Eur. J. Math. Anal. 3 (2023) 5
doi: 10.28924/ada/ma.3.5

Updated and Weaker Convergence Criteria of Newton Iterates for Equations

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Abstract. Newton iteration is often used as a solver for nonlinear equations in abstract spaces. Some of the main concerns are general: Criteria for convergence, error estimations on consecutive iterates, and the location of a solution. A plethora of authors has addressed these concerns by presenting results based on the celebrated Kantorovich theory. This article contributes in this direction by extending earlier results but without additional conditions. These extensions become possible using a more precise majorization than the one given in earlier articles. Numerical experimentation complements the theoretical results involving a partial differential and an integral equation.

1. Introduction

Nonlinear equation

$$F(x) = 0, \tag{1.1}$$

plays a important role due to the fact that many applications can be brought to look like it. The celebrated Newton Iteration (NI) in the following form

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),\ \forall\ n = 0, 1, 2, \dots \tag{1.2}$$

is often applied to solve equation (1.1) iteratively. Here, $F : \Omega \subset M_1 \longrightarrow M_2$ is differentiable per Fréchet and operates between Banach spaces M_1 and M_2, whereas set $\Omega \neq \emptyset$.

Kantorovich inaugurated the semi-local convergence of NI (SLCNI) analysis of NI in abstract spaces by applying the contraction mapping principle due to Banach. He presented two different

Received: 29 Apr 2022.
Key words and phrases. iterative processes; Newton iteration; Banach space; semi-local convergence.

proofs based on majorization and recurrent relations [12]. The Newton-Kantorovich Theorem gives the SLCNI. Numerous authors applied this result, in applications and also as a theoretical tool.

Even a simple equation given in [1–4, 7, 10, 11] shows that convergence criteria may not be satisfied. However, NI may be convergent (see the numerical Section, Example 4.1). That is why these criteria are weakened in [2–4]. But no new conditions are added. In this study two additional features are presented. One involves an explicit upper bound on the smallness of initial approximation. Moreover by choosing a bit larger bound the convergence order of NI is recovered. Consequently, new results can always replace corresponding ones by Kantorovich [7] and others [5, 8–11], since preceding results imply the one in this study but not necessarily vice versa. Method in this study uses smaller Lipschitz or Hölder parameters to achieve these extensions which are specializations of earlier ones. That is no additional effort is needed. The generality of this idea allows its application on other processes [3, 4, 11].

Contributions by others can be found in Section 4, where comparisons take place. The majorization of NI is discussed in Section 2. SLCNI appears in Section 3. The numerical experimentation is given in Section4. Conclusions complete this study in Section 5.

2. Majorization of NI

Let K_0, K, L_0, L denote positive numbers, $q \in (0, 1]$ and t stand for a positive variable. These parametrs are connected in Section 3 to initial data $D = (\Omega, y, F, F', x_0)$. Define sequence $\{s_n\}$ by $s_0 = 0, s_1(t) = s_1 = t$

$$\begin{aligned} s_2(t) &= s_2 = s_1 + \frac{K(s_1 - s_0)^{1+q}}{(1+q)(1 - K_0 s_1^q)}, \\ s_{n+2}(t) &= s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)^{1+q}}{(1+q)(1 - L_0 s_{n+1}^q)}, \ \forall n = 1, 2, \dots. \end{aligned} \tag{2.1}$$

Sequence $\{x_n\}$ is majorized by $\{s_n\}$ (see Section 3). That is why convergence is studied first for sequence $\{s_n\}$.

LEMMA 2.1. *Suppose*

$$K_0 t^q < 1 \ \text{ and } \ L_0 s_{n+1}^q < 1 \ \forall n = 0, 1, 2, \dots. \tag{2.2}$$

Then, sequence $\{s_n\}$ is strictly increasing and converges to some limit point $s_* \in (0, (\frac{1}{L_0})^{\frac{1}{q}}]$. The point s_* is the unique least upper bound of sequence $\{s_n\}$.

Proof. The result follows from definition of sequence $\{s_n\}$ and hypothesis (2.1). □

Let ϵ be a positive constant. Moreover, introduce parameters by $\alpha = 1+\epsilon$, $\beta = \frac{L}{(1+q)}(1+\epsilon)$, $\gamma = \frac{\epsilon}{(1+\epsilon)L_0}$, $\delta = \beta(s_2 - s_1)$, $\lambda = \gamma^{\frac{1}{q}}$, $h = \delta^{1+q}$ and $u = (\frac{1}{K_0})^{\frac{1}{q}}$. Furthermore, consider functions with common domain in $T = [0, u)$ given as

$$f_1(t) = \left(\frac{Kt^q}{(1+q)(1-K_0t^q)} + t\right)^q - \gamma,$$

$$f_2(t)\frac{KL\alpha t^{1+q}}{(1+q)^2(1-K_0t^q)} - 1$$

and

$$f_3(t) = (s_2 + \frac{\beta^{-\frac{1}{q}}h}{1-h})^q - \gamma.$$

It follows by these definitions $f_1(0) = -\gamma < 0, f_2(0) = -1 < 0, f_3(0) = -\gamma < 0$ and $f_1(t) \longrightarrow \infty, f_2(t) \longrightarrow \infty$ and $f_3(t) \longrightarrow \infty$ as $t \longrightarrow u^-$. So, function $f_i, i = 1, 2, 3$ have zeros in interval T by IVT (Intermediate Value Theorem). Let η_i denote the smallest such zero of functions f_i in interval $T_0 = (0, u)$, respectively.

It also follows by these choices of zeros η_i

$$K_0s_1^q < 1,\ s_2^q < \gamma, f_1(t) < 0 \text{ at } t = \eta_1 \tag{2.3}$$

$$\delta < 1,\ f_2(t) < 0 \text{ at } t = \eta_2 \tag{2.4}$$

and

$$f_3(t) < 0 \text{ at } t = \eta_3. \tag{2.5}$$

Define parameter

$$\eta_0 = \min\{\eta_i\}. \tag{2.6}$$

Suppose

$$\eta \leq \eta_0. \tag{2.7}$$

If $\eta_0 = \eta_1$ or $\eta_0 = \eta_2$, suppose hypothesis (2.7) holds as a strict inequality.

A second stronger convergence result follows. But hypotheses are easier to verify.

LEMMA 2.2. *Suppose hypothesis (2.7) holds. Then, sequence $\{s_n\}$ is strictly increasing and convergent to some $s_* \in (0, \gamma_0)$, where $\gamma_0 = s_2 + \frac{\beta^{-\frac{1}{q}}h}{1-h}$. Moreover, for $\sigma_{n+2} = s_{n+2} - s_{n+1}\ \forall n = 0, 1, 2, \ldots$*

$$\sigma_{n+2} \leq \beta\sigma_{n+1}^{1+q} \leq \beta^{-\frac{1}{q}}\delta^{(1+)^n} \tag{2.8}$$

and

$$s_* - s_{n+1} \leq \beta^{-\frac{1}{q}}\frac{\delta^{(1+q)^n}}{1-\delta^{1+q}}. \tag{2.9}$$

Proof. The assertions

$$(I_j): \qquad 0 < \frac{1}{1-L_0s_{j+1}^q} \leq \alpha \tag{2.10}$$

is shown using induction. Assertion (I_1) is true by the choice of η_1 and estimates (2.3). It follows by (I_1) and sequence $\{s_n\}$ that

$$0 < s_3 - s_2 \leq \beta(s_2 - s_1)^{1+q},$$

or

$$s_3 < s_2 + \beta^{-\frac{1}{q}}\delta^{(1+q)^1} \leq \gamma.$$

So, assertion (2.8) holds for $n = 1$. Suppose assertion (2.10) holds $\forall j = 1, 2, \dots n$. Then,

$$0 < \sigma_{n+1} \leq \beta\sigma_n^{1+q}$$

and

$$\begin{aligned}
s_{n+1} &\leq s_n + \beta\sigma_n^{1+q} \leq \dots \\
&\leq s_2 + \beta^{\frac{(1+q)-1}{1+q-1}}\sigma_2^{1+q} + \beta^{\frac{(1+q)^2-1}{1+q-1}}\sigma_2^{(1+q)^2} + \dots + \beta^{\frac{(1+q)^{n-1}-1}{1+q-1}}\sigma_2^{(1+q)^{n-1}} \\
&= s_2 + \beta^{-\frac{1}{q}}(\delta^{1+q} + \delta^{2(1+q)} + \dots + \delta^{(n-1)(1+q)})\ (\delta < 1) \\
&= s_2 + \beta^{-\frac{1}{q}}\delta^{1+q}\frac{1 - (\delta^{1+q})^{n-1}}{1 - \delta^{1+q}} \\
&< s_2 + \beta^{-\frac{1}{q}}\frac{\delta^{1+q}}{1 - \delta^{1+q}} \\
&= s_2 + \beta^{-\frac{1}{q}}\frac{h}{1 - h} = \gamma_0.
\end{aligned}$$

Hence, estimate

$$s_{n+1}^q \leq \gamma$$

holds if $f_3(t) \leq 0$ at $t = \eta_3$, which is estimate (2.5). The induction for assertion (2.10) is completed. It follows that estimate (2.8) holds. Notice

$$\delta = \beta\sigma_2 = \frac{L\alpha}{1+q}\frac{K\sigma_1^{1+q}}{(1+q)(1 - K_0 s_1^q)} < 1 \tag{2.11}$$

also holds since it is equivalent to the second estimate in (2.4). Let $n = 2, 3, \dots$. Then, it follows in turn by assertion (2.8)

$$\begin{aligned}
s_{j+n} - s_{j+1} &\leq \sigma_{j+n} + \sigma_{j+n-1} + \dots + \sigma_{j+2} \\
&\leq \beta^{-\frac{1}{q}}(\delta^{(1+q)^{j+n-2}} + \delta^{(1+q)^{j+n-1}} + \dots + \delta^{(1+q)^j}) \\
&\leq \beta^{-\frac{1}{q}}\delta^{(1+q)^n}\delta^{1+q}\frac{1 - \delta^{2n-1}}{1 - \delta^{1+q}}.
\end{aligned} \tag{2.12}$$

Then, assertion (2.9) follows from estimate (2.12) if $n \longrightarrow \infty$.

□

REMARK 2.3. *An at least as large parameter as η_3 can replace it in condition (2.7) as follows. Define sequences of functions φ_n on the interval T by*

$$\varphi_n(t) = (s_2(t) + \beta^{-\frac{1}{q}}(\delta(t)^{1+q} + \delta(t)^{(1+q)^2} + \dots + \delta(t)^{(1+q)^{n-1}})^q - \gamma. \tag{2.13}$$

It follows by these definition that

$$\varphi_{n+1}(t) - \varphi_n(t) \geq 0,$$

so

$$\varphi_n(t) \leq \varphi_{n+1}(t) \ \forall t \in T. \tag{2.14}$$

Moreover these functions have zeros in T_0. *These zeros are assured to exist by (IVT), since by the definitions of fucntions* φ_n *give* $\varphi_n(0) = -\gamma < 0$ *and* $\varphi_n(t) \longrightarrow \infty$ *as* $t \longrightarrow u^-$. *Denote the smallest such zeros of functions* φ_n *in* T *by* r_n, *respectively. According to the proof of Lemma 2.2,*

$$\lim_{n \longrightarrow \infty} \varphi_n(t) \leq f_3(t) \ \forall t \in T. \tag{2.15}$$

So, this limit exists as a well defined function denoted by ψ. *Then, this function has zeros in* T_0, *since* $\psi(0) = -\gamma$ *and* $\psi(t) \longrightarrow \infty$ *as* $t \longrightarrow u^-$. *Denote by* η_4 *the smallest such zero in* $(0, u)$. *Clearly, the proof of Lemma 2.2 goes through if instead of* $f_3(t) \leq 0$, *it is shown that*

$$\psi_3(t) \leq 0 \ \forall t \in T. \tag{2.16}$$

Define parameter

$$\bar{\eta}_0 = \min\{\eta_1, \eta_2, \eta_4\}.$$

Notice that by (2.15) $\psi(\eta_3) \leq f_3(\eta_3) = 0$, *so* $\eta_3 \leq \eta_4$ *and consequenlty* $\eta_0 \leq \bar{\eta}_0$. *If* $\bar{\eta}_0$ *replaces* η_0 *in condition (2.6), then assertion (2.16) follows. Condition (2.7) becomes*

$$\eta \leq \bar{\eta}_0. \tag{2.17}$$

Hence, the range of initial approximations η *is further extended.*

3. Convergence of NM

The notation $U(w, \rho), U[w, \rho]$ means the open and closed balls with radius $\rho > 0$ and center $w \in X$, respectively. The parameters K_0, L_0, K, L and t are connected with operator F as follows. Consider conditions (A):

Suppose

(A1) There exist $x_0 \in \Omega$, $t \geq 0$ such that $F'(x_0)^{-1} \in L(M_2, M_1)$,

$$\|F'(x_0)^{-1}F(x_0)\| \leq t.$$

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq K_0\|x_1 - x_0\|^q$$

and

$$\|F'(x_0)^{-1}(F'(x_0 + \tau(x_1 - x_0)) - F'(x_0))\| \leq K\|\tau(x_1 - x_0)\|^q.$$

(A2) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|^q, \ \ \forall x \in \Omega$. Set $B_1 = U(x_0, (\frac{1}{L_0})^{\frac{1}{q}}) \cap \Omega$.

(A3) $\|F'(x_0)^{-1}(F'(x + \tau(y - x)) - F'(x))\| \leq L\|\tau(y - x)\|^q \ \forall x, y \in B_1$ and $\forall \tau \in [0, 1)$.

(A4) Conditions of Lemma 2.1 or Lemma 2.2 hold

(A5) $U[x_0, t^*] \subset \Omega$.

Notice that $K_0 \leq K \leq L_0$.

Next, conditions A are applied to show the main convergence result for NI.

THEOREM 3.1. *Under conditions A sequence NI is convergent to a solution* $x^* \in U[x_0, s^*]$ *of equation* $F(x^*) = 0$. *Moreover, upper bounds*

$$\|x^* - x_n\| \leq s^* - s_n \tag{3.1}$$

hold $\forall n = 0, 1, 2, \ldots$.

Proof. The items

$$\|x_{i+1} - x_i\| \leq s_{i+1} - s_i, \tag{3.2}$$

and

$$U[x_{i+1}, s^* - s_{i+1}] \subseteq U[x_i, s^* - s_i], \tag{3.3}$$

are shown by induction $\forall i = 0, 1, 2, \ldots$. Let $u \in U[x_1, s^* - s_1]$. It follows by condition (A1)

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq t = s_1 - s_0,$$

$$\|u - x_0\| \leq \|u - x_1\| + \|x_1 - x_0\| \leq s^* - s_1 + s_1 - s_0 = s^*.$$

Hence, point $u \in U[x_0, s^* - s_0]$. That is items (3.2) and (3.3) hold for $i = 0$. Assume these assertions hold if $i = 0, 1, \ldots, n$. It follows for each $\xi \in [0, 1]$

$$\|x_i + \xi(x_{i+1} - x_i) - x_0\| \quad \leq \quad s_i + \xi(s_{i+1} - s_i) \leq s^*,$$

and

$$\|x_{i+1} - x_i\| \leq \sum_{j=1}^{i+1} \|x_j - x_{j-1}\| \leq \sum_{j=1}^{i+1} (s_j - s_{j-1}) = s_{i+1}.$$

It follows by induction hypotheses, Lemmas and conditions (A1) and (A2)

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x_{i+1}) - F'(x_0))\| &\leq \bar{K}\|x_{i+1} - x_0\|^q, \\ &\leq \bar{K}(s_{i+1} - s_0)^q \leq \bar{K}s_{i+1}^q < 1.\end{aligned}$$

Hence, the inverse of linear operator $F'(x_{i+1})$ exists. Therefore, Hence, $F'(v)^{-1} \in L(M_2, M_1)$ and

$$\|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - \bar{K}s_{i+1}^q)}, \tag{3.4}$$

follows as a consequence of a Lemma on invertible linear operators due to Banach [2, 7], where $\bar{K} = \begin{cases} K_0, & i = 0 \\ L_0, & i = 1, 2, \ldots. \end{cases}$

NI gives

$$\begin{aligned} F(x_{i+1}) &= F(x_{i+1}) - F(x_i) - F'(x_i)(x_{i+1} - x_i), \\ &= \int_0^1 (F'(x_i + \xi(x_{i+1} - x_i))d\xi - F'(x_i))(x_{i+1} - x_i). \end{aligned} \tag{3.5}$$

Then, using induction hypotheses, identity (A3) and condition (??)

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{i+1})\| &\leq \bar{L}\int_0^1 (\|x_{i+1} - x_i\|)^q \\ &\leq \frac{\bar{L}}{1+q}(s_{i+1} - s_i)^{1+q}, \end{aligned} \tag{3.6}$$

where $\bar{L} = \begin{cases} K, & i = 0 \\ L, & i = 1, 2, \ldots. \end{cases}$

It follows by NI, estimates (3.4), (3.6) and the definition (2.1) of sequence $\{s_n\}$

$$\begin{aligned} \|x_{i+2} - x_{i+1}\| &\leq \|F'(x_{i+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{i+1})\|, \\ &\leq \frac{\tilde{K}(s_{i+1} - s_i)^2}{2(1 - \tilde{L}s_{i+1})} = s_{i+2} - s_{i+1}, \end{aligned}$$

where $\tilde{K} = \begin{cases} K, & i = 0 \\ L, & i = 1, 2, \ldots. \end{cases}$ and $\tilde{L} = \begin{cases} K_0, & i = 0 \\ L_0, & i = 1, 2, \ldots. \end{cases}$ Moreover, if $v \in U[x_{i+2}, s^* - s_{i+2}]$ it follows

$$\begin{aligned} \|v - x_{i+1}\| &\leq \|v - x_{i+2}\| + \|x_{i+2} - x_{i+1}\| \\ &\leq s^* - s_{i+2} + s_{i+2} - s_{i+1} = s^* - s_{i+1}. \end{aligned}$$

Hence, point $w \in U[x_{i+1}, s^* - s_{i+1}]$ completing the induction for items (3.2) and (3.3). Notice that scalar majorizing sequence $\{s_i\}$ is fundamental as convergent. Hence, the sequence $\{x_i\}$ is also convergent to some $x^* \in U[x_0, s^*]$. Furthermore, let $i \longrightarrow \infty$ in estimate (3.6), to conclude $F(x^*) = 0$.

□

Next, the uniqueness ball for a solution is presented. Notice that not all condition A are used.

PROPOSITION 3.2. *Under center-Lipschitz condition (A2) further suppose the existence of a solution $p \in U(x_0, r) \subset \Omega$ of equation (1.1) such that operator $F'(p)$ is invertible for some $r > 0$; a parameter $r_1 \geq r$ given by*

$$r_1 = \left(\frac{1+q}{L_0} - r^q\right)^{\frac{1}{q}}. \tag{3.7}$$

Then, the poiny p solves uniquely equation $F(x) = 0$ in the domain $B_2 = U(x_0, r_1) \cap \Omega$.

Proof. Define linear operator $Q = \int_0^1 F'(\bar{p} + \xi(p - \bar{p}))d\xi$ for some point $\bar{p} \in B_2$ satisfying $F(\bar{p}) = 0$. By using the definition of r_1, set B_2 and condition (A2)

$$\begin{aligned} \|F'(x_0)^{-1}(F'(x_0) - Q)\| &\leq \int_0^1 L_0((1-\xi)\|x_0 - p\|^q + \xi\|x_0 - \bar{p}\|^q)d\xi, \\ &< \frac{L_0}{1+q}(r_1^q + r^q) = 1, \end{aligned}$$

concluding that $p = \bar{p}$, where the invertability of linear operator is also used together with the approximation $0 = F(p) - F(\bar{p}) = Q(p - \bar{p})$.

□

REMARK 3.3. *(1) If conditions A hold, set $p = x^*$ and $r = s^*$ in Proposition 3.2.*
(2) Lipschitz condition (A3) can be replaced by

$$\|F'(x_0)^{-1}(F'(z_1 + \tau(z_2 - z_1)) - F'(z_1))\| \leq d\|\tau(z_1 - z_2)\|^q \tag{3.8}$$

for all $z_1 \in B_1$ and $z_2 = z_1 - F'(z_1)^{-1}F(z_1) \in B_1$. This even smaller parameter d can replace L in the previous results. The existence of iterate z_2 is assured by (A2).

4. Numerical Experimentation

Three experimenta are considered in this section.

EXAMPLE 4.1. *The parameters using example of the introduction are $K_0 = \frac{\mu+5}{3}$, $K = L_0 = \frac{\mu+11}{6}$. Moreover,$\Omega_0 = U(1, 1-\mu) \cap U(1, \frac{1}{L_0}) = U(1, \frac{1}{L_0})$. Set $L = 2(1 + \frac{1}{3-\mu})\ L_0 < L_1$ and $L < L_1$ for all $\mu \in (0, 0.5)$. The Kantorovich criterion $\eta \leq \frac{1}{L_1}$ is violated, since $\eta > \frac{1}{L_1} \forall \mu \in (0, 0.5)$, where L_1 is the Lipschitz constant on Ω. Interval can be enlarged if condition of Lemma 2.1 is verified. Then, for $\mu = 0.4$, we have the following; $\frac{1}{L_0} = 0.3846$,*

TABLE 1. Sequence (2.1)

n	1	2	3	4	5	6	7
s_{n+1}	0.2000	0.2594	0.2744	0.2755	0.2755	0.2755	0.2755

Hence conditions of Lemma 2.1 hold.

Hence condition (2.2) holds, and the interval is extended form $\emptyset$ to $[0.4, o.5]$.

EXAMPLE 4.2. *Let us consider the two point PBVP(TPBVP)*

$$\begin{aligned} u'' + u^{\frac{3}{2}} &= 0 \\ u(0) = u(1) &= 0. \end{aligned}$$

The interval $[0,1]$ *is divided into* j *subintervals. Set* $m=\frac{1}{j}$. *Denote by* $w_0=0<w_1<\ldots<w_j=1$ *the points of subdivision with corresponding values of the function* $u_0=u(w_0),\ldots,u_j=u(w_j)$. *Then, the discretization of* u'' *is given by*

$$u_k''\approx\frac{u_{k-1}-2u_k+u_{k+1}}{m^2},\quad \forall k=2,3,\ldots j-1.$$

Notice that $u_0=u_j=0$. *It follows that the following system of equations is obtained*

$$\begin{aligned}m^2u_1^{\frac{3}{2}}-2u_1+u_2 &= 0,\\ u_{k-1}+m^2u_k^{\frac{3}{2}}-2u_k+u_{k+1} &= 0,\quad \forall k=2,3,\ldots,j-1\\ u_{j-2}+m^2u_{j-1}^{\frac{3}{2}}-2u_{j-1} &= 0.\end{aligned}$$

This system can be converted into an operator equation as follows: Define operator $G:\mathbb{R}^{j-1}\longrightarrow\mathbb{R}^{j-1}$ *whose derivative is given as*

$$G'(u)=\begin{bmatrix}\frac{3}{2}m^2u_1^{\frac{1}{2}}-2 & 1 & 0 & \ldots & 0\\ 1 & \frac{3}{2}m^2u_2^{\frac{1}{2}}-2 & 1 & 0 & \ldots\\ 0 & \ddots & \ddots & \ddots & \ddots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 1 & 0 & \frac{3}{2}m^2u_{j-1}^{\frac{1}{2}}-2\end{bmatrix}.$$

Let $z\in\mathbb{R}^{j-1}$ *be arbitrary. The norm is* $\|z\|=\max_{1\le k\le j-1}\|z_k\|$, *where as the norm for* $G\in\mathbb{R}^{j-1}\times\mathbb{R}^{j-1}$ *is given as*

$$\|G\|=\max_{1\le k\le j-1}\sum_{i=1}^{j-1}\|g_{k,i}\|.$$

Then, if $u,z\in\mathbb{R}^{j-1}$ *for* $|u_k|>0,|z_k|>0$, $\forall k=1,2,\ldots,j-1$ *to obtain in turn*

$$\begin{aligned}\|G'(u)-G'(z)\| &= \|diag\{\frac{3}{2}(u_k^{\frac{1}{2}}-z_k^{\frac{1}{2}})\}\|\\ &= \frac{3}{2}m^2\left[\max_{1\le k\le j-1}|u_k-z_k|\right]^{\frac{1}{2}}\\ &= \frac{3}{2}m^2\|u-z\|^{\frac{1}{2}}.\end{aligned}$$

Choose as an initial guess vector $130\sin\pi x$ *to obtain after four iterations* $u_0=[3.35740e+01, 6.5202e+01, 9.15664e+01, 1.09168e+02, 1.15363e+02, 1.09168e+02, 9.15664e+01, 6.52027e+01, 3.35740e+01]^{tr}]$. *Then, the parameters are* $\|Q'(u_0)^{-1}\|\le 2.5582e+01, \eta=9.15311E-05$, $q=0.5, K_0=L_0=K=L=\frac{3}{200}=0.015$. *Then,* $K_0\eta^p=1.4351e-04$ *and the following table shows that the conditions of Lemma 2.1 are satisfied.*

EXAMPLE 4.3. *Let* $M_1=M_2=C[0,1]$ *be the set of continuous real functions on* $[0,1]$. *The norm-max is used. Set* $\Omega=U[x_0,3]$. *Consider Hammerstein nonlinear integral operator* H *[3,6] on*

Table 2. Sequence (2.1)

n	1	2	3	4	5	6
v_{n+1}	0.1435e-03	0.1435e-03	0.1435e-03	0.1435e-03	0.1435e-03	0.1435e-03

Ω *as*

$$H(v)(z_1) = v(z_1) - y(z_1) - \int_0^1 \mathcal{V}(z_1, z_2)v^3(z_2)dz_1 = 2,\ v \in C[0,1], z_1 \in [0,1]. \tag{4.1}$$

where function $y \in C[0,1]$, *and* $\mathcal{V}$ *is a kernel related by Green's function*

$$\mathcal{V}(z_1, z_2) = \begin{cases} (1 - z_1)z_2, & z_2 \leq z_1 \\ z_2(1 - z_1), & z_1 \leq z_2. \end{cases} \tag{4.2}$$

It follows by this definition that H' *is*

$$[H'(v)(z)](z_1) = z(z_1) - 3\int_0^1 \mathcal{V}(z_1, z_2)v^2(z_2)z(z_2)dz_2 \tag{4.3}$$

$z \in C[0,1], z_1 \in [0,1]$. *Pick* $x_0(z_1) = y(z_1) = 1$. *It then follows from (4.1)-(4.3) that* $H'(x_0)^{-1} \in L(M_2, M_1)$,

$$\|I - H'(x_0)\| < 0.375,\ \|H'(x_0)^{-1}\| \leq 1.6,$$

$$\eta = 0.2,\ L_0 = 2.4,\ L_1 = 3.6,$$

and $\Omega_0 = U(x_0, 3) \cap U(x_0, 0.4167) = U(x_0, 0.4167)$, *so* $L = 1.5$. *Notice that* $L_0 < L_1$ *and* $L < L_1$. *Set* $K_0 = K = L_0$. *The Kantorovich convergence criterion (A3) is not satisfied, since* $2L_1\eta = 1.44 > 1$. *Therefore convergence of NI is not guaranteed. However, the new condition (2.7) is satisfied, since* $2L\eta = 0.6 < 1$.

5. Conclusions

An updated and weaker unified framework is presented for NI. The new analysis is finer than before. Convergence order $1 + q$ is also recovered by choosing a larger upper bound on t. New Lipschitz or Hölder parameters are smaller and specilizations of previous parameters. The new theory can always replace previous ones due to weaker criterion. The strategy can be applied on other iterations [2,3,7,11].

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