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On Certain Properties of a Degenerate Sigmoid Function

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Abstract. In this paper, we introduce a degenerate sigmoid function. By employing analytical techniques, we present some properties such as logarithmic concavity, monotonicity and inequalities of the new function.

1. Introduction

It is known that, what is currently referred to as the logistic equation or the S-shaped curve was first introduced by Verhulst (see [17]). It maps a very large input domain to a small range of output of real numbers between 0 and 1. It is a one - to- one functioon and increases monotonically (see [8]). The sigmoid function, also known in the literature as the Sigmoidal curve or standard logistic function is defined as (see [13]),

$$S(t) = \frac{e^t}{1+e^t} = \frac{1}{1+e^{-t}}, \quad t \in (-\infty, \infty), \tag{1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{t}{2}\right), \quad t \in (-\infty, \infty). \tag{2}$$

It has the following as its first and second derivatives

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$$S'(t) = \frac{e^t}{(1+e^t)^2} = S(t)(1-S(t)), \tag{3}$$

$$S''(t) = \frac{e^t\left(1-e^t\right)}{(1+e^t)^3} = S(t)(1-S(t))(1-2S(t)), \tag{4}$$

for all $t \in (-\infty, \infty)$.

The sigmoid function is used in a wide range of scientific disciplines, including probability and statistics, biology, demography, machine learning, population dynamics, ecology, and mathematical psychology(see [7], [16]). In the business sector, the sigmoid function has been utilized to analyze performance growth in manufacturing and service management (see [10]). At each neuron's output, the function serves as an activation function in artificial neural networks (see [12], [18], [15]) and the references therein.

In addition, the function is used in medicine to research pharmacokinetic responses and mimic tumor development (see [11]). In [5], the site index of unmanaged loblolly and slash pine plantations in East Texas is predicted using a generic variant of the sigmoid function. It is also used in computer graphics and image processing to improve picture contrast (see [4], [9], [6]). It is clear from the above applications of the sigmoid function that, further research needs to be conducted on this very important function to unearth more of its properties and potential applications.

Recently, in [13], the author studied properties such as super multiplicativity, subadditivity, convexity and inequalities of the sigmoid function.

In this paper, a degenerate sigmoid function is introduced and properties such as logarithmic concavity, monotonicity and inequlities involving the function are provided. We start with the following definitions and lemmas.

2. Some Definitions and Lemmas

Definition 2.1. [1] A function $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a mean function if it satisfies the following.

(1) $M(r, t) = M(t, r)$,
(2) $M(t, t) = t$,
(3) $r < M(r, t) < t$, for $r < t$,
(4) $M(\eta r, \eta t) = \eta M(r, t)$, for $\eta > 0$.

There are many well-known mean functions in the literature. Amongst them are the following.

(1) Arithmetic mean: $A(r, t) = \frac{r+t}{2}$,
(2) Geometric mean: $G(r, t) = \sqrt{rt}$,

(3) Harmonic mean: $H(r,t) = \frac{1}{A\left(\frac{1}{r},\frac{1}{t}\right)} = \frac{2rt}{r+t}$,

(4) Logarithmic mean: $L(r,t) = \frac{r-t}{\ln r - \ln t}$, for $r \neq t$ and $L(t,t) = t$,

(5) Identric mean: $I(r,t) = \frac{1}{e}\left(\frac{r^r}{t^t}\right)^{\frac{1}{r-t}}$, for $r \neq t$ and $I(t,t) = t$.

Definition 2.2. [1] Let $g : I \subseteq (0,\infty) \to (0,\infty)$ be a continuous function and U and V be any two mean functions. Then, g is said to be $UV-$convex ($UV-$concave) if

$$g(U(r,t)) \leq (\geq) V(g(r), g(t)),$$

for all $r, t \in I$.

Lemma 2.3. *[1] Let $f : I \subseteq (0,\infty) \to (0,\infty)$ be a differentiable function. Then*

(1) *f is AG-convex(or concave) if and only if $\frac{f'(t)}{f(t)}$ is increasing(or decreasing) for all $t \in I$.*

(2) *f is AH-convex(or concave) if and only if $\frac{f'(t)}{f(t)^2}$ is increasing(or decreasing) for all $t \in I$.*

Lemma 2.4. *[2] Let $f : I \subseteq (b,\infty) \to (-\infty,\infty)$ with $b \geq 0$. If the function defined by $g(t) = \frac{f(t)-1}{t}$ is increasing on (b,∞), then the function $h(t) = f\left(t^2\right)$ satisfies the Grumbaum-type inequality*

$$1 + h\left(z^2\right) \geq h\left(r^2\right) + h\left(t^2\right), \tag{5}$$

where $r, t \geq b$ and $z^2 = r^2 + t^2$. If g is decreasing, then the inequality (5) is reversed.

3. Main Results

Definition 3.1. The degenerate sigmoid function is defined for $\lambda \in (0,\infty)$ and $t \in (-\infty,\infty)$ as

$$S_\lambda(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}} \tag{6}$$

$$= \frac{1}{1+(1+\lambda t)^{-\frac{1}{\lambda}}} \tag{7}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh_\lambda\left(\frac{t}{2}\right). \tag{8}$$

It is clear that, taking the limit of $S_\lambda(t)$ as $\lambda \to 0$, then $S_\lambda(t) \to S(t)$.
The first derivative of the degenerate sigmoid function is given as

$$S'_\lambda(t) = \frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2} > 0, \tag{9}$$

for all $t \in (-\infty,\infty)$ and $\lambda \in (0,\infty)$.

The degenerate sigmoid function satisfies the following identities.

$$S_\lambda(t)+S_\lambda(-t)=1, \tag{10}$$

$$S_\lambda(t)\,S_\lambda(-t)=(1+\lambda t)\,S_\lambda'(t)\,, \tag{11}$$

$$S_\lambda'(t)=S_\lambda'(-t)\,, \tag{12}$$

$$\lim_{t\to\infty} S_\lambda(t)=1, \tag{13}$$

$$\lim_{t\to 0} S_\lambda(t)=\frac{1}{2}, \tag{14}$$

$$\lim_{t\to 0} S_\lambda'(t)=\frac{1}{4}, \tag{15}$$

$$\lim_{t\to\infty} S_\lambda'(t)=0. \tag{16}$$

Theorem 3.2. *The function $S_\lambda(t)$ is AG-concave on $(0,\infty)$. In other words, for all $r,t,\lambda\in(0,\infty)$, the inequality*

$$S_\lambda\left(\frac{r+t}{2}\right)\geq\left[S_\lambda(r)\,S_\lambda(t)\right]^{\frac{1}{2}} \tag{17}$$

is satisfied.

Proof. We have

$$\begin{aligned}\frac{S_\lambda'(t)}{S_\lambda(t)}=&\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}\right)\left(\frac{1+(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}}\right)\\ =&\frac{1}{(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}}\end{aligned}$$

and

$$\left(\frac{S_\lambda'(t)}{S_\lambda(t)}\right)'=-\frac{\lambda+(1+\lambda)\,(1+\lambda t)^{\frac{1}{\lambda}}}{\left[(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}\right]^2}<0, \tag{18}$$

which imlplies that $\frac{S_\lambda'(t)}{S_\lambda(t)}$ is decreasing on $(0,\infty)$. Hence, by Lemma 2.3(1), we obtain the desired result (17). ∎

Theorem 3.3. *The function $S_\lambda(t)$ is AH-concave on $(0,\infty)$. In other words, for all $r,t,\lambda\in(0,\infty)$, the inequality*

$$S_\lambda\left(\frac{r+t}{2}\right)\geq\frac{2S_\lambda(r)\,S_\lambda(t)}{S_\lambda(r)+S_\lambda(t)} \tag{19}$$

is valid.

Proof. Now we have

$$\begin{aligned}\frac{S_\lambda'(t)}{S_\lambda(t)^2} &= \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}\right)\left(\frac{\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}{(1+\lambda t)^{\frac{2}{\lambda}}}\right)\\ &=\frac{1}{(1+\lambda t)(1+\lambda t)^{\frac{1}{\lambda}}}\\ &=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}+1}}\end{aligned}$$

and

$$\left(\frac{S_\lambda'(t)}{S_\lambda(t)^2}\right)' = -\frac{(1+\lambda)(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{2}{\lambda}+2}} < 0.$$

By Lemma 2.3(2), we conclude that $S_\lambda(t)$ is AH-concave on $(0,\infty)$. This implies inequality (19). □

Theorem 3.4. *The function $S_\lambda(t)$, for $r,t,\lambda\in(0,\infty)$ and $z^2=r^2+t^2$, satisfies the Grunbaum-type inequality*

$$1+S_\lambda\left(z^2\right)\geq S_\lambda\left(r^2\right)+S_\lambda\left(t^2\right). \tag{20}$$

Proof. Let $h(t)$ be defined for $t,\lambda\in(0,\infty)$ as $h(t)=\frac{S_\lambda(t)-1}{t}$. This implies

$$\begin{aligned}h(t) &=\frac{\frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}-1}{t}\\ &=-\frac{1}{t+t(1+\lambda t)^{\frac{1}{\lambda}}}.\end{aligned}$$

Differentiating $h(t)$, we have

$$h'(t)=\frac{1+(1+\lambda t)^{\frac{1}{\lambda}}+t(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left[t+t(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}>0,$$

which implies that $h(t)$ is increasing. By applying Lemma 2.4, we obtain the desired result (20). □

Theorem 3.5. *For $\lambda\in(0,\infty)$, the function $S_\lambda(t)$ satisfies the inequalities*

$$S_\lambda^2(r+t)\geq S_\lambda(r)\,S_\lambda(t),\quad r,t\in[0,\infty) \tag{21}$$

and

$$S_\lambda^2(r+t)\leq S_\lambda(r)\,S_\lambda(t),\quad r,t\in(-\infty,0]. \tag{22}$$

Equality holds if $r=t=0$.

Proof. Let $r, t \in [0, \infty)$ and $\lambda \in (0, \infty)$. Recall that $S_\lambda(t)$ is increasing. Thus we have

$$S_\lambda(r+t) \geq S_\lambda(r) > 0, \tag{23}$$

$$S_\lambda(r+t) \geq S_\lambda(t) > 0, \tag{24}$$

since $r+t \geq r$ and $r+t \geq t$. Now by multiplying (23)and (24), we obtain the desired result (21).
Next, let $r, t \in (-\infty, 0]$ and $\lambda \in (0, \infty)$, we have

$$0 < S_\lambda(r+t) \leq S_\lambda(r), \tag{25}$$

$$0 < S_\lambda(r+t) \leq S_\lambda(t), \tag{26}$$

since $r+t \leq r$ and $r+t \leq t$. By multiplying the inequalities (25) and (26), we have the desired result. □

Theorem 3.6. *The function $S_\lambda(t)$, for $\lambda \in (0, \infty)$, satisfies the inequalities*

$$S_\lambda^2(rt) \leq S_\lambda(r) S_\lambda(t), \quad r, t \in [0, 1] \tag{27}$$

and

$$S_\lambda^2(rt) \geq S_\lambda(r) S_\lambda(t), \quad r, t \in [1, \infty). \tag{28}$$

Equality holds if $r = t = 1$.

Proof. Let $r, t \in [0, 1]$ and $\lambda \in (0, \infty)$. Recall that $S_\lambda(t)$ is increasing. Thus we have

$$0 < S_\lambda(rt) \leq S_\lambda(r), \tag{29}$$

$$0 < S_\lambda(rt) \leq S_\lambda(t), \tag{30}$$

since $rt \leq r$ and $rt \leq t$. Now by multiplying (29)and (30), we obtain the result (27).
Next, let $r, t \in [1, \infty,)$ and $\lambda \in (0, \infty)$, we have

$$S_\lambda(rt) \geq S_\lambda(r) > 0, \tag{31}$$

$$S_\lambda(rt) \geq S_\lambda(t) > 0, \tag{32}$$

since $rt \geq r$ and $rt \geq t$. By multiplying the inequalities (31) and (32), the desired result is obtained (28). □

Theorem 3.7. *For $r, t \in (-\infty, \infty)$ and $\lambda \in (0, \infty)$, the function $S_\lambda(t)$ is logarithmically concave. In other words, the inequality*

$$S_\lambda\left(\frac{r}{a} + \frac{t}{b}\right) \geq \left[S_\lambda(r)\right]^{\frac{1}{a}} \left[S_\lambda(t)\right]^{\frac{1}{b}} \tag{33}$$

is satisfied. Where $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$.

Proof. Let $Q(t) = \ln S_\lambda(t)$. Then,

$$Q'(t) = \frac{S'_\lambda(t)}{S_\lambda(t)} = \frac{\frac{(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}}{\frac{(1+\lambda t)^{\frac{1}{\lambda}}}{1+(1+\lambda t)^{\frac{1}{\lambda}}}}$$
$$= \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)\left[1+(1+\lambda t)^{\frac{1}{\lambda}}\right]^2}\right)\left(\frac{1+(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}}\right)$$
$$= \frac{1}{(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}}.$$

Taking the second derivative of $Q(t)$, we have

$$Q''(t) = -\frac{\lambda + (1+\lambda)(1+\lambda t)^{\frac{1}{\lambda}}}{\left[(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}\right]^2} < 0,$$

and this completes the proof. □

Corollary 3.8. *For $\lambda \in (0,\infty)$ and $t \in (-\infty,\infty)$, the inequalities*

$$S''_\lambda(t)\, S_\lambda(t) \le \left[S'_\lambda(t)\right]^2 \tag{34}$$

and

$$S_\lambda(1+u)\, S_\lambda(1-u) \le \left[\frac{(1+\lambda)^{\frac{1}{\lambda}}}{1+(1+\lambda)^{\frac{1}{\lambda}}}\right]^2 \tag{35}$$

are valid.

Proof. Since $S_\lambda(t)$ is logarithmically concave, then $[\ln(S_\lambda(t))]'' \le 0$, for all $t \in (-\infty,\infty)$ and $\lambda \in (0,\infty)$. This implies that,

$$[\ln(S_\lambda(t))]'' = \left[\frac{S'_\lambda(t)}{S_\lambda(t)}\right]' = \frac{S''_\lambda(t)\, S_\lambda(t) - S'_\lambda(t)\, S'_\lambda(t)}{[S_\lambda(t)]^2}$$
$$= \frac{S''_\lambda(t)\, S_\lambda(t) - \left[S'_\lambda(t)\right]^2}{[S_\lambda(t)]^2} \le 0.$$

Hence, $S''_\lambda(t)\, S_\lambda(t) - \left[S'_\lambda(t)\right]^2 \le 0$, which yields equation (34).

Next, let $a=b=2, t=1+u$ and $r=1-u$ in equation (33). we have

$$S_\lambda\left(\frac{1+u}{2}+\frac{1-u}{2}\right)\geq\left[S_\lambda(1+u)\right]^{\frac{1}{2}}\left[S_\lambda(1-u)\right]^{\frac{1}{2}}$$

$$S_\lambda(1)\geq\left(\left[S_\lambda(1+u)\right]\left[S_\lambda(1-u)\right]\right)^{\frac{1}{2}}$$

$$\left[\frac{(1+\lambda)^{\frac{1}{\lambda}}}{1+(1+\lambda)^{\frac{1}{\lambda}}}\right]^2\geq S_\lambda(1+u)\,S_\lambda(1-u)\,,$$

resulting in equation (35). This concludes the proof. □

Theorem 3.9. *For* $t,\lambda\in(0,\infty)$, *the function* $S_\lambda(t)$ *satisfies the inequality*

$$1<\frac{S_\lambda(t+1)}{S_\lambda(t)}<\frac{2(1+\lambda)^{\frac{1}{\lambda}}}{1+(1+\lambda)^{\frac{1}{\lambda}}}. \tag{36}$$

Proof. Recall from equation (18), that

$$\left(\frac{S_\lambda'(t)}{S_\lambda(t)}\right)'=-\frac{\lambda+(1+\lambda)(1+\lambda t)^{\frac{1}{\lambda}}}{\left[(1+\lambda t)+(1+\lambda t)^{\frac{1}{\lambda}+1}\right]^2}<0,$$

for all $t,\lambda\in(0,\infty)$. This implies, the function $\frac{S_\lambda'(t)}{S_\lambda(t)}$ is decreasing on the given interval.
Now, let

$$\begin{aligned}P(t)=\frac{S_\lambda(t+1)}{S_\lambda(t)}&=\left(\frac{\left[1+\lambda(t+1)\right]^{\frac{1}{\lambda}}}{1+\left[1+\lambda(t+1)\right]^{\frac{1}{\lambda}}}\right)\left(\frac{1+(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}}\right)\\&=\frac{\left[1+\lambda(t+1)\right]^{\frac{1}{\lambda}}+(1+\lambda t)^{\frac{1}{\lambda}}\left[1+\lambda(t+1)\right]^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}+(1+\lambda t)^{\frac{1}{\lambda}}\left[1+\lambda(t+1)\right]^{\frac{1}{\lambda}}}\end{aligned}$$

and

$$\begin{aligned}\Omega(t)&=\ln P(t)\\&=\ln S_\lambda(t+1)-\ln S_\lambda(t)\,.\end{aligned}$$

Then,

$$\Omega'(t)=\frac{S_\lambda'(t+1)}{S_\lambda(t+1)}-\frac{S_\lambda'(t)}{S_\lambda(t)}<0,$$

since $\frac{S_\lambda'(t)}{S_\lambda(t)}$ is decreasing. This implies $\Omega(t)$ and consequently $P(t)$ are decreasing. Hence, for all $t,\lambda\in(0,\infty)$, we have

$$1=\lim_{t\to\infty}P(t)<P(t)<\lim_{t\to 0}P(t)=\frac{2(1+\lambda)^{\frac{1}{\lambda}}}{1+(1+\lambda)^{\frac{1}{\lambda}}},$$

which yields the desired result (36). □

4. conclusion

We have introduced a degenerate sigmoid function. Properties such as concavity, monotonicity and inequalities involving the new function have been established. These established properties can be applied in several areas of mathematics.

5. Conflicts of interest

The corresponding author affirms on behalf of all authors that there is no conflict of interest for the publication of this research.

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