MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 34, 1996

RANDOM VIBRATIONS OF DISCRETE SYSTEMS UNDER KINEMATIC WAVE EXCITATIONS

ZBIGNIEW ZEMBATY

Faculty of Civil Engineering
Technical University, Opole

The paper presents a matrix formulation of linear response of discrete dynamic systems to propagating random excitations. Joint effects of pseudo-static and dynamic response are considered. Characteristic features of displacement and force response are pointed out. Direct and mode superposition methods, respectively are considered. A simple, practical example is analyzed for illustration.

1. Introduction

Theory of random vibrations can be applied in various fields of civil, mechanical and aeronautical engineering. Standard algorithms are now available for response statistics of dynamic systems under random excitations (e.g., Lin (1976), Sobczyk (1973) and (1991), Chmielewski (1982)). During the last two decades these standard techniques were developed to include: first passage problem, peak response, non-stationarity, non-linear random vibrations and many other subjects. Among them there is a problem of random vibrations of structures under propagating excitations. Such effects can be observed in civil engineering for seismic excitations of bridges, life-lines, dams or for off-shore structures excited by ocean waves. In aeronautical engineering these can be excitations of aircraft components by pressure waves. In recent years some solutions for more or less complicated structural systems and specific types of structures have been proposed and analyzed (e.g., Harichandran, Wang (1988) and (1990), Harichandran (1992), Hao (1991), Nadim et al. (1991), Zerva (1991), Heredia Zavoni, Vanmarcke (1994)).

In this paper a compact, direct, matrix formulation of linear response of discrete dynamic systems to random, propagating excitations is presented.

In addition, the mode superposition method which proves to be particularly efficient in numerical computations, is presented using index notation. Both methods are derived in terms of the classic, correlation theory of stochastic processes. A simple example illustrates an application of derived formulas.

2. Equations of motion

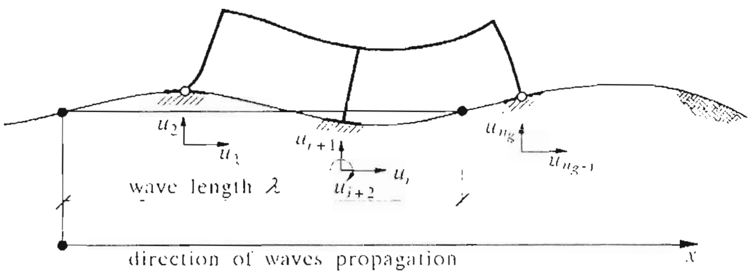

Fig. 1. A structure under multi-support kinematic wave excitations

Consider a structure with n degrees of freedom for which $u_1, u_2, \dots, u_{n_g}$ are degrees associated with support motion and the remaining n_s degrees of freedom represent the response of the structure (Fig.1)

$$n = n_g + n_s \tag{2.1}$$

A matrix equation of motion of such a system takes the following form

$$\mathbf{M}\ddot{q}^t + \mathbf{C}\dot{q}^t + \mathbf{K}q^t = \mathbf{0} \tag{2.2}$$

where **M**, **C**, **K** are mass, damping and stiffness matrices, respectively. The vector

$$q^t = \begin{Bmatrix} q_1^t \\ q_2^t \\ \vdots \\ q_n^t \end{Bmatrix} \tag{2.3}$$

represents total displacements with a reference to fixed coordinates. Applying Eq (2.1), Eq (2.2) can be written in the sub-matrix form

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{sg} \\ \mathbf{M}_{gs} & \mathbf{M}_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{q}_s^t \\ \ddot{u} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sg} \\ \mathbf{C}_{gs} & \mathbf{C}_{gg} \end{bmatrix} \begin{Bmatrix} \dot{q}_s^t \\ \dot{u} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sg} \\ \mathbf{K}_{gs} & \mathbf{K}_{gg} \end{bmatrix} \begin{Bmatrix} q_s^t \\ u \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{2.4}$$

where vectors

$$\boldsymbol{u} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_{n_g} \end{array} \right\} \qquad \boldsymbol{q}_s^t = \left\{ \begin{array}{c} q_1^t \\ q_2^t \\ \vdots \\ q_{n_s}^t \end{array} \right\} \tag{2.5}$$

represent the free field motion (excitations) and structural displacements (response) respectively. It should be noted that when excitations in some directions are not applied (e.g., support rotations or vertical excitations) it results *in removing respective rows and columns in matrices* M_{gs}, M_{sg}, M_{gg}, C_{gs}, C_{sg}, C_{gg}, K_{gs}, K_{sg}, K_{gg}. The total response $\boldsymbol{q}^t$ can be divided into dynamic displacements $\boldsymbol{q}$ and pseudo-static displacements $\boldsymbol{q}^p$ (also called quasistatic)

$$\boldsymbol{q}^t = \left\{ \begin{array}{c} \boldsymbol{q}^p \\ \boldsymbol{u} \end{array} \right\} + \left\{ \begin{array}{c} \boldsymbol{q} \\ \boldsymbol{0} \end{array} \right\} \tag{2.6}$$

Thus the total motion of the supports is determined only by the vector $\boldsymbol{u}$, while the total structural displacements consist of two parts: pseudo-static $\boldsymbol{q}^p$, reflecting kinematic motion of the structure and displacements $\boldsymbol{q}$ representing dynamic response of the structure. In case of classic, uniform support excitations the pseudo-static motion of the structure usually is not analyzed, as it reflects only undeformed motion of the structure. For non-uniform excitations however the pseudo-ststic motion leads to deformations of the structure. Thus the effects of pseudo-static motion are present even if one assumes that all the masses are equal to zero (no dynamic effects). Substituting Eq (2.6) into the equation of motion (2.4) and neglecting dynamic terms one obtains the vector of pseudo-static motion

$$\boldsymbol{q}^p = -\mathsf{K}_{ss}^{-1}\mathsf{K}_{sg}\boldsymbol{u} \tag{2.7}$$

where K_{ss}^{-1} is the inverse matrix of K_{ss}. Substituting again Eqs (2.7) and (2.6) into Eq (2.4) leads to the following equation of motion with n_s degrees of freedom

$$\begin{aligned} &\mathsf{M}_{ss}\ddot{\boldsymbol{q}} + \mathsf{C}_{ss}\dot{\boldsymbol{q}} + \mathsf{K}_{ss}\boldsymbol{q} = \\ &= (\mathsf{M}_{ss}\mathsf{K}_{ss}^{-1}\mathsf{K}_{sg} - \mathsf{M}_{sg})\ddot{\boldsymbol{u}} + (\mathsf{C}_{ss}\mathsf{K}_{ss}^{-1}\mathsf{K}_{sg} - \mathsf{C}_{sg})\dot{\boldsymbol{u}} = \boldsymbol{p}_{\text{eff}} \end{aligned} \tag{2.8}$$

For lightly damped structural systems or for the stiffness proportional damping the contribution of the second term (on the right-hand side of Eq (2.8) is negligible in the "effective" vector of excitations $\boldsymbol{p}_{\text{eff}}$. Thus the equation of motion takes the form

$$\mathsf{M}_{ss}\ddot{\boldsymbol{q}} + \mathsf{C}_{ss}\dot{\boldsymbol{q}} + \mathsf{K}_{ss}\boldsymbol{q} \cong (\mathsf{M}_{ss}\mathsf{K}_{ss}^{-1}\mathsf{K}_{sg} - \mathsf{M}_{sg})\ddot{\boldsymbol{u}} \tag{2.9}$$

Analysis of Eq (2.9) results in a conclusion that the response of a structure to non-uniform excitations is a response to weighted excitations of all n_g moving support directions. The contribution of an individual excitation component u_i to dynamic response (e.g., in displacements q_i) depends on stiffness and mass geometry of the structure.

The situation is however not so clear when analyzing the force response. For all the analyzed degrees of freedom one may calculate the corresponding generalized elastic forces. Assume again little contribution of damping forces to total forces. In this case the generalized force vector depends only on the stiffness matrix and structural displacements

$$\boldsymbol{f} = \mathbf{K}\boldsymbol{q}^t \tag{2.10}$$

The forces $\boldsymbol{f}$ can be divided into structural $\boldsymbol{f}_s$ and support $\boldsymbol{f}_g$, analogously as it has been done for displacements in Eq (2.6)

$$\left\{ \begin{array}{c} \boldsymbol{f}_s \\ \boldsymbol{f}_g \end{array} \right\} = \left[\begin{array}{cc} \mathbf{K}_{ss} & \mathbf{K}_{sg} \\ \mathbf{K}_{gs} & \mathbf{K}_{gg} \end{array} \right] \left\{ \begin{array}{c} \boldsymbol{q}_s^t \\ \boldsymbol{u} \end{array} \right\} \tag{2.11}$$

where

$$\boldsymbol{f}_s = \left\{ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_{n_s} \end{array} \right\} \qquad \boldsymbol{f}_g = \left\{ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_{n_{fg}} \end{array} \right\} \tag{2.12}$$

It should be pointed out that the number of support forces n_{fg} can generally be greater or equal to the number of excitation directions n_g. This is due to the fact that one may associate generalized forces also with inactive support degrees of freedom, which are omitted when formulating equations of motion. In this case the stiffness matrices $\mathbf{K}_{gs}$, $\mathbf{K}_{sg}$ and $\mathbf{K}_{gg}$, should be enlarged in such a way that the global matrix $\mathbf{K}$ is modified without changing structural submatrix $\mathbf{K}_{ss}$. The inactive degrees of freedom can be e.g., rotational, torsional or vertical excitations when they are omitted in the analysis. From Eq (2.11) the structural forces are equal to

$$\boldsymbol{f}_s = \mathbf{K}_{ss}\boldsymbol{q}_s^t + \mathbf{K}_{sg}\boldsymbol{u} \tag{2.13}$$

while the support forces are equal to

$$\boldsymbol{f}_g = \mathbf{K}_{gs}\boldsymbol{q}_s^t + \mathbf{K}_{gg}\boldsymbol{u} \tag{2.14}$$

Substituting from Eq (2.6) $\boldsymbol{q}_s^t = \boldsymbol{q}^p + \boldsymbol{q}$ and applying Eq (2.7) one obtains

$$\boldsymbol{f}_s = \mathbf{K}_{ss}\boldsymbol{q} \tag{2.15}$$

$$\boldsymbol{f}_g = \mathbf{K}_{gs}\boldsymbol{q} + (\mathbf{K}_{gg} - \mathbf{K}_{gs}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg})\boldsymbol{u} \tag{2.16}$$

It is interesting to note from Eq (2.15) that the forces associated with structural degrees of freedom depend only on dynamic displacements of the structure which, in turn, depend on "averaged" excitations of the structure. On the other hand the forces associated with support degrees of freedom, Eq (2.16), depend on two terms: dynamic and pseudo-static. The generalized elastic forces may be applied to calculate any desired inner forces (shear, axial or moments). In practice however it is better to calculate the forces in the structure using classic formulae of the Finite Element Method (FEM) which are more effective in numerical computations than the formulae (2.10)÷(2.16). Regardless of the applied method the calculated forces are linear combinations of structural displacements and the stiffness properties of the structure. Unlike the structural forces $\boldsymbol{f}_s$, the support forces $\boldsymbol{f}_g$ are directly affected by pseudo-static motion. The combination of dynamic and pseudo-static motion of the structure plays particular role in the wave passage effects on multi-support structures.

3. Mean square response by direct method

Substituting

$$\mathbf{A} = \mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg} - \mathbf{M}_{sg} \tag{3.1}$$

into the equation of motion (2.9) gives

$$\mathbf{M}_{ss}\ddot{q} + \mathbf{C}_{ss}\dot{q} + \mathbf{K}_{ss}q = \mathbf{A}\ddot{u} \tag{3.2}$$

The solution of this equation for zero initial conditions is given by the Duhamel integral

$$q(t) = \int_0^t \mathbf{h}(\tau)\mathbf{A}\ddot{u}(t-\tau)\, d\tau \tag{3.3}$$

where $\mathbf{h}(t)$ is the matrix of impulse response functions of the system.

Assume now that the system has constant deterministic coefficients (matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$) and that the excitation vector $\ddot{u}$ is a stationary stochastic process with zero mean and the following, classic spectral representation

$$\ddot{\boldsymbol{u}}(t) = \left\{ \begin{array}{c} \ddot{u}_1(t) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\, d\hat{\ddot{u}}_1(\omega) \\ \ddot{u}_2(t) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\, d\hat{\ddot{u}}_2(\omega) \\ \vdots \\ \ddot{u}_{n_g}(t) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\, d\hat{\ddot{u}}_{n_g}(\omega) \end{array} \right\} \tag{3.4}$$

where $\hat{\ddot{u}}_1(\omega)$, $\hat{\ddot{u}}_2(\omega)$,..., $\hat{\ddot{u}}_{n_g}(\omega)$ are random functions in frequency domain with orthogonal increments, i.e.

$$\langle d\hat{\ddot{u}}_j(\omega_1) d\hat{\ddot{u}}_k^*(\omega_2)\rangle = \left\{ \begin{array}{lll} \langle d\hat{\ddot{u}}_j(\omega) d\hat{\ddot{u}}_k^*(\omega)\rangle = S_{jk}(\omega)\, d\omega & \text{for} & \omega_1 = \omega_2 = \omega \\ 0 & \text{for} & \omega_1 \neq \omega_2 \end{array} \right. \tag{3.5}$$

$\mathrm{i} = \sqrt{-1}$, asterisk denotes complex conjugate, operator $\langle\ \rangle$ stands for mathematical expectation and $S_{jk}(\omega)$ is a complex cross-spectral density function. Substituting the spectral decomposition (3.4) into Eq (3.3) gives

$$\boldsymbol{q}(t) = \int\limits_0^t \mathbf{h}(\tau)\mathbf{A} \int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(t-\tau)}\, d\hat{\ddot{\boldsymbol{u}}}(\omega) d\tau \tag{3.6}$$

where $d\hat{\ddot{\boldsymbol{u}}}(\omega) = \{d\hat{\ddot{u}}_1(\omega), d\hat{\ddot{u}}_2(\omega), ..., d\hat{\ddot{u}}_{n_g}(\omega)\}^\top$ and the symbol $\top$ stands for transposition. Changing the order of integration, assuming stationarity and taking advantage of the fact that $\mathbf{h}(t) = \mathbf{0}$ for $t < 0$ leads to the response in frequency domain

$$\boldsymbol{q}(t) = \int\limits_{-\infty}^{\infty} \mathbf{H}(\omega)\mathbf{A}\mathrm{e}^{\mathrm{i}\omega t}\, d\hat{\ddot{\boldsymbol{u}}}(\omega) \tag{3.7}$$

where

$$\mathbf{H}(\omega) = \int\limits_{-\infty}^{\infty} \mathbf{h}(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\, d\tau \tag{3.8}$$

is a transmittance matrix (matrix of frequency response functions of the system). Eq (3.7) together with the orthogonality condition (3.5) can be used to obtain any required response statistics. For example, the covariance matrix of dynamic displacements can be written as follows

$$\mathbf{D}_q = \int\limits_{-\infty}^{\infty} \mathbf{H}(\omega)\mathbf{A}\mathbf{S}(\omega)\mathbf{A}^\top\mathbf{H}^*(\omega)\, d\omega \tag{3.9}$$

where $\mathbf{S}(\omega) = \mathbf{S}_{\ddot{u}}(\omega)$ is a matrix of complex cross spectra of accelerations $\ddot{\boldsymbol{u}}$. It should be noted that the spectral densities without sub- or superscripts $\ddot{u}$ denote here spectral densities of accelerations.

Now one can introduce the matrix $\boldsymbol{\Gamma}(\omega)$ of complex coherency functions $\gamma_{ij}(\omega)$

$$\gamma_{ij}(\omega) = \frac{S_{ij}(\omega)}{\sqrt{S_i(\omega)S_j(\omega)}} \tag{3.10}$$

where $S_i(\omega)$ and $S_j(\omega)$ are real point spectra. Assuming spatial stationarity (homogeneity) of excitations i.e., $S_i(\omega) = S_j(\omega) = S(\omega)$ we have

$$\gamma_{ij}(\omega) = \frac{S_{ij}(\omega)}{S(\omega)} \tag{3.11}$$

$$\mathbf{D}_q^{\cdot} = \int\limits_{-\infty}^{\infty} \mathbf{H}(\omega)\mathbf{A}\boldsymbol{\Gamma}(\omega)\mathbf{A}^{\top}\mathbf{H}^*(\omega)S(\omega)\,d\omega \tag{3.12}$$

Similar analysis can be made for the pseudo-static response resulting in

$$\mathbf{D}_{q^p} = \int\limits_{-\infty}^{\infty} \mathbf{K}_{ss}^{-1}\mathbf{K}_{sg}\mathbf{S}_u(\omega)\mathbf{K}_{sg}^{\top}\mathbf{K}_{ss}^{-1}\,d\omega \tag{3.13}$$

Taking into account that the elements of spectral matrix of excitation displacements $\mathbf{S}_u(\omega)$ can be calculated from the acceleration spectral densities

$$S_{ij}^{uu}(\omega) = \frac{1}{\omega^4}S_{ij}^{\ddot{u}\ddot{u}}(\omega) = \frac{1}{\omega^4}S_{ij}(\omega) = \frac{1}{\omega^4}\gamma_{ij}(\omega)S(\omega) \tag{3.14}$$

and including Eq (3.11) yields

$$\mathbf{D}_{q^p} = \int\limits_{-\infty}^{\infty} \mathbf{K}_{ss}^{-1}\mathbf{K}_{sg}\boldsymbol{\Gamma}(\omega)\mathbf{K}_{sg}^{\top}\mathbf{K}_{ss}^{-1}\frac{1}{\omega^4}S(\omega)\,d\omega \tag{3.15}$$

Finally consider mean square support forces. Introducing into Eq (2.16) matrix

$$\mathbf{B} = \mathbf{K}_{gg} - \mathbf{K}_{gs}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg} \tag{3.16}$$

yields

$$\boldsymbol{f}_g = \mathbf{K}_{gs}\boldsymbol{q} + \mathbf{B}\boldsymbol{u} \tag{3.17}$$

and then

$$\begin{aligned}\mathbf{D}_{fg} &= \int\limits_{-\infty}^{\infty} \Big(\mathbf{K}_{gs}\mathbf{H}(\omega)\mathbf{A}\mathbf{S}_{\ddot{u}}(\omega)\mathbf{A}^{\top}\mathbf{H}^{*}(\omega)\mathbf{K}_{gs}^{\top} + \mathbf{K}_{gs}\mathbf{H}(\omega)\mathbf{A}\mathbf{S}_{\ddot{u}u}(\omega)\mathbf{B}^{\top} + \\ &+ \mathbf{B}\mathbf{S}_{u\ddot{u}}(\omega)\mathbf{A}^{\top}\mathbf{H}^{*}(\omega)\mathbf{K}_{gs}^{\top} + \mathbf{B}\mathbf{S}_{u}(\omega)\mathbf{B}^{\top}\Big)\, d\omega\end{aligned} \tag{3.18}$$

Taking into account Eq (3.14) and having in mind that the elements of the matrix of cross spectral densities displacements-accelerations are equal to

$$S_{ij}^{u\ddot{u}}(\omega) = S_{ij}^{\ddot{u}u}(\omega) = -\frac{1}{\omega^2}S_{ij}^{\ddot{u}\ddot{u}}(\omega) = -\frac{1}{\omega^2}S_{ij}(\omega) = -\frac{1}{\omega^2}\gamma_{ij}(\omega)S(\omega) \tag{3.19}$$

one obtains

$$\begin{aligned}\mathbf{D}_{fg} &= \int\limits_{-\infty}^{\infty} \Big[\mathbf{K}_{gs}\mathbf{H}(\omega)\mathbf{A}\boldsymbol{\Gamma}(\omega)\mathbf{A}^{\top}\mathbf{H}^{*}(\omega)\mathbf{K}_{gs}^{\top} - \frac{1}{\omega^2}\mathbf{K}_{gs}\mathbf{H}(\omega)\mathbf{A}\boldsymbol{\Gamma}(\omega)\mathbf{B}^{\top} + \\ &- \frac{1}{\omega^2}\mathbf{B}\boldsymbol{\Gamma}(\omega)\mathbf{A}^{\top}\mathbf{H}^{*}(\omega)\mathbf{K}_{gs}^{\top} + \frac{1}{\omega^4}\mathbf{B}\boldsymbol{\Gamma}(\omega)\mathbf{B}^{\top}\Big] S(\omega)\, d\omega\end{aligned} \tag{3.20}$$

4. Mean square response by mode superposition method

Assume now that the eigenproblem of the analyzed structure is solved and the natural circular frequencies of the structure $\omega_1, \omega_2, ..., \omega_{n_s}$ are known together with corresponding eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_{n_s}$ which form eigenmatrix $\mathbf{W}$. Applying modal transformation

$$q = \mathbf{W}\overline{q} \tag{4.1}$$

to the equation of motion (2.9) and pre-multiplying it by $\boldsymbol{w}_i^{\top}$ yields

$$\boldsymbol{w}_i^{\top}\mathbf{M}_{ss}\mathbf{W}\ddot{\overline{q}} + \boldsymbol{w}_i^{\top}\mathbf{C}_{ss}\mathbf{W}\dot{\overline{q}} + \boldsymbol{w}_i^{\top}\mathbf{K}_{ss}\mathbf{W}\overline{q} = \boldsymbol{w}_i^{\top}(\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg} - \mathbf{M}_{sg})\ddot{u} \tag{4.2}$$

where $\overline{q}$ stands for response vector in normal coordinates. Taking into account the orthogonality conditions

$$\begin{aligned}\boldsymbol{w}_i^{\top}\mathbf{M}_{ss}\boldsymbol{w}_j &= 0 \\ \boldsymbol{w}_i^{\top}\mathbf{K}_{ss}\boldsymbol{w}_j &= 0\end{aligned} \qquad \text{for } i \neq j \tag{4.3}$$

and assuming the same for damping matrix

$$\boldsymbol{w}_i^\top \mathbf{C}_{ss} \boldsymbol{w}_j = 0 \qquad \text{for } i \neq j \tag{4.4}$$

reduces Eq (4.2) to following system of uncoupled equations of motion

$$m_i \ddot{\overline{q}}_i + c_i \dot{\overline{q}}_i + k_i \overline{q}_i = \mathcal{P}_i(t) \qquad \text{for } 1 \leq i \leq n_s \tag{4.5}$$

where

$$\begin{aligned} m_i &= \boldsymbol{w}_i^\top \mathbf{M}_{ss} \boldsymbol{w}_i & c_i &= \boldsymbol{w}_i^\top \mathbf{C}_{ss} \boldsymbol{w}_i = 2\xi_i \omega_i m_i \\ k_i &= \boldsymbol{w}_i^\top \mathbf{K}_{ss} \boldsymbol{w}_i = \omega_i^2 m_i & \mathcal{P}_i &= \boldsymbol{w}_i^\top (\mathbf{M}_{ss} \mathbf{K}_{ss}^{-1} \mathbf{K}_{sg} - \mathbf{M}_{sg}) \ddot{u} \end{aligned} \tag{4.6}$$

For real structures, the assumption (4.4) can only be treated as a certain approximation. It gives however a possibility of specifying damping ratios ξ_i separately for various modes of vibration of the structure.

Thus Eqs (4.5) can be re-written in the standard form

$$\ddot{\overline{q}}_i + 2\xi_i \omega_i \dot{\overline{q}}_i + \omega_i^2 \overline{q}_i = \frac{\mathcal{P}_i}{m_i} \qquad \text{for } 1 \leq i \leq n_s \tag{4.7}$$

Introducing the matrix

$$\mathbf{G} = \operatorname{diag}\left[\frac{1}{m_i}\right] \mathbf{W}^\top (\mathbf{M}_{ss} \mathbf{K}_{ss}^{-1} \mathbf{K}_{sg} - \mathbf{M}_{sg}) \tag{4.8}$$

where $\operatorname{diag}[1/m_i]$ represents a diagonal matrix with the elements $1/m_i$ on its diagonal, gives the modal response in the following form

$$\ddot{\overline{q}}_i + 2\xi_i \omega_i \dot{\overline{q}}_i + \omega_i^2 \overline{q}_i = \sum_{j=1}^{n_g} G_{ij} \ddot{u}_j \qquad \text{for } 1 \leq i \leq n_s \tag{4.9}$$

The elements G_{ij} of matrix $\mathbf{G}$ may be called modal excitation participation factors. They control the participation of jth component of excitation in the vibrations of ith mode.

The response in the ith normal coordinate can be given by the Duhamel integral

$$\overline{q}_i(t) = \sum_{j=1}^{n_g} \int_0^t h_i(\tau) G_{ij} \ddot{u}_j(t-\tau)\, d\tau \tag{4.10}$$

where h_i is an impulse response function of the ith normal mode

$$h_i(t) = \frac{1}{\omega_i \sqrt{1-\xi_i^2}} \exp(-\xi_i \omega_i t) \sin(\omega_i t) \tag{4.11}$$

The corresponding frequency response function of the ith normal mode takes the form

$$H_i(\omega) = \int_{-\infty}^{\infty} h_i(\tau)e^{-i\omega\tau}\, d\tau = \frac{1}{\omega_i^2 - \omega^2 + 2i\xi_i\omega_i\omega} \tag{4.12}$$

Substituting the spectral decomposition

$$\ddot{u}_j(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d\hat{\ddot{u}}_j(\omega) \tag{4.13}$$

from Eq (3.4) into Eq (4.10) gives

$$\overline{q}_i(t) = \sum_{j=1}^{n_g} \int_0^t h_i(\tau)G_{ij} \int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\, d\hat{\ddot{u}}_j(\omega)d\tau \tag{4.14}$$

Taking into account the left part of Eq (4.12), after similar transformation as in Section 3, one obtains the response in normal coordinates

$$\overline{q}_i(t) = \sum_{j=1}^{n_g} G_{ij} \int_{-\infty}^{\infty} H_i(\omega)e^{i\omega t}\, d\hat{\ddot{u}}_j(\omega) \tag{4.15}$$

Consider now again the modal transformation (4.1) in index notation

$$q_k \cong \sum_{i=1}^{n_s} W_{ki}\overline{q}_i \tag{4.16}$$

where k is the nodal coordinate number. Substituting Eq (4.15) into (4.16) gives the following formula for the spectral decomposition of the response

$$q_k \cong \sum_{i=1}^{n_w} W_{ki} \sum_{j=1}^{n_g} G_{ij} \int_{-\infty}^{\infty} H_i(\omega)e^{i\omega t}\, d\hat{\ddot{u}}_j(\omega) \tag{4.17}$$

where n_s has been replaced by $n_w \leq n_s$ which makes it possible to apply only a necessary number of modes, reducing numerical effort in practical computations.

This formula, together with Eq (3.5) can be used to obtain any required statistics of the nodal displacements. For example, mean square displacements are equal to

$$\sigma_{q_k}^2 \cong \sum_{i=1}^{n_w}\sum_{p=1}^{n_w} W_{ki}W_{kp} \int_{-\infty}^{\infty} H_i(\omega)H_p^*(\omega) \sum_{j=1}^{n_g}\sum_{r=1}^{n_g} G_{ij}G_{pr}S_{jr}(\omega)\, d\omega \tag{4.18}$$

Taking into account Eqs (2.16) and (4.17) gives the spectral decomposition of force response

$$f_k \cong \sum_{p=1}^{n_w} K_{kp}^{gs} \sum_{i=1}^{n_w} W_{pi} \sum_{j=1}^{n_g} G_{ij} \int\limits_{-\infty}^{\infty} H_i(\omega)e^{i\omega t}\, d\widehat{\ddot{u}}_j(\omega) + \sum_{r=1}^{n_g} K_{kr}^{g} \int\limits_{-\infty}^{\infty} e^{i\omega t}\, d\widehat{u}_r(\omega) \tag{4.19}$$

where K_{kp}^{gs} denotes the elements of matrix $\mathbf{K}_{gs}$ and K_{kr}^{g} are the elements of matrix $\mathbf{K}_g$ given by the following formula

$$\mathbf{K}_g = \mathbf{K}_{gg} - \mathbf{K}_{gs}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg} \tag{4.20}$$

Applying Eqs (4.19) and (3.5) one may derive the mean square force response

$$\begin{aligned}
\sigma_{f_k}^2 \cong& \\
\cong& \sum_{p=1}^{n_s}\sum_{m=1}^{n_s} K_{kp}^{gs}K_{km}^{gs} \sum_{i=1}^{n_w}\sum_{n=1}^{n_w} W_{pi}W_{mn} \int\limits_{-\infty}^{\infty} H_i(\omega)H_n^*(\omega) \sum_{j=1}^{n_g}\sum_{r=1}^{n_g} G_{ij}G_{nr}S_{jr}^{\ddot{u}\ddot{u}}(\omega)d\omega + \\
&+\sum_{p=1}^{n_s} K_{kp}^{gs} \sum_{i=1}^{n_w} W_{pi} \sum_{j=1}^{n_g} G_{ij} \sum_{r=1}^{n_g} K_{kr}^{g} \int\limits_{-\infty}^{\infty} \mathrm{Re}\Big[H_i(\omega)S_{jr}^{\ddot{u}\ddot{u}}(\omega)\Big]\, d\omega + \\
&+\sum_{j=1}^{n_g}\sum_{r=1}^{n_g} K_{kj}^{g}K_{kr}^{g} \int\limits_{-\infty}^{\infty} S_{jr}^{uu}(\omega)\, d\omega
\end{aligned} \tag{4.21}$$

5. Example

Consider a simple multi-support structure; a dynamic one degree of freedom system with double support excitations (Fig.2). Two support points A and B are excited horizontally by plane waves propagating with the same apparent velocity in the entire frequency domain. In addition to propagation effects a loss of coherency due to wave refraction and attenuation occurs at the distance $|AB| = d$. These two effects are described by the coherency matrix of two excitation accelerations $\ddot{u}_A$ and $\ddot{u}_B$

$$\mathbf{\Gamma}(\omega) = \begin{bmatrix} 1 & \gamma_{AB} \\ \gamma_{AB}^* & 1 \end{bmatrix} \tag{5.1}$$

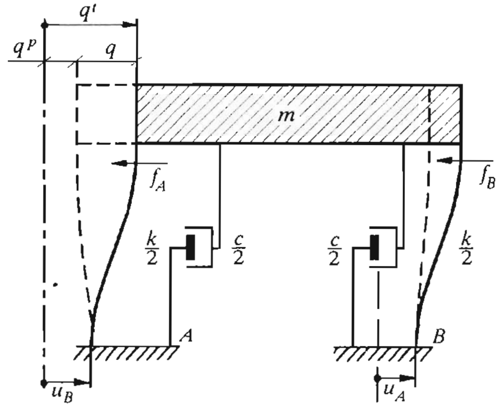

Fig. 2. One degree of freedom structure under double support excitations

where γ_{AB} is defined by Eqs (3.10) and (3.11). The coherency function can be written in terms of modulus and phase as well as real and imaginary parts

$$\gamma_{AB} = |\gamma_{AB}|e^{i\varphi}$$
$$\gamma_{AB} = \mathrm{Re}\gamma_{AB} + i\mathrm{Im}\gamma_{AB} \tag{5.2}$$

The modulus of coherency $|\gamma_{AB}|$ is called the loss of coherency (lagged coherency) and is a measure of similarity of signals at the points A and B excluding the effect of travelling waves. The effect of wave propagation is included in the phase φ. Real value of coherency $\mathrm{Re}\gamma_{AB}$ is called unlagged coherency. It includes both the effect of signal attenuation and wave propagation. Assuming plane waves propagating at the same apparent velocity for all frequencies one may write the phase term in the following form

$$\varphi = \varphi(\omega, d, v_g) = \frac{\omega d}{v_g} \tag{5.3}$$

where v_g is the apparent wave velocity.

The complex coherency function takes the following form

$$\gamma_{AB}(\omega, d, v_g) = |\gamma_{AB}(\omega, d, v_g)|e^{i\omega d/v_g} \tag{5.4}$$

Introducing the reduced apparent wave velocity $v_r = v_g/d$ which represents wave propagation in terms of support distances yields

$$\gamma_{AB}(\omega, d, v_r) = |\gamma_{AB}(\omega, d, v_r)|e^{i\omega/v_r} \tag{5.5}$$

The point spectral density function of horizontal accelerations will be a broad-band spectrum taken from seismic engineering following Kanai (1957), Tajimi (1960) as well as Ruiz and Penzien (1969)

$$S(\omega) = \frac{\omega_g^4 + 4\xi_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\xi_g^2\omega_g^2\omega^2} \frac{\omega^4}{(\omega_1^2 - \omega^2)^2 + 4\xi_1^2\omega_1^2\omega^2} S_0 \tag{5.6}$$

where S_0 is an intensity factor and $\omega_g = 4\pi, \xi_g = 0.6, \omega_1 = 1.636, \xi_1 = 0.619$.

To calculate the mean square response the direct method from Section 3 will be applied. For this system $n_s = 1$, $n_g = 2$ and $n = 3$. The main equation of motion (2.4) takes form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}^t \\ \ddot{u}_A \\ \ddot{u}_B \end{bmatrix} + \begin{bmatrix} c & -\frac{c}{2} & -\frac{c}{2} \\ -\frac{c}{2} & \frac{c}{2} & 0 \\ -\frac{c}{2} & 0 & \frac{c}{2} \end{bmatrix} \begin{bmatrix} \dot{q}^t \\ \dot{u}_A \\ \dot{u}_B \end{bmatrix} + \\ + \begin{bmatrix} k & -\frac{k}{2} & -\frac{k}{2} \\ -\frac{k}{2} & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & \frac{k}{2} \end{bmatrix} \begin{bmatrix} q^t \\ u_A \\ u_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.7}$$

with the structural sub-matrices $\mathbf{M}_{ss} = [m]$, $\mathbf{C}_{ss} = [c]$, $\mathbf{K}_{ss} = [k]$. Separating dynamic q and pseudo-static q^p motion according to Eqs (2.6) ÷ (2.9) and introducing the matrix $\mathbf{A}$ from Eq (3.1)

$$\mathbf{A} = \mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sg} - \mathbf{M}_{sg} = m\frac{1}{k}\left[-\frac{k}{2};\ -\frac{k}{2}\right] - [0;\ 0] = \left[-\frac{m}{2};\ -\frac{m}{2}\right] \tag{5.8}$$

gives the following equation of motion

$$m\ddot{q} + c\dot{q} + kq = \mathbf{A}\ddot{\boldsymbol{u}} = \left[-\frac{m}{2};\ -\frac{m}{2}\right] \begin{bmatrix} \ddot{u}_A \\ \ddot{u}_B \end{bmatrix} = -\frac{m}{2}(\ddot{u}_A + \ddot{u}_B) \tag{5.9}$$

Introducing natural frequency $\omega_0 = \sqrt{k/m}$, and damping ratio $\xi = c/(2m\omega_0)$ gives the equation of motion normalized with respect to unit mass

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2 q = -\frac{1}{2}(\ddot{u}_A + \ddot{u}_B) \tag{5.10}$$

For the above equation the frequency response function takes the form

$$H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2\mathrm{i}\xi\omega_0\omega} \tag{5.11}$$

Thus, following Eq (3.12), the mean square response is given by the following formula

$$\mathbf{D}_q = \left[\sigma_q^2\right] = \int_{-\infty}^{\infty} \left[\frac{H(\omega)}{m}\right] \left[-\frac{m}{2}; -\frac{m}{2}\right] \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma_{AB}^*(\omega) & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{m}{2} \\ -\frac{m}{2} \end{bmatrix} \left[\frac{H^*(\omega)}{m}\right] S(\omega)\, d\omega \tag{5.12}$$

and then

$$\sigma_q^2 = \frac{1}{2} \int_{-\infty}^{\infty} |H(\omega)|^2 \Big[1 + \text{Re}\gamma_{AB}(\omega)\Big] S(\omega)\, d\omega \tag{5.13}$$

The pseudo-static response q^p is equal (in this example) to the mean value of support displacements

$$\boldsymbol{q}^p = q^p = -\frac{1}{k} \left[-\frac{k}{2}; -\frac{k}{2}\right] \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \frac{1}{2}(u_A + u_B) \tag{5.14}$$

From Eq (3.13) the mean square pseudo-static response is

$$\mathbf{D}_{q^p} = \left[\sigma_{q^p}^2\right] = \int_{-\infty}^{\infty} \frac{1}{k} \left[-\frac{k}{2}; -\frac{k}{2}\right] \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma_{AB}^*(\omega) & 1 \end{bmatrix} \begin{bmatrix} -\frac{k}{2} \\ -\frac{k}{2} \end{bmatrix} \frac{1}{k} \frac{1}{\omega^4} S(\omega)\, d\omega \tag{5.15}$$

and then

$$\sigma_{q^p}^2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\omega^4} \Big[1 + \text{Re}\gamma_{AB}(\omega)\Big] S(\omega)\, d\omega) \tag{5.16}$$

Now, consider the mean square support forces. The support forces in this example are equal to the respective column shear forces f_A, f_B (Fig.2). Calculating the matrix $\mathbf{B}$

$$\mathbf{B} = \begin{bmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{bmatrix} - \frac{1}{k} \begin{bmatrix} -\frac{k}{2} \\ -\frac{k}{2} \end{bmatrix} \left[-\frac{k}{2}; -\frac{k}{2}\right] = \begin{bmatrix} \frac{k}{4} & -\frac{k}{4} \\ -\frac{k}{4} & \frac{k}{4} \end{bmatrix} \tag{5.17}$$

and applying Eq (3.18) yields

$$\mathbf{D}_{fg} = \int_{-\infty}^{\infty} \left\{ \begin{bmatrix} -\frac{k}{2} \\ -\frac{k}{2} \end{bmatrix} \left[\frac{H(\omega)}{m}\right] \left[-\frac{m}{2}; -\frac{m}{2}\right] \cdot \right.$$
$$\cdot \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma^*_{AB}(\omega) & 1 \end{bmatrix} \begin{bmatrix} -\frac{m}{2} \\ -\frac{m}{2} \end{bmatrix} \left[\frac{H^*(\omega)}{m}\right] \left[-\frac{k}{2}; -\frac{k}{2}\right] +$$
$$-\frac{1}{\omega^2} \begin{bmatrix} -\frac{k}{2} \\ -\frac{k}{2} \end{bmatrix} \left[\frac{H(\omega)}{m}\right] \left[-\frac{m}{2}; -\frac{m}{2}\right] \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma^*_{AB}(\omega) & 1 \end{bmatrix} \begin{bmatrix} \frac{k}{4} & -\frac{k}{4} \\ -\frac{k}{4} & \frac{k}{4} \end{bmatrix} + \quad (5.18)$$
$$-\frac{1}{\omega^2} \begin{bmatrix} \frac{k}{4} & -\frac{k}{4} \\ -\frac{k}{4} & \frac{k}{4} \end{bmatrix} \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma^*_{AB}(\omega) & 1 \end{bmatrix} \begin{bmatrix} -\frac{m}{2} \\ -\frac{m}{2} \end{bmatrix} \left[\frac{H^*(\omega)}{m}\right] \left[-\frac{k}{2}; -\frac{k}{2}\right] +$$
$$\left. + \frac{1}{\omega^2} \begin{bmatrix} \frac{k}{4} & -\frac{k}{4} \\ -\frac{k}{4} & \frac{k}{4} \end{bmatrix} \begin{bmatrix} 1 & \gamma_{AB}(\omega) \\ \gamma^*_{AB}(\omega) & 1 \end{bmatrix} \begin{bmatrix} \frac{k}{4} & -\frac{k}{4} \\ -\frac{k}{4} & \frac{k}{4} \end{bmatrix} \right\} S(\omega)\, d\omega$$

After some algebra the covariance matrix of force response is equal to

$$\begin{aligned} \mathbf{D}_{fg} &= \frac{k^2}{8} \int_{-\infty}^{\infty} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} |H(\omega)|^2 \Big(1 + \mathrm{Re}\gamma_{AB}(\omega)\Big) + \right. \\ &- \frac{2}{\omega^2} \mathrm{Im}\gamma_{AB}(\omega) \begin{bmatrix} \mathrm{Im} H(\omega) & \mathrm{iRe} H(\omega) \\ -\mathrm{iRe} H(\omega) & -\mathrm{Im} H(\omega) \end{bmatrix} + \\ &+ \left. \frac{1}{\omega^4} \Big[1 - \mathrm{Re}\gamma_{AB}(\omega)\Big] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\} S(\omega)\, d\omega \end{aligned} \quad (5.19)$$

In Fig.3 and Fig.4 the spectral density of displacements q (Fig.3) and force f_A (Fig.4) of the analyzed system (integrands from Eqs (5.13) and (5.19) are shown for no loss of correlation ($|\gamma_{AB}| = \text{const} = 1$) and two values of $v_r = 0.5$ and $v_r = 4.0$ (a,b) as well as for the total loss of correlation (c) i.e., for $|\gamma_{AB}| = \text{const} = 0$ and for coherent excitations (d), the natural frequency $\omega_0 = 2\pi$ rad/s and damping ratio $\xi = 0.05$. It can be seen from these two figures that unlike the displacements the force spectra display a low frequency hump left to the resonance peak. It results from pseudo-static contribution in the force response. Its contribution increases with decreasing velocity and is greatest when the total loss of coherency is assumed (Fig.4c). In Fig.5 the root mean square (RMS) displacements (Fig.5a) and force f_A (Fig.5b) are shown vs. the apparent wave velocity for no loss of coherency ($|\gamma_{AB}| = 1$). In addition two values not depending on velocity are shown:

– Result for total loss of coherency (solid straight line)

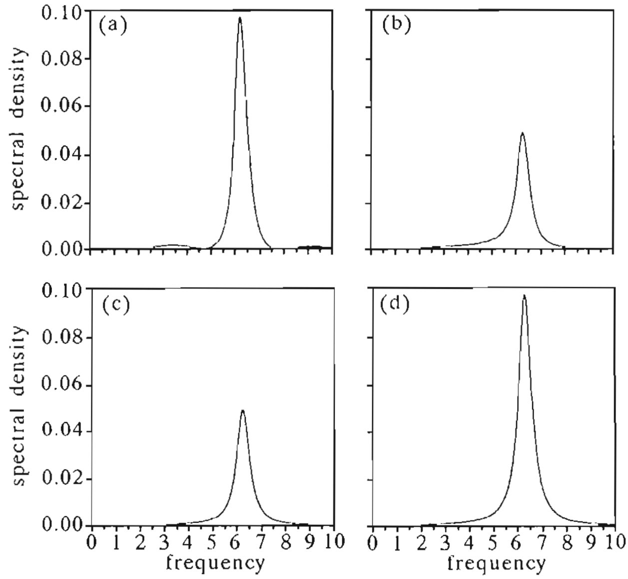

Fig. 3. Spectral densities of the displacement response q for no loss of coherency and $v_r = 0.5$ (a), $v_r = 4.0$ (b) as well as for total loss of coherency (c) and coherent excitations (d)

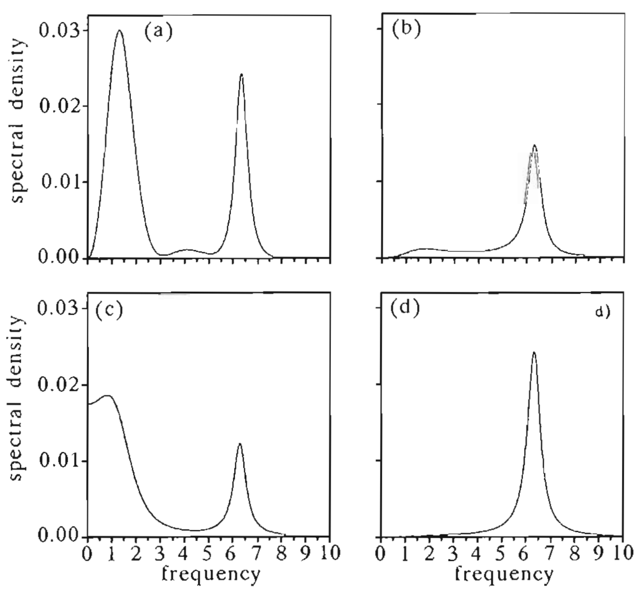

Fig. 4. Spectral densities of the force f_A response for no loss of coherency and $v_r = 0.5$ (a), $v_r = 4.0$ (b) as well as for total loss of coherency (c) and coherent excitations (d)

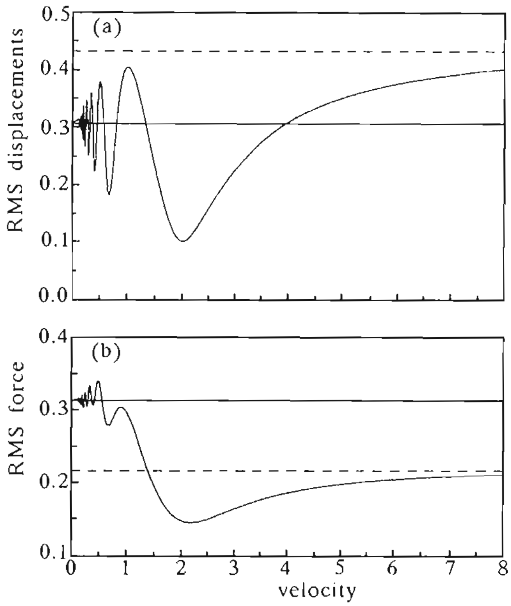

Fig. 5. RMS displacements (a) and force (b) vs. the apparent wave velocity for no loss of coherency, total loss of coherency (horizontal solid line) and for coherent excitations (horizontal dashed line). respectively

– Result for coherent excitations (dashed straight line).

It can be seen from Fig.5 that for low propagation velocity the RMS response oscillates about the results obtained for the total loss of coherency. As the velocity increases the RMS response goes asymptotically to the classic results obtained for coherent excitations (dashed line) as could be expected.

6. Conclusions

A matrix formulation of the linear response of discrete dynamic systems to propagating, random excitations has been presented. A simple structural system has been analyzed for illustration. Joint action of pseudo-static motion and dynamic response results in unique phenomenona which are not present for coherent excitations. It is interesting to note that the classic "coherent" results are conservative for the RMS displacements and may become unconservative for force response and low propagation velocity (Fig.5).

Acknowledgements

A partial support of State Committee for Scientific Research (KBN), Warsaw – grant number PB 0373/P4/94/06 – is greatfully acknowledged.

References

1. CHMIELEWSKI T., 1982, Metody probabilistyczne w dynamice konstrukcji, *Z.N WSI w Opolu*, seria: *Studia i Monografie*, z.1, Opole
2. HAO H., 1991, Response of Multiply Supported Rigid Plate to Spatially Correlated Seismic Excitations, *Earthquake Engineering and Structural Dynamics*, **20**, 821-838
3. HARICHANDRAN R.S., 1992, Random Vibration under Propagating Excitation: Closed-Form Solutions, *Journal of Engineering Mechanics, ASCE*, **118**, 575-586
4. HARICHANDRAN R.S., WANG W., 1988, Response of Simple Beam to Spatially Varying Earthquake Excitation, *Journal of Engineering Mechanics, ASCE*, **114**, 1526-1541
5. HARICHANDRAN R.S., WANG W., 1990, Response of Indeterminate Two-Span Beam to Spatially Varying Seismic Excitation, *Earthquake Engineering and Structural Dynamics*, **19**, 173-187
6. HEREDIA ZAVONI E., VANMARCKE E.H., 1994, Seismic Random-Vibration Analysis of Multi-Support Structural Systems, *Journal of Engineering Mechanics, ASCE*, **120**, 1107- 1128
7. KANAI K., 1957, Semi-Empirical Formula for the Seismic Characteristic of the Ground, *Bulletin of Earthquake Research Institute*, **35**, Tokyo, 309-325
8. LIN Y.K., 1976, *Probabilistic Theory of Structural Dynamics*, Krieger Publishing Company, New York
9. NADIM N., VANMARCKE E.H., GUDMESTEAD O.T., HETLAND S., 1991, Influence of Spatial Variation of Earthquake Motion on Response of Gravity Based Platforms, *Structural Safety*, **10**, 113-128
10. RUIZ. P., PENZIEN J., 1969, Probabilistic Study of the Behavior of Structures During Earthquakes, *Report No. EERC 69-03*, Earthquake Engineering Research Center, University of California, Berkeley, CA
11. SOBCZYK K., 1973, *Metody dynamiki statystycznej*, PWN, Warszawa
12. SOBCZYK K., 1991, *Stochastic Differential Equations with Application to Physics and Engineering*, Kluwer, Amsterdam
13. TAJIMI H., 1960, A Statistical Method of Determining the Maximum Response of a Building Structure During an Earthquake, *Proceedings of 2nd World Conference on Earthquake Engineering*, **2**, Tokyo, 781-798
14. ZEMBATY Z., KRENK S., 1993, Spatial Seismic Excitations and Response Spectra, *Journal of Engineering Mechanics, ASCE*, **119**, 2449-2460

15. ZERVA A., 1991, Effect of Spatial Variability and Propagation of Seismic Ground Motions on the Response of Multiply Supported Structures, *Probabilistic Engineering Mechanics*, **6**, 212-221

16. ZEMBATY Z., 1996, Spatial Seismic Coefficients, *Journal of Engineering Mechanics, ASCE*, **122**, 3, (in printing)

17. ZEMBATY Z., Vibrations of a Bridge Structure under Kinematic Wave Excitations, (to appear in Journal of Structural Engineering, ASCE)

Drgania losowe układów dyskretnych przy kinematycznym wymuszeniu falowym

Streszczenie

W pracy przedstawiono macierzowe sformułowanie dynamiki układów dyskretnych poddanych propagującym się wymuszeniom losowym. Rozważono łączne efekty pseudostatycznej i dynamicznej odpowiedzi układów. Podkreślono różnice w rozwiązaniach dla średniokwadratowych przemieszczeń i sił. Zastosowano metodę bezpośrednią oraz metodę superpozycji postaci drgań. Pracę zilustrowano prostym, praktycznym przykładem.

Manuscript received January 9, 1995; accepted for print October 25, 1995