A new perspective on the Ermakov-Pinney and scalar wave equations

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Abstract

The first part of the paper proves that a subset of the general set of Ermakov-Pinney equations can be obtained by differentiation of a first-order non-linear differential equation. The second part of the paper proves that, similarly, the equation for the amplitude function for the parametrix of the scalar wave equation can be obtained by covariant differentiation of a first-order non-linear equation. The construction of such a first-order non-linear equation relies upon a pair of auxiliary 1-forms (ψ, ρ). The 1-form ψ satisfies the divergenceless condition $\mathrm{div}(\psi) = 0$, whereas the 1-form ρ fulfills the non-linear equation $\mathrm{div}(\rho) + \rho^2 = 0$. The auxiliary 1-forms (ψ, ρ) are evaluated explicitly in Kasner space-time, hence, amplitude and phase function in the parametrix are obtained. Thus, the novel method developed in this paper can be used with profit in physical applications.

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1. INTRODUCTION

Although the modern theoretical description of gravitational interactions [1] has completely superseded Newtonian gravity, the investigation of ordinary differential equations provides an invaluable tool in the analysis of chaotic dynamical systems [2] and in studying the interplay between linear and non-linear differential equations [3]. Moreover, ordinary differential equations may prove useful in developing methods which are part of the framework necessary to solve more difficult cases of partial differential equations. We are going to provide a concrete example of application: the scalar wave equation in curved space-time. This is of special interest because, as was proved by Cohen and Kegeles [4], the evaluation of electromagnetic fields in curved space-times can be reduced to solving a complex linear scalar wave equation. Several space-times of astrophysical relevance can be studied in this way, e.g., black-hole and neutron-star spaces and cosmological models [4].

In our paper particularly, we are interested in the Ermakov-Pinney [5, 6] non-linear differential equation

$$y'' + py = qy^{-3}, \tag{1}$$

which has found, along the years, many applications in theoretical physics, including quantum mechanics [7, 8] and relativistic cosmology [9]. The first aim of our work is to provide another perspective on the way of arriving at equations of type 1. For this purpose, section 2 provides a concise summary of well-established results on the canonical form of second-order linear differential equations. Section 3 applies an ansatz based on amplitude and phase functions, and proves eventually equivalence between Eqs. of type 1 with $p = 0$ and our Eq. 17, which is a nonlinear equation with only first derivatives of the desired solution. Section 4 studies the correspondence between sections 2 and 3 on the one hand, and the parametrix construction for scalar wave equation on the other hand. Section 5 obtains a first-order non-linear equation for the amplitude function occurring in such a parametrix. Section 6 evaluates in Kasner space-time the auxiliary 1-forms that are needed for a successful application of our method. Section 7 solves the first-order equations for amplitude and phase function in Kasner space-time. Concluding remarks are then made in section 8.

2. CANONICAL FORM OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

In the theory of ordinary differential equations, it is well-known that every linear second-order equation

$$\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right] u(x) = 0 \tag{2}$$

can be solved by expressing the unknown function u in the form of a product

$$u(x) = \varphi(x)\chi(x), \tag{3}$$

where [10, 11]

$$\varphi(x) = \exp\left(-\frac{1}{2}\int P(x)dx\right), \tag{4}$$

while χ solves the linear equation

$$\left[\frac{d^2}{dx^2} + J(x)\right] \chi(x) = 0, \tag{5}$$

having defined

$$J(x) \equiv Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x). \tag{6}$$

All complications arising from the variable nature of coefficient functions P and Q in Eq. 2 are encoded into the potential term $J(x)$ of Eq. 5 defined in Eq. 6. One can therefore hope to gain insight by the familiar solution of linear second-order equations solved by $\sin(x), \cos(x)$ or real-valued exponentials. More precisely, a theorem [12] guarantees that, if $J(x)$ is continuous on the closed interval $[a, b]$, and if there exist real constants ω, Ω such that

$$0 < \omega^2 < J(x) < \Omega^2, \tag{7}$$

one can compare the zeros of solutions of Eq. 5 with the zeros of solutions of the equations

$$\eta'' + \mu^2 \eta = 0, \; \mu = \omega \text{ or } \Omega. \tag{8}$$

Equations 8 are solved by periodic functions $\sin(\mu(x - x_0))$ which have zeros at $x_0 + \frac{k\pi}{\mu}$, k being an integer and μ being equal to ω or Ω as in Eq. 8. One can then prove that the difference δ between two adjacent zeros of a solution of Eq. 5 lies in between $\frac{\pi}{\Omega}$ and $\frac{\pi}{\omega}$ [12]. Equation 5 is therefore regarded as the canonical form of every linear second-order differential equation [10].

3. AN ANSATZ IN TERMS OF AMPLITUDE AND PHASE FUNCTIONS

The work in Ref. [13] has shown long ago that, upon looking for solutions of Eq. 5, one can use with profit the ansatz

$$\chi(x) = u(x) \exp\left(i \int \pi(x) u^{-\lambda}(x) dx \right). \tag{9}$$

By doing so, we find

$$\begin{aligned} \chi''(x) &= \Big[u''(x) + i\pi'(x) u^{1-\lambda}(x) \\ &+ i(2-\lambda)\pi(x) u'(x) u^{-\lambda}(x) - \pi^2(x) u^{1-2\lambda}(x) \Big] \\ &\times \exp\left(i \int \pi(x) u^{-\lambda}(x) dx \right). \end{aligned} \tag{10}$$

If the potential term $J(x)$ vanishes in Eq. 5, we therefore find (A and B being integration constants)

$$\chi(x) = A + Bx. \tag{11}$$

On the other hand, by virtue of Eq. 11, Eq. 10 yields

$$\begin{aligned} & u''(x) + i\pi'(x) u^{1-\lambda}(x) + i(2-\lambda)\pi(x) u'(x) u^{-\lambda}(x) \\ - & \pi^2(x) u^{1-2\lambda}(x) = 0. \end{aligned} \tag{12}$$

Equation 12 suggests setting $\lambda = 2$, hence we find

$$u^3(x) u''(x) - \pi^2(x) = -i\pi'(x) u^2(x). \tag{13}$$

Thus, if $\pi(x) = \text{constant} = \tau$, we obtain a particular case of the Ermakov-Pinney equation 1 with $p = 0$ and $q = \tau^2$ therein, i.e.

$$u^3(x) u''(x) = \tau^2. \tag{14}$$

Furthermore, we can write that

$$\chi(x) = A + Bx = u(x) \exp\left(i\tau \int \frac{dx}{u^2(x)} \right), \tag{15}$$

which implies

$$\log(A + Bx) = \log(u(x)) + i\tau \int \frac{dx}{u^2(x)}. \tag{16}$$

By differentiation, this eventually yields the non-linear equation

$$\frac{u'(x)}{u(x)} + i \frac{\tau}{u^2(x)} = \frac{B}{(A + Bx)}. \tag{17}$$

In other words, the Ermakov-Pinney equations with $p = 0$ in Eq. 1 are equivalent to the non-linear equation 17, *provided* that u is a function of class C^2. On the other hand, Eq. 17 allows for solutions for u which are just of class C^1. A useful check of our calculation is obtained by differentiating with respect to x both sides of Eq. 17 when u is of class C^2, and then using 17 in order to re-express the square of $\frac{u'(x)}{u(x)}$. One then recovers the Ermakov-Pinney Eq. 14, which therefore originates from Eq. 17, and in turn from Eq. 16.

4. AMPLITUDE-PHASE ANSATZ FOR THE PARAMETRIX OF THE SCALAR WAVE EQUATION

The work in Ref. [14] has studied the parametrix for the scalar wave equation in curved spacetime. The topic is relevant both for the mathematical theory of hyperbolic equations on manifolds [11] and for the modern trends in mathematical relativity [15]. For our purposes, we can limit ourselves to the following outline.

In a pseudo-Riemannian manifold (M, g) endowed with a Levi-Civita connection ∇, the wave operator

$$\Box \equiv \sum_{\mu,\nu=1}^{4} (g^{-1})^{\mu\nu} \nabla_\nu \nabla_\mu \tag{18}$$

is a variable-coefficient operator, and the homogeneous wave equation $\Box \phi = 0$, for given Cauchy data

$$\phi(x, t = 0) = \zeta_0(x), \; \frac{\partial \phi}{\partial t}(x, t = 0) = \zeta_1(x), \tag{19}$$

can be solved in the form

$$\phi(x, t) = \sum_{j=0}^{1} E_j(t) \zeta_j(x), \tag{20}$$

where E_j are the Fourier-Maslov integral operators [14, 16]

$$\begin{aligned} E_j(t)\zeta_j(x) &= \sum_{k=1}^{2} (2\pi)^{-3} \int e^{i\varphi_k(x,t,\xi)} \alpha_{jk}(x, t, \xi) \tilde{\zeta}_j(\xi) d^3\xi \\ &+ R_j(t) \zeta_j(x), \end{aligned} \tag{21}$$

the $R_j(t)$ being regularizing operators which smooth out the singularities upon which they act [16]. In the simplest possible terms, the meaning of Eq. 21 is that the integral operators which generalize the Fourier transform to pseudo-Riemannian manifolds involve again an integrand proportional to amplitude $\times$ eponential of (i times a phase function), but, unlike flat spacetime, the amplitude function depends explicitly on all cotangent bundle coordinates, while the phase function is no longer linear in these variables [16].

The amplitude and phase functions, denoted by α (of class C^2) and φ (of class C^1), respectively, can be obtained by solving the coupled equations [14]

$$\sum_{\gamma,\beta=1}^{4} (g^{-1})^{\gamma\beta} \nabla_\beta \left(\alpha^2 \nabla_\gamma \varphi \right) = 0, \tag{22}$$

$$\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}(\nabla_\beta\varphi)(\nabla_\gamma\varphi)=\frac{\Box\alpha}{\alpha}. \tag{23}$$

These equations lead in turn to the following recipe [14]. First, find a divergenceless covector ψ_γ, i.e.

$$\mathrm{div}\psi=\sum_{\gamma=1}^{4}\nabla^\gamma\psi_\gamma=\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\nabla_\beta\psi_\gamma=0, \tag{24}$$

then solve the non-linear equation

$$\alpha^3\Box\alpha=\sum_{\gamma=1}^{4}\psi_\gamma\psi^\gamma=\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\psi_\beta\psi_\gamma, \tag{25}$$

and eventually obtain the phase from the equation

$$\nabla_\gamma\varphi=\alpha^{-2}\psi_\gamma. \tag{26}$$

Interestingly, upon defining

$$q\equiv\sum_{\gamma=1}^{4}\psi_\gamma\psi^\gamma, \tag{27}$$

Eq. 25 becomes of the type 1 with $p=0$. Thus, bearing in mind our finding in Sec. III, we remark that a simple but non-trivial correspondence exists between a subset of the general set of Ermakov-Pinney equations and their tensor-calculus counterpart for the analysis of the scalar wave equation, expressed by the following recipes:

$$\pi(x)=\tau=\text{constant in Eq. 13}\;\leftrightarrow\;\text{Eq. 24},$$

$$\text{Eq. 14}\;\leftrightarrow\;\text{Eq. 25 with constant value } q$$
$$\text{of the right}-\text{hand side},$$
$$u''\text{ in Eq. 14}\;\leftrightarrow\;\Box\alpha\text{ in Eq. 25}.$$

Moreover, we know that Eq. 14 is solved by a function solving the possibly simpler equation 17. This implies in turn that Eq. 25 for the amplitude α must be obtainable from the as yet unknown solution $\mathcal{U}$ of an unknown nonlinear equation involving at most first-order derivatives of $\mathcal{U}$. This is the topic of next section.

5. A FIRST-ORDER NON-LINEAR EQUATION FOR THE AMPLITUDE IN THE PARAMETRIX

Now, we are going to prove that not only does our approach shed new light on the Ermakov-Pinney equation as resulting from differentiation of the non-linear equation 17, which is therefore more fundamental (allowing also for solutions which are only of class C^1, but not C^2), but that also the second-order equation for the amplitude α in the parametrix can be replaced by a first-order equation. For this purpose, since $\Box\alpha$ should be the counterpart of $u''(x)$, and the divergenceless condition, Eq. 24, the counterpart of $\pi'=0$ in Section 3, we are led to consider the first-order non-linear equation

$$\frac{(\nabla_\gamma\alpha)}{\alpha}+i\frac{\psi_\gamma}{\alpha^2}=\rho_\gamma, \tag{28}$$

where ρ_γ are the components of a suitable covector that should generalize the behaviour of $R(x)=\frac{B}{(A+Bx)}$ on the right-hand side of Eq. 17.[1] At this stage, inspired by Section 3, we perform covariant differentiation ∇^γ of both sides of Eq. 28, finding first the equation

$$-\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\frac{(\nabla_\beta\alpha)}{\alpha}\frac{(\nabla_\gamma\alpha)}{\alpha}+\frac{\Box\alpha}{\alpha}-2i\frac{\psi_\gamma}{\alpha^2}\frac{(\nabla^\gamma\alpha)}{\alpha}$$
$$=\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\nabla_\beta\rho_\gamma, \tag{29}$$

because the divergenceless condition 24 holds by assumption. Next, we exploit Eq. 28 by re-expressing all first covariant derivatives of α in Eq. 29 in the form

$$\frac{(\nabla_\beta\alpha)}{\alpha}=\rho_\beta-i\frac{\psi_\beta}{\alpha^2},$$

hence finding

$$-\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\left(\rho_\beta-i\frac{\psi_\beta}{\alpha^2}\right)\left(\rho_\gamma-i\frac{\psi_\gamma}{\alpha^2}\right)+\frac{\Box\alpha}{\alpha}$$
$$-\;2i\sum_{\gamma=1}^{4}\frac{\psi_\gamma}{\alpha^2}\left(\rho^\gamma-i\frac{\psi^\gamma}{\alpha^2}\right)=\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\nabla_\beta\rho_\gamma. \tag{30}$$

In this equation, the terms proportional to $\sum_{\gamma=1}^{4}\rho^\gamma\psi_\gamma$ add up to 0, and hence we obtain

$$\frac{\Box\alpha}{\alpha}-\sum_{\gamma=1}^{4}\frac{\psi_\gamma\psi^\gamma}{\alpha^4}=\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\Big(\nabla_\beta\rho_\gamma+\rho_\beta\rho_\gamma\Big). \tag{31}$$

Thus, provided that

$$\sum_{\gamma,\beta=1}^{4}(g^{-1})^{\gamma\beta}\Big(\nabla_\beta\rho_\gamma+\rho_\beta\rho_\gamma\Big)=0, \tag{32}$$

we obtain eventually the second-order equation 25 for the amplitude α in the parametrix for the scalar wave equation. Remarkably, Eq. 32 is precisely the tensorial generalization of the differential equation obeyed by the right-hand side $R(x)=\frac{B}{(A+Bx)}$ of Eq. 17, because

$$\frac{d}{dx}R(x)+R^2(x)=-B^2(A+Bx)^{-2}+B^2(A+Bx)^{-2}=0.$$

6. EVALUATION OF THE AUXILIARY 1-FORMS ψ AND ρ

So far, the critical reader might think that our method, despite being elegant and correct, does not offer any concrete advantage with respect to the direct investigation of the coupled equations 22 and 23, or 25 and 26. The aim of the present section is to prove that the 1-forms ψ and ρ fulfilling Eqs. 24 and 32 are explicitly computable in a non-trivial case of physical interest.

For this purpose, inspired again by our Ref. [14], we consider Kasner spacetime, whose metric in $c=1$ units reads as [1]

$$g=-dt\otimes dt+t^{2p_1}dx\otimes dx+t^{2p_2}dy\otimes dy+t^{2p_3}dz\otimes dz, \tag{33}$$

[1] Note that, strictly speaking, since α, φ and ψ_γ are real-valued, we are dealing with a complex-valued vector field $\sum_{\gamma=1}^{4}\rho^\gamma\frac{\partial}{\partial x^\gamma}$ [17], with the associated dual concept of complex-valued 1-form field.

where the real numbers p_1, p_2, p_3 satisfy the condition

$$\sum_{k=1}^{3} p_k = 1, \tag{34}$$

as well as the unit 2-sphere condition

$$\sum_{k=1}^{3} (p_k)^2 = 1. \tag{35}$$

Let us assume for simplicity that the only non-vanishing component of the desired 1-form ψ is $\psi_1 = \psi_0(t)$. Hence, we find (since the Christoffel coefficients $\Gamma^0_{kk} = p_k t^{2p_k-1}, \forall k = 1, 2, 3$)

$$\begin{aligned} \mathrm{div}(\psi) &= \sum_{\mu,\nu=1}^{4} (g^{-1})^{\mu\nu} \nabla_\nu \psi_\mu \\ &= -\partial_0 \psi_0 + \psi_0 \left(\Gamma^0_{00} - \sum_{k=1}^{3} t^{-2p_k} \Gamma^0_{kk} \right) \\ &= -\frac{d\psi_0}{dt} - \frac{\psi_0}{t} \sum_{k=1}^{3} p_k = -\frac{d\psi_0}{dt} - \frac{\psi_0}{t}. \end{aligned} \tag{36}$$

The vanishing divergence condition 24 is therefore satisfied by

$$\psi_0 = \frac{\kappa}{t}, \ \kappa = \text{constant}. \tag{37}$$

Similarly, assuming that also the auxiliary 1-form ρ has only one non-vanishing component $\rho_0(t)$, one finds

$$\begin{aligned} &\mathrm{div}(\rho) + \rho^2 = -\frac{d\rho_0}{dt} - \frac{\rho_0}{t} + (g^{-1})^{00}(\rho_0)^2 = -\frac{d\rho_0}{dt} \\ &- \frac{\rho_0}{t} - (\rho_0)^2. \end{aligned} \tag{38}$$

But we know from section 5 that ρ should be complex-valued, hence we set

$$\rho_0(t) = \beta_1(t) + i\beta_2(t), \tag{39}$$

β_1 and β_2 being the real and imaginary part of ρ_0, respectively. Thus, by virtue of the identity 38, Eq. 32 leads to the non-linear coupled system

$$\frac{d\beta_1}{dt} + \frac{\beta_1(t)}{t} + (\beta_1(t))^2 - (\beta_2(t))^2 = 0, \tag{40}$$

$$\frac{d\beta_2}{dt} + \frac{\beta_2(t)}{t} + 2\beta_1(t)\beta_2(t) = 0. \tag{41}$$

Equations 40 and 41 suggest re-expressing them in terms of the unknown function

$$B(t) \equiv \frac{\beta_1(t)}{\beta_2(t)}. \tag{42}$$

This leads to the equivalent system

$$\begin{aligned} &\frac{dB}{dt}\beta_2(t) + B(t)\left(\frac{d\beta_2}{dt} + \frac{\beta_2(t)}{t} \right) \\ &+ (B^2(t) - 1)(\beta_2(t))^2 = 0, \end{aligned} \tag{43}$$

$$\frac{d\beta_2}{dt} + \frac{\beta_2(t)}{t} + 2B(t)(\beta_2(t))^2 = 0. \tag{44}$$

By insertion of Eq. 44 into Eq. 43, we find

$$\beta_2(t) = \frac{B'(t)}{(1 + B^2(t))}, \tag{45}$$

$$\beta_1(t) = B(t)\beta_2(t) = \frac{B(t)B'(t)}{(1 + B^2(t))}, \tag{46}$$

and hence Eq. 44 yields for $B(t)$ the equation

$$\begin{aligned} &\frac{B''(t)}{(1 + B^2(t))} - 2\frac{B(t)(B'(t))^2}{(1 + B^2(t))^2} + \frac{1}{t}\frac{B'(t)}{(1 + B^2(t))} \\ &+ 2\frac{B(t)(B'(t))^2}{(1 + B^2(t))^2} = 0, \end{aligned} \tag{47}$$

which is equivalent to the linear differential equation[2]

$$B''(t) + \frac{1}{t}B'(t) = 0. \tag{48}$$

Equation 48 implies that $B'(t)$ is proportional to $\frac{1}{t}$, and hence, upon introducing the real parameter σ, one can write that (κ being the same parameter used in 37)

$$\frac{dB}{dt} = \frac{\sigma}{\kappa}\frac{1}{t}, \tag{49}$$

hence

$$B(t) = B(T) + \frac{\sigma}{\kappa}\log\left(\frac{t}{T}\right). \tag{50}$$

Hereafter we set $B(T) = 0$ for simplicity. By virtue of 45, 46 and 50 we obtain eventually, upon defining[3]

$$D_\sigma(t) \equiv \frac{\kappa^2}{\sigma} + \sigma\log^2\left(\frac{t}{T}\right), \tag{51}$$

the exact formulae

$$\beta_1(t) = \frac{\sigma}{t}\frac{\log\left(\frac{t}{T}\right)}{D_\sigma(t)}, \tag{52}$$

$$\beta_2(t) = \frac{\kappa}{t}\frac{1}{D_\sigma(t)}. \tag{53}$$

Remarkably, the exact solution of the non-linear equations 40 and 41 has been obtained from the general solution of the linear equation 48.

7. AMPLITUDE AND PHASE FUNCTIONS IN KASNER SPACE-TIME

In light of 28, we can now evaluate the amplitude function α from the equation

$$\nabla_\gamma(\log(\alpha)) + i\frac{\psi_\gamma}{\alpha^2} = \rho_\gamma, \tag{54}$$

and eventually the phase function φ from Eq. 26, which reads in our case

$$\frac{d\varphi}{dt} = \frac{\kappa}{t\alpha^2}. \tag{55}$$

[2]Note also that $(1 + B^2(t))$ in Eq. 47 can never vanish, bearing in mind the real nature of $B(t)$ from the definition 42.

[3]The work in Ref. [18] arrives instead at Eq. 51 by solving directly for the amplitude α, without making any use of the auxiliary 1-form ρ and of our Eq. 28.

From Eq. 54 we obtain, in Kasner space-time, the ordinary differential equation

$$\frac{d}{dt}\log(\alpha(t)) + i\frac{\kappa}{t\alpha^2(t)} = \beta_1(t) + i\beta_2(t), \tag{56}$$

i.e., upon separating real and imaginary part, the pair of equations

$$\frac{d}{dt}\log(\alpha(t)) = \beta_1(t), \tag{57}$$

$$\frac{\kappa}{t\alpha^2(t)} = \beta_2(t) = \varphi(t). \tag{58}$$

Hence we find in Kasner space-time the amplitude function

$$\alpha(t) = \alpha(T)\exp\int_T^t \frac{\sigma}{\tau}\frac{\log\left(\frac{\tau}{T}\right)}{D_\sigma(\tau)}d\tau = \sqrt{\frac{\kappa^2}{\sigma} + \sigma\log^2\left(\frac{t}{T}\right)}, \tag{59}$$

for which $\alpha(T) = \sqrt{\frac{\kappa^2}{\sigma}}$, as well as the phase function

$$\varphi(t) = \varphi(T) + \kappa\int_T^t \frac{d\tau}{\tau D_\sigma(\tau)} = \varphi(T) + \arctan\left(\frac{\sigma}{\kappa}\log\left(\frac{t}{T}\right)\right), \tag{60}$$

which holds for all positive values of the real ratio $\frac{T}{(t-T)}$. It should be stressed that, in a generic space-time without any symmetry, the amplitude and phase, if computable, will depend on all cotangent bundle local coordinates [16] (see further comments in Section 8).

8. CONCLUDING REMARKS

In our paper, starting from well known properties in the theory of linear differential equations, we have first proved that the Ermakov-Pinney equations with $p = 0$ in Eq. 1 result from differentiation of the more fundamental equation 17, provided that the function u solving 17 is taken to be at least of class C^2.

By comparison with the construction of amplitude and phase in the scalar parametrix, we have then proved that finding the amplitude α for which Eq. 25 holds with a divergenceless covector ψ_γ, is equivalent to finding also a covector ρ_γ for which Eq. 32 holds. One can then obtain the amplitude α from the first-order non-linear equation 54. Our successful calculations of sections 6 and 7, have evaluated the auxiliary 1-forms with components

$$\psi_\mu(t) = \left(\frac{\kappa}{t}, 0, 0, 0\right), \tag{61}$$

$$\rho_\mu(t) = \left(\frac{1}{tD_\sigma(t)}\left(\sigma\log\left(\frac{t}{T}\right) + i\kappa\right), 0, 0, 0\right), \tag{62}$$

prove that our original method leads to a powerful tool for studying the scalar wave equation with the associated parametrix. This will be of concrete interest in applied mathematics and in the theoretical physics of fundamental interactions.

Note also that, in principle, there might exist solutions of Eq. 54 which are of class C^1 but not C^2. Thus, the consideration of Eq. 28 is closer to the modern emphasis on finding new solutions of partial differential equations under weaker differentiability properties. Of course, the corresponding physical interpretation is a relevant open problem.

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References

[1] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Clarendon Press, Oxford, 2009).

[2] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*. Tomes 1, 2, 3 (Gauthier-Villars, Paris, 1892; ibid. 1893; ibid. 1899).

[3] J. Carinena, A. Ibort, G. Marmo and G. Morandi, *Geometry from Dynamics, Classical and Quantum* (Springer, Berlin, 2015).

[4] J. M. Cohen and L. S. Kegeles, Electromagnetic fields in curved spaces: a constructive procedure, Phys. Rev. D **10**, 1070 (1974).

[5] V. P. Ermakov, Univ. Izv. Kiev Ser. III **9**, 1 (1880).

[6] E. Pinney, The nonlinear differential equation $y'' + p(x)y + cy^{-3} = 0$, Proc. Amer. Math. Soc. **1**, 681 (1950).

[7] H. R. Lewis and W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. **10**, 1458 (1969).

[8] P. G. Kevrekidis and Y. Drossimos, Nonlinearity from linearity: the Ermakov-Pinney equation revisited, Math. Comp. Sim. **74**, 196 (2007).

[9] R. M. Hawkins and J. E. Lidsey, Ermakov-Pinney equation in scalar-field cosmologies, Phys. Rev. D **66**, 023523 (2002).

[10] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[11] G. Esposito, *From Ordinary to Partial Differential Equations*, Unitext **106** (Springer, Berlin, 2017).

[12] G. Valiron, *Équations Fonctionnelles. Applications* (Masson, Paris, 1945).

[13] E. Majorana, *Notes on Theoretical Physics*, edited by S. Esposito and E. Recami (Springer, Berlin, 2003).

[14] G. Esposito, E. Battista and E. Di Grezia, Bicharacteristics and Fourier integral operators in Kasner spacetime, Int. J. Geom. Methods Mod. Phys. **12**, 1550060 (2015).

[15] M. Minucci, *Hyperbolic Equations and General Relativity* (Nova Science, New York, 2019).

[16] F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators. Fourier Integral Operators*, Vol. 2 (Plenum Press, New York, 1980).

[17] O. Stormark, On the theorem of Frobenius for complex vector fields, Ann. Sc. Norm. Sup. Serie 4, **9**, 57 (1982).

[18] D. Bini and G. Esposito, Ermakov-Pinney-like non-linear equation in curved space-time, work in progress (2019).