|w_n - \xi\|, \|s_n - \xi\|)\|y_n - \xi\|}{1 - \varphi_0(\delta_1(\|x_n - \xi\|), \delta_2(\|x_n - \xi\|))} \\ &\leq h_2(\|x_n - \xi\|)\|x_n - \xi\| \leq \|x_n - \xi\|, \\ x_{n+1} - \xi &= z_n - \xi - A_n^{-1}G(z_n) - [(p + q + r + d - 1)I + (q + 2r + 3d)(A_n^{-1}Q_n - I) \\ &\quad + (r + 3d)(A_n^{-1}Q_n - I)^2 + d(A_n^{-1}Q_n - I)^3]A_n^{-1}G(z_n)\end{aligned}$$

which can be shortened for

$$\begin{aligned}D_n &= A_n^{-1}(Q_n - A_n), \\ T_n &= (p + q + r + d - 1)I + (p + 2r + 3d)D_n + (r + 3d)D_n^2 + dD_n^3.\end{aligned}$$

Thus

$$x_{n+1} - \xi = A_n^{-1}(A_n - [z_n, \xi; G])(z_n - \xi) - T_nA_n^{-1}G(z_n).$$

But,

$$\begin{aligned}\|D_n\| &\le \|A_n^{-1}\mathscr{P}\|\|\mathscr{P}^{-1}(Q_n - A_n)\| \\ &\le \frac{\varphi_1(\|w_n-\xi\|,\|s_n-\xi\|,\|u_n-\xi\|,\|v_n-\xi\|)}{1-\varphi_0(\|w_n-\xi\|,\|s_n-\xi\|)} = \epsilon_n, \\ \|T_n\| &\le |p+q+r+d-1| + |p+2r+3d|\epsilon_n + |r+3d|\epsilon_n^2 + |d|\epsilon_n^3 = \lambda_n,\end{aligned}$$

leading to

$$\begin{aligned}\|x_{n+1}-\xi\| &\le \left[\frac{\varphi(\|z_n-\xi\|,\|w_n-\xi\|,\|s_n-\xi\|)}{1-\varphi_0(\delta_1(\|x_n-\xi\|),\delta_2(\|x_n-\xi\|)} + \frac{\lambda_n(1+\varphi_2(\|z_n-\xi\|))}{1-\varphi_0(\delta_1(\|x_n-\xi\|),\delta_2(\|x_n-\xi\|))}\right]\|z_n-\xi\| \\ &\le h_3(\|x_n-\xi\|)\|x_n-\xi\| < \|x_n-\xi\|.\end{aligned}$$

Hence, the iterates $\{x_n\}, \{y_n\}, \{z_n\} \subset S(\xi, r)$ and there exists $c = h_3(\|x_0-\xi\|) \in [0,1)$ such that

$$\|x_{n+1}-\xi\| \le c\|x_n-\xi\| < r,$$

from which it follows that $\lim_{n\to\infty} x_n = \xi$.

Therefore, we achieve the following local convergence result for the method (2).

THEOREM 2.1. *Under the assumptions* (C_1)*-*(C_5)*,* $\{x_n\} \subset S(\xi, r)$ *and* $\lim_{n\to+\infty} x_n = \xi$ *provided that* $x_0 \in S(\xi,r) - \{\xi\}$*.*

REMARK 2.2. *The functions* δ_j*,* $j = 1,2,3,4$ *are left uncluttered in the Theorem 2.1. A possible choice for the first function* δ_1 *is motivated by the estimate*

$$\begin{aligned}w-\xi = x-\xi+aF(x) &= (I + a[x,\xi;F])(x-\xi) \\ &= (I + a\mathscr{P}\mathscr{P}^{-1}([x,\xi;G] - \mathscr{P} + \mathscr{P}))(x-\xi), \\ &= [(I+a\mathscr{P}) + a\mathscr{P}\mathscr{P}^{-1}([x,\xi;G]-\mathscr{P})](x-\xi), \\ \|w-\xi\| &\le [\|I+a\mathscr{P}\| + |a|\|\mathscr{P}\|\varphi_0(\|x-\xi\|)]\|x-\xi\|.\end{aligned}$$

Thus, we can choose

$$\delta_1(t) = [\|I+a\mathscr{P}\| + |a|\|\mathscr{P}\|\varphi_0(t)]t.$$

Similarly, we can choose

$$\delta_2(t) = [\|I-a\mathscr{P}\| + |a|\|\mathscr{P}\|\varphi_0(t)]t,$$

$$\delta_3(t) = [\|I+b\mathscr{P}\| + |b|\|\mathscr{P}\|\varphi_0(h_2(t)t)]h_2(t)t$$

and

$$\delta_4(t) = [\|I-b\mathscr{P}\| + |b|\|\mathscr{P}\|\varphi_0(h_2(t)t)]h_2(t)t.$$

Two possible choices for the linear operator $\mathscr{P}$ *are:*

The differentiable option : $\mathscr{P} = G'(\xi)$ *and*

The non-differentiable option : $\mathscr{P} = [x_0, x_{-1}; G]$*. Other choices are possible [18].*

3. Convergence 2: Semi-Local

The role of ξ, "φ" is replaced by x_0, "ψ" as follows. Assume:

(H_1) There exist CNF $\psi_0 : M \times M \to M$, $x_0 \in \mathcal{B}$, $g_1 : M \to M, g_2 : M \to M$ and a linear operator $\mathscr{P}$ such that for $x \in \mathcal{B}$

$$w = x + aG(x), \quad s = x - aG(x),$$
$$\|w - x_0\| \leq g_1(\|x - x_0\|), \quad \|s - x_0\| \leq g_2(\|x - x_0\|)$$
$$\|\mathscr{P}^{-1}([w, s; G] - \mathscr{P})\| \leq \psi_0(\|w - x_0\|, \|s - x_0\|).$$

(H_2) The equation $\psi_0(g_1(t), g_2(t)) - 1 = 0$ has a smallest positive solution denoted by ρ.
Let $M_2 = [0, \rho)$ and $\mathcal{B}_1 = \mathcal{B} \cap S(x_0, \rho)$.
Notice that $\|\mathscr{P}^{-1}([w_0, s_0; G] - \mathscr{P})\| \leq \psi(0, 0) < 1$.
Thus, $A_0^{-1} \in \mathscr{W}(\mathcal{U})$ and the iterate y_0 is well-defined by the first sub-step of the method (2).

(H_3) There exists CNF $g_3 : M_2 \to M$, $g_4 : M_2 \to M$, $\psi_1, \psi_2 : M_2 \times M_2 \times M_2 \times M_2 \to M$ such that for each $x, y \in \mathcal{B}_1$

$$\|u - x_0\| \leq g_3(\|z - x_0\|, \quad \|v - x_0\| \leq g_4(\|z - x_0\|)$$
$$\|\mathscr{P}^{-1}([y, x; G] - [w, s; G])\| \leq \psi_1(\|x - x_0\|, \|y - x_0\|, \|w - x_0\|, \|s - x_0\|)$$
and
$$\|\mathscr{P}^{-1}([w, s; G] - [u, v; G])\| \leq \psi_2(\|w - x_0\|, \|s - x_0\|, \|u - x_0\|, \|v - x_0\|).$$

Define the real sequence $\{\alpha_n\}$ for $\alpha_0 = 0, \beta_0 \geq \|A_0^{-1}G(x_0)\|$, and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned}
\gamma_n &= \beta_n + \frac{\psi_1(\alpha_n, \beta_n, g_1(\alpha_n), g_2(\alpha_n))(\beta_n - \alpha_n)}{1 - \psi_0(g_1(\alpha_n), g_2(\alpha_n))},\\
\epsilon_{n,1} &= \frac{\psi_2(g_1(\alpha_n), g_2(\alpha_n), g_3(\gamma_n), g_4(\gamma_n))}{1 - \psi_0(g_1(\alpha_n), g_2(\alpha_n))},\\
\lambda_{n,1} &= |p + q + r + d| + |p + 2r + 3d|\epsilon_{n,1} + |r + 3d|\epsilon_{n,1}^2 + |d|\epsilon_{n,1}^3,\\
\alpha_{n+1} &= \gamma_n + \frac{\psi_1(\beta_n, \gamma_n, g_1(\alpha_n), g_2(\alpha_n))(\gamma_n - \beta_n)\lambda_{n,1}}{1 - \psi_0(g_1(\alpha_n), g_2(\alpha_n)},\\
\delta_{n+1} &= \psi_1(\alpha_n, \alpha_{n+1}, g_1(\alpha_n), g_2(\alpha_n))(\alpha_{n+1} - \alpha_n) + (1 + \psi_0(g_1(\alpha_n), g_2(\alpha_n))(\alpha_{n+1} - \beta_n)\\
\beta_{n+1} &= \alpha_{n+1} + \frac{\delta_{n+1}}{1 - \psi_0(g_1(\alpha_{n+1}, g_2(\alpha_{n+1})}.
\end{aligned} \tag{3}$$

A convergence set of conditions for the sequence $\{\alpha_n\}$ is given for each $n = 0, 1, 2, \ldots$.

(H_4) $\psi_0(g_1(\alpha_n), g_2(\alpha_n)) < 1$ and $\alpha_n \leq \alpha < \rho$.

It follows by this condition and (3) that $0 \leq \alpha_n \leq \beta_n \leq \gamma_n \leq \alpha_{n+1}$ and there exists $\alpha^* \in [0, \alpha]$ such that $\lim_{n\to\infty} \alpha_n = \alpha^*$.

and

(H_5) $S[x_0, \alpha^*] \subset \mathcal{B}$.

As in the local case the motivation for the introduction of the sequence $\{\alpha_n\}$ follows in turn to form the estimates:

$$z_n - y_n = -A_n^{-1}G(y_n),$$

but

$$G(y_n) = G(y_n) - G(x_n) - A_n(y_n - x_n) = ([y_n, x_n; G] - A_n)(y_n - x_n),$$

so

$$\begin{aligned}
\|z_n - y_n\| &\leq \frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|w_n - x_0\|, \|s_n - x_0\|)\|y_n - x_n\|}{1 - \psi_0(\|w_n - x_0\|, \|s_n - x_0\|)} \\
&\leq \gamma_n - \beta_n, \\
\|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \leq \gamma_n - \beta_n + \beta_n - \alpha_0 = \gamma_n < a^*, \\
x_{n+1} - z_n &= -T_n A_n^{-1} G(z_n), \\
\|x_{n+1} - z_n\| &\leq \frac{\lambda_{n,1}\psi_1(\|y_n - x_0\|, \|z_n - x_0\|, \|w_n - x_0\|, \|s_n - x_0\|)\|z_n - y_n\|}{1 - \psi_0(\|w_n - x_0\|, \|s_n - x_0\|)} \\
&\leq \alpha_{n+1} - \gamma_n,
\end{aligned}$$

since

$$\begin{aligned}
T_{n,1} &= (p + q + r + d)I + (q + 2r + 3d)D_n + (r + 3d)D_n^2 + dD_n^3, \\
\|D_n\| &\leq \frac{\psi_2(\|w_n - x_0\|, \|s_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|)}{1 - \psi_0(\|w_n - x_0\|, \|s_n - x_0\|)}, \\
\|T_{n,1}\| &\leq \lambda_{n,1}
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x_0\| &\leq \|x_{n+1} - z_n\| + \|z_n - x_0\| \leq \alpha_{n+1} - \gamma_n + \gamma_n - \alpha_0 \\
&= \alpha_{n+1} < \alpha^*.
\end{aligned}$$

Also,

$$\begin{aligned}
G(x_{n+1}) &= G(x_{n+1}) - G(x_n) - A_n(y_n - x_n) \\
&= G(x_{n+1}) - G(x_n) - A_n(x_{n+1} - x_n) + A_n(x_{n+1} - y_n), \\
\|\mathscr{P}^{-1}G(x_{n+1})\| &\leq \psi_1(\|x_n - x_0\|, \|x_{n+1} - x_0\|, \|w_n - x_0\|, \|s_n - x_0\|)\|x_{n+1} - x_n\| \\
&\quad + (1 + \psi_0(\|w_n - x_0\|, \|s_n - x_0\|))\|x_{n+1} - y_n\| = \bar{\delta}_{n+1} \leq \delta_{n+1}, \qquad (4) \\
\|y_{n+1} - x_{n+1}\| &\leq \|A_{n+1}^{-1}\mathscr{P}\|\|\mathscr{P}^{-1}G(x_{n+1}\| \\
&\leq \frac{\bar{\delta}_{n+1}}{1 - \psi_0(\|w_{n+1} - x_0\|, \|s_{n+1} - x_0\|)} \leq \beta_{n+1} - \alpha_{n+1}
\end{aligned}$$

and

$$\|y_{n+1} - x_0\| \le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \le \beta_{n+1} - \alpha_{n+1} + \alpha_{n+1} - \alpha_0$$
$$= \beta_{n+1} < \alpha^*.$$

Therefore, the sequence $\{x_n\}$ is complete in Banach space $\mathcal{U}$. Hence, there exists $\xi = \lim_{n\to\infty} x_n$ and by (4) $G(\xi) = 0$.

Then, we achieve the following semi-local convergence result for the method (2).

THEOREM 3.1. *Under the conditions (H_1)-(H_5) the sequence $\{x_n\}$ converges to a solution $\xi \in S[x_0, a^*]$ of the equation $G(x) = 0$.*

REMARK 3.2. *A possible choice for the functions g_j, $j = 1, 2, 3, 4$ follows as in the local case. We have in turn*

$$w - x_0 = x - x_0 + a(G(x) - G(x_0) + G(x_0))$$
$$= [(I + a\mathscr{P}) + a\mathscr{P}\mathscr{P}^{-1}([x, x_0; G] - \mathscr{P})](x - x_0) + aG(x_0),$$

lead to the choice

$$g_1(t) = [\|I + a\mathscr{P}\| + |a|\|\mathscr{P}\|\psi_3(t)]t + |a|\|G(x_0)\|$$

provided that for some CNF $\psi_3 : M_1 \to M$, $x \in \mathcal{B}$

$$\|\mathscr{P}^{-1}([x, x_0; G] - \mathscr{P})\| \le \psi_3(\|x - x_0\|).$$

Similarly, we define

$$g_2(t) = [\|I - a\mathscr{P}\| + |a|\|\mathscr{P}\|\psi_3(t)]t + |a|\|G(x_0)\|,$$
$$g_3(t) = [\|I + b\mathscr{P}\| + |a|\|\mathscr{P}\|\psi_3(t)]t + |b|\|G(x_0)\|,$$

and

$$g_4(t) = [\|I - b\mathscr{P}\| + |b|\|\mathscr{P}\|\psi_3(t)]t + |b|\|G(x_0)\|.$$

The options for $\mathscr{P}$ are:

$$\mathscr{P} = G'(x_0) \quad or \quad \mathscr{P} = [x_0, x_{-1}; G].$$

Other options exist [10].

4. Isolation of a Solution

We first present the uniqueness result for the local convergence case.

PROPOSITION 4.1. *There exists a solution $v^* \in S(\xi, \rho_2)$ of the equation $G(x) = 0$ for some $\rho_2 > 0$;*
The last condition in (C_3) holds in the ball $S(\xi, \rho_2)$ and there exists $\rho_3 \ge \rho_2$ such that

$$\psi_2(\rho_3) < 1. \tag{5}$$

Set $\mathcal{B}_3 = \mathcal{B} \cap S[\xi, \rho_3]$. Then, ξ is the only solution of the equation G(x)=0 in the set $\mathcal{B}_3$.

Proof. Let $v^* \neq \xi$. Then, the divided difference $V = [\xi, v^*; G]$ is well-defined. Using the last condition in (C_3) and (5), we obtain in turn that

$$\|\mathscr{P}^{-1}(V - \mathscr{P})\| \leq \psi_2(\|v^* - \xi\|) \leq \psi_2(\rho_3) < 1,$$

so, $V^{-1} \in \mathscr{W}(\mathcal{U})$ and from the approximation

$$v^* - \xi = V^{-1}(G(v^*) - G(\xi)) = V^{-1}(0) = 0,$$

we deduce $v^* = \xi$. □

PROPOSITION 4.2. *Assume:*

There exists a solution $v^* \in S(x_0, \rho_4)$ of the equation $G(x) = 0$ for some $\rho_4 > 0$;

The condition (H_1) holds on the ball $S(x_0, \rho_4)$ and there exist $\rho_5 \geq \rho_4$ such that

$$\varphi_0(\rho_4, \rho_5) < 1. \tag{6}$$

Set $\mathcal{B}_4 = \mathcal{B} \cap S[x_0, \rho_5]$. Then, v^* is the only solution of the equation $G(x) = 0$ in the set $\mathcal{B}_4$.

Proof. Let $z^* \in \mathcal{B}_4$ with $G(z^*) = 0$ and $z^* \neq v^*$. Define the linear operator $F = [v^*, z^*; G]$. Then, by the condition (H_1) and (6)

$$\|\mathscr{P}^{-1}(F - \mathscr{P})\| \leq \varphi_0(\|v^* - x_0\|, \|z^* - x_0\|) \leq \varphi_0(\rho_4, \rho_5) < 1,$$

thus, again $v^* = z^*$. □

REMARK 4.3. (i) *The limit point α^* can be replaced by ρ in the condition (H_5).*

(ii) *Under all the assumptions (H_1)-(H_5), let $v^* = \xi$ and $\rho_4 = \alpha^*$ in Proposition 4.2.*

5. Experiments

EXAMPLE 5.1. *Consider the system of differential equations with*

$$G_1'(w_1) = e^{w_1}, \quad G_2'(w_2) = (e-1)w_2 + 1, \quad G_3'(w_3) = 1$$

subject to $G_1(0) = G_2(0) = G_3(0) = 0$. Let $G = (G_1, G_2, G_3)$. Let $\mathcal{U} = \mathbb{R}^3$ and $\mathcal{B} = U[0, 1]$. Then $\xi = (0, 0, 0)^T$ is a root. Let function G on $\mathcal{B}$ for $w = (w_1, w_2, w_3)^T$ be

$$G(w) = (e^{w_1} - 1, \frac{e-1}{2} w_2^2 + w_2, w_3)^T.$$

This definition gives

$$G'(w) = \begin{bmatrix} e^{w_1} & 0 & 0 \\ 0 & (e-1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, by the definition of G it follows that $G'(\xi) = 1$. Let $\mathscr{P} = G'(\xi)$ and $[x, y; G] = \int_0^1 G'(x + \theta(y - x))d\theta$. Then, for $a = b = 1$, the conditions (C_1)-(C_5) are validated by Remark 2.2 provided that

$$\begin{aligned}
\delta_1(t) &= (2 + \frac{1}{2}(e-1)t)t,\\
\delta_2(t) &= \frac{1}{2}(e-1)t^2,\\
\varphi_0(t_1, t_2) &= \frac{1}{2}(e-1)(\delta_1(t_1) + \delta_2(t_2))\\
\delta_3(t) &= (2 + \frac{1}{2}(e-1)h_2(t))h_2(t)t,\\
\delta_4(t) &= \frac{1}{2}(e-1)h_2(t)^2t^2,\\
\varphi(t_1, t_2, t_3) &= \frac{1}{2}(e-1)(t_1 + \delta_1(t_2) + \delta_2(t_3))\\
\varphi_1(t_1, t_2, t_3, t_4) &= \frac{1}{2}(e-1)[\delta_1(t_1) + \delta_2(t_2) + \delta_3(t_3) + \delta_4(t_4)]\\
&\text{and}\\
\varphi_2(t) &= \frac{1}{2}(e-1)t.
\end{aligned}$$

By solving, we get $\rho_0 = 0.426037$ and hence $M_0 = [0, \rho_0)$. The radii are obtained as $r_1 = 0.204146$, $r_2 = 0.134409$ and $r_3 = 0.126891$. Therefore, by the definition $r = min\{r_i\}$, we get the radius of convergence, $r = 0.126891$.

REMARK 5.2. *A non-differentiable non-linear system is solved using the method (2), where the divided difference is defined by the 2×2 matrix given for $\bar{t} = (t_1, t_2) \in \mathbb{R} \times \mathbb{R}$, $\tilde{t} = (t_3, t_4) \in \mathbb{R} \times \mathbb{R}$ and $G = (G_1, G_2)$ by*

$$[\bar{t}, \tilde{t}; G]_{i,1} = \frac{G_i(t_3, t_4) - G_i(t_1, t_4)}{t_3 - t_1}, \quad t_3 \neq t_1$$

and

$$[\bar{t}, \tilde{t}; G]_{i,2} = \frac{G_i(t_1, t_4) - G_i(t_1, t_2)}{t_4 - t_2}, \quad t_4 \neq t_2.$$

Otherwise, we set $[\cdot, \cdot; G] = \mathbf{0}$.

The actual example is given below

EXAMPLE 5.3. *Let us solve the non-linear and non-differentiable system given as*

$$\begin{aligned}
3t_1^2t_2 + t_2^2 - 1 + |t_1 - 1| &= 0\\
t_1^4 + t_1t_2^3 - 1 + |t_2| &= 0.
\end{aligned}$$

Then, we set $G = (G_1, G_2)$*, where*

$$G_1(t_1, t_2) = 3t_1^2 t_2 + t_2^2 - 1 + |t_1 - 1|$$
$$G_2(t_1, t_2) = t_1^4 + t_1 t_2^3 - 1 + |t_2|$$

Choose the initial points $(5, 5)$ *and* $(1, 0)$*. Then, using the aforementioned divided difference and the method (2), we obtain the solution* $\xi = (x_1^*, x_2^*)$ *after three iterations with*

$$x_1^* = 0.894655074977661$$

and

$$x_2^* = 0.327826643198819.$$

EXAMPLE 5.4. *We consider the system of* 25 *equations*

$$\sum_{j=1, j\neq i}^{25} x_j - e^{-x_i} = 0, \quad 1 \leq i \leq 25,$$

with initial point $x_0 = \{1.5, 1.5, \ldots, 1.5\}^T$*. Then, applying method (2) we get the solution* $\xi = \{0.04003162719010837\cdots, 0.04003162719010837\cdots, \ldots, 0.04003162719010837\cdots\}^T$ *after* 4 *iterations.*

6. Conclusion

A new procedure has been developed to demonstrate both local and semi-local convergence analysis of high-order convergence methods, using only derivatives that appear on the methodology. Previous works have proven convergence based on the existence of high-order derivatives that may not be present in the methodology. Hence, it has been a limitation of their applicability. This procedure also offers error limits and uniqueness results that were not available before. Moreover, this procedure is general in the sense that it is not dependent on the method itself. This is the reason why it may be used in the same way to broaden the scope of other methods of higher order, such as single and multi-step methods [1, 2, 4, 10–17, 19, 21, 22].

References

[1] A. Cordero, J. L. Hueso, E. Martínez, J. R. Torregrosa, A modified Newton-Jarratt's composition, Numer. Algorithms 55 (2010) 87-99. https://doi.org/10.1007/s11075-009-9359-z.

[2] A. M. Ostrowski, Solutions of Equations and System of Equations, Academic Press (1960) New York.

[3] F. A. Potra, V. Ptak, Nondiscrete Induction and Iterarive Processes. Pitman Publishing (1984) Boston.

[4] H. Ren, Q. Wu, W. Bi, A class of two-step Steffensen type methods with fourth-order convergence, Appl. Math. Comput. 209 (2009) 206–210. https://doi.org/10.1016/j.amc.2008.12.039.

[5] I. K. Argyros, J. A. John, J. Jayaraman, On the semi-local convergence of a sixth order method in Banach space, J. Numer. Anal. Approx. Theory 51 (2022) 144–154. https://doi.org/10.33993/jnaat512-1284.

[6] I. K. Argyros, J. A. John, J. Jayaraman, S. Regmi, Extended local convergence for the Chebyshev method under the Majorant condition, Asian Res. J. Math. 18 (2022) 102–109. https://doi.org/10.9734/arjom/2022/v18i12629.

[7] I. K. Argyros, S. Regmi, J. A. John, J. Jayaraman, Extended convergence for two sixth order methods under the same weak conditions, Foundations 3 (2023) 127–139. https://doi.org/10.3390/foundations3010012.

[8] J. A. John, J. Jayaraman, I. K. Argyros, Local Convergence of an Optimal Method of Order Four for Solving Non-Linear System, Int. J. Appl. Comput. Math. 8 (2022) 194. https://doi.org/10.1007/s40819-022-01404-3.

[9] J. F. Steffensen, Remarks on iteration, Scand. Actuar. J. 16 (1933) 64–72. https://doi.org/10.1080/03461238.1933.10419209.

[10] J. F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, New Jersey (1964).

[11] J. M Ortega, W. C Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, (1970). https://doi.org/10.1137/1.9780898719468.fm.

[12] J. R. Sharma, H. Arora, An efficient derivative free iterative method for solving systems of nonlinear equations, Appl. Anal. Discrete Math. 7 (2013) 390–403. https://doi.org/10.2298/AADM130725016S.

[13] J. R. Sharma, H. Arora, A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, Numer. Algorithms 4 (2014) 917–933. https://doi.org/10.1007/s11075-014-9832-1.

[14] J. R. Sharma, H. Arora, Efficient derivative-free numerical methods for solving systems of nonlinear equations, Comput. Appl. Math. 35 (2016) 269–284. https://doi.org/10.1007/s40314-014-0193-0.

[15] J. R. Sharma, P. Gupta, Efficient family of Traub-Steffensen-type methods for solving systems of nonlinear equations. Adv. Numer. Anal. 2014 (2014) 152187. https://doi.org/10.1155/2014/152187.

[16] M. Grau-Sánchez, À. Grau, M. Noguera, Frozen divided difference scheme for solving systems of nonlinear equations, J. Comput. Appl. Math. 235 (2011) 1739-1743. https://doi.org/10.1016/j.cam.2010.09.019.

[17] M. Grau-Sánchez, M. Noguera, S. Amat, On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods, J. Comput. Appl. Math. 237 (2013) 363-372. https://doi.org/10.1016/j.cam.2012.06.005.

[18] M. Narang, S. Bhatia, V. Kanwar, New efficient derivative free family of seventh-order methods for solving systems of nonlinear equations, Numer. Algorithms 76 (2017) 283-307. https://doi.org/10.1007/s11075-016-0254-0.

[19] Q. Zheng, P. Zhao, F. Huang, A family of fourth-order Steffensen-type methods with the applications on solving nonlinear ODEs, Appl. Math. Comput. 217 (2011) 8196–8203. https://doi.org/10.1016/j.amc.2011.01.095.

[20] S. Regmi, I. K. Argyros, J. A. John, J. Jayaraman, Extended convergence of two multi-step iterative methods, Foundations 3 (2023) 140–153. https://doi.org/10.3390/foundations3010013.

[21] X. Wang, T. Zhang, A family of Steffensen type methods with seventh-order convergence, Numer. Algorithms 62 (2013) 429–444. https://doi.org/10.1007/s11075-012-9597-3.

[22] Z. Liu, Q, Zheng, P. Zhao, A variant of Steffensen's method of fourth-order convergence and its applications, Appl. Math. Comput. 216 (2010) 1978-1983. https://doi.org/10.1016/j.amc.2010.03.028.