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HYPERBOLIC HEAT CONDUCTION WITH VARIABLE RELAXATION TIME [1]

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There are several competing phenomenological models for the propagation of heat pulses. The semi-empirical heat conduction model uses the internal variable approach. It was previously shown that this model is in accordance with observed behaviour. The aim of the present paper is to choose material functions in a way to obtain good quantitative agreement with experimental data over the whole admissible temperature range. New approximations of experimental data are introduced and discussed together with the Clausius-Duhem inequality. The new model is verified by numerical results.

1. Introduction

The main idea of hyperbolic heat transfer models is that the heat flux q is not given as an immediate response to a temperature gradient $\nabla\theta$, but rather as a somehow delayed reaction. This delay may be introduced by history dependence (materials with memory, cf Gurtin and Pipkin (1968)), by an ordinary differential equation for the heat flux q with the classical Fourier law as forcing term (velocity type materials, cf Maxwell (1867); Cattaneo (1948); Vernotte (1958); Morro and Ruggeri (1987)), or by an internal (hidden)

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state variable (material with internal variables, cf Coleman and Gurtin (1967); Perzyna and Kosiński (1972)).

We will study here the latter case, and introduce one internal variable β – called the semi-empirical temperature. In that approach, the heat flux is related to the gradient of β by a Fourier-type law, rather than to that of the classical absolute temperature θ. The main problem is now definition of the kinetic equation which governs the time evolution of β. Further, the coefficient in the constitutive low of the heat flux and the specific heat have to be specified, respectively.

We have to study the following system of partial differential equations:

– constitutive law

$$q = -\alpha(\theta, \beta)\nabla\beta \tag{1.1}$$

– kinetic equation

$$\dot{\beta} = f(\theta, \beta) \tag{1.2}$$

– energy balance

$$\dot{\epsilon} = -\mathrm{div} q + r \tag{1.3}$$

The easiest possible choice is the following

ϵ	-	internal energy, $\epsilon = \epsilon(\theta)$
$\kappa(\theta)$	-	heat conductivity, $\alpha(\theta, \beta) = \kappa(\theta)$
τ	-	relaxation time, $f(\theta, \beta) = (\theta - \beta)/\tau$.

Note, that for constant κ and $c_v = \epsilon'$ the above system is equivalent to the classical Maxwell-Cattaneo-Vernotte equation. Hence it is natural to support the present assumptions by comparison with that classical case.

Further, for relaxed states, e.g., for stationary solutions, β coincides with θ.

In a more general approach, also dependencies on the gradients of θ and β are considered (cf Frischmuth and Cimmelli (1994); Kosiński and Wojno (1995)). Coleman and Neumann (1988), allow ϵ to be a function of q. In the present paper, we remain within the framework given by Cimmelli and Kosiński (1991). So, we assume the internal energy ϵ and the conductivity α to be functions of θ only, the free energy is allowed to depend on θ and $\nabla\beta$. The form of the right-hand side of the kinetic equation is now not assumed a priori, but identified from measurements of wave speeds and thermodynamical restrictions. Hence, we put aside the assumption that for relaxed states it holds $\beta = \theta$. Consequently, $\alpha(\theta)$ does no longer coincide with its counterpart for the stationary case $\kappa(\theta)$.

2. Experimental basis

We found NaF to be the best measured material which shows heat pulses. But even here, we have only three curves at our disposal from which we have to induce all constitutive functions. Of course, this is only possible, if we assume reasonable a priori restrictions, i.e., we must focus on the dependencies we believe to be essential.

The measured data are (cf Coleman and Neumann (1988); Jackson and Walker (1971) and papers cited there):

- The specific heat conductivity κ in the critical temperature range where heat waves are observed, i.e., between 10 and 20 K.

  An approximation[2] of measured data is given by

$$\kappa(\theta) = e^{e^{\kappa_0 + \kappa_1 \ln\theta + \kappa_2 \ln^2\theta}} \tag{2.1}$$

  where

$$\kappa_0 = -7.150703 \qquad \kappa_1 = 6.530065 \qquad \kappa_2 = -1.204074$$

- The wave velocity, derived from the appearance time of heat waves.

  We denote by U_E the speed of a wave travelling through material at an equilibrium state, i.e., into a domain where $q \equiv 0$.

  Approximation of measured data is given by

$$U_E = u_0 + u_1\theta \tag{2.2}$$

  where

$$u_0 = 0.41943 \qquad u_1 = -0.0127398$$

- The specific heat c_v from (modified) Debye's Law, approximated by

$$c_v(\theta) = (\epsilon_0 + \epsilon_1\theta)\theta^3 \tag{2.3}$$

  where

$$\epsilon_0 = 2.154452 \qquad \epsilon_1 = 0.023421$$

All theses three approximations are valid in the interval $[10, 20]$ (and should not be applied out of that range or to different materials).

[2]Throughout this paper we use only dimensionless variables. The values of constants and parameters of functions correspond to the following units: temperature (θ and β) in K, length in cm, time in μs, speed in cm/(μs), energy in J.

3. Clausius-Duhem inequality

We exploit now thermodynamics in order to obtain additional relations which are useful for the identification of constitutive functions.

First, we define the Helmholtz free energy ψ by

$$\psi = \epsilon - \theta\eta \tag{3.1}$$

with the entropy η.

The second law of thermodynamics takes the form

$$\dot{\eta} \geq -\mathrm{div}\left(\frac{q}{\theta}\right) + \frac{r}{\theta} \tag{3.2}$$

This, together with the energy balance, gives the Clausius-Duhem inequality

$$-\dot{\psi} - \eta\dot{\theta} - \frac{q\nabla\theta}{\theta} \geq 0 \tag{3.3}$$

For the present version of the semi-empirical model, i.e., for the system

$$q = -\alpha(\theta)\nabla\beta \qquad \dot{\beta} = f(\theta,\beta) \qquad c_v(\theta)\dot{\theta} + \mathrm{div}q = r\ , \tag{3.4}$$

we obtain, on the assumption that the Helmoltz free energy ψ depends on θ and $\nabla\beta$ only, i.e.,

$$\psi = \psi(\theta, \nabla\beta) \tag{3.5}$$

the following relations

$$\eta = -\psi_\theta \qquad q = -\psi_{\nabla\beta} f_\theta \theta \qquad 0 \geq \psi_{\nabla\beta} f_\beta \nabla\beta \tag{3.6}$$

We denote $\tau^{-1} = f_\theta$ and $\sigma^{-1} = -f_\beta$. From Eqs $(3.4)_1$ and $(3.6)_2$ it follows

$$\begin{aligned} \psi(\theta,\nabla\beta) &= \frac{1}{2}\psi_2(\theta)(\nabla\beta)^2 + \psi_0(\theta) \\ \psi_2(\theta) &= \frac{\tau\alpha}{\theta} \end{aligned} \tag{3.7}$$

We conclude from Eq $(3.7)_2$ and from $f = f(\theta,\beta)$ that $\tau = \tau(\theta)$. Further, $f_{\theta\beta} = 0$.

So we have

$$f(\theta,\beta) = f_1(\theta) + f_2(\beta)\ , \tag{3.8}$$

with

$$f_1(\theta) = \int_{\theta_0}^{\theta} \frac{d\theta}{\tau(\theta)} + C_1 \tag{3.9}$$

From Eqs $(3.6)_1$ and (3.1) it follows

$$\begin{aligned} &\epsilon(\theta) = \psi - \theta\psi_\theta = \frac{1}{2}\psi_2(\theta)(\nabla\beta)^2 + \psi_0(\theta) - \frac{1}{2}\theta\psi_2'(\theta)(\nabla\beta)^2 - \theta\psi_0'(\theta) \quad \Rightarrow \\ &\Rightarrow \quad \psi_2(\theta) = \theta\psi_2'(\theta) \quad \Rightarrow \quad \psi_2(\theta) = \psi_{20}\theta = \frac{\tau(\theta)\alpha(\theta)}{\theta} \quad \Rightarrow \\ &\Rightarrow \quad \tau = \psi_{20}\frac{\theta^2}{\alpha(\theta)} \end{aligned} \tag{3.10}$$

4. Relaxed states

A discussion of stationary solutions leads to the following consistency condition

$$\alpha\nabla\beta = \kappa\nabla\theta \tag{4.1}$$

In this case the semi-empirical temperature β is in its relaxed state $\beta_E(\theta)$ which is defined as the (unique) solution to

$$\dot{\beta} = 0 = f(\theta, \beta) \tag{4.2}$$

The implicit function theorem gives $\beta_E'(\theta) = \sigma_E/\tau$ with $\sigma_E(\theta) = -\partial f(\theta, \beta_E(\theta))/\partial\beta$. Hence, using (4.1) and the chain rule, we arrive at

$$\alpha = \frac{\tau}{\sigma_E}\kappa \tag{4.3}$$

Note that σ_E is still allowed to depend on θ while σ is a function of β only.

On the other hand from the hyperbolic system (3.4) we have the wave velocity U_E given by

$$U_E = \sqrt{\frac{\alpha}{c_v\tau}} = \sqrt{\frac{\kappa}{c_v\sigma_E}}\ , \tag{4.4}$$

which gives, together with (4.3) a formula for σ_E in terms of the three given curves

$$\sigma_E = \frac{\kappa}{c_v U_E^2} \tag{4.5}$$

Simple calculation yields now formulas for α and τ

$$\alpha = \theta\sqrt{\psi_{20} c_v U_E^2} \qquad \tau = \theta\sqrt{\frac{\psi_{20}}{c_v U_E^2}} \tag{4.6}$$

where ψ_{20} is the constant from (3.10). Let us pursue our analysis under the additional hypothesis $\sigma =$ const.

The right-hand side of the kinetic equation reads now as follows

$$f(\theta,\beta) = \int_{\theta_0}^{\theta} \frac{d\theta}{\tau(\theta)} - \frac{\beta}{\sigma} + C \tag{4.7}$$

Since, the calculation of σ from (4.5) does not give exactly a constant function, we define hence σ as the integral mean value of the function (4.5).

Now, we have to choose the two constants ψ_{20} and C. Note that $C = \theta_0/\sigma_E$ renders β_0 relaxed at θ_0. The choice of C is hence equivalent to the choice of the intersection θ_0 between the classical and the relaxed semi-empirical temperature scales, respectively. We choose θ_0 from the interval $[10, 20]$ where our approximations of the experimental data are valid.

Finally, the present version of the semi-empirical model can be re-written in a form very similar to the classic case

$$\dot{\beta} = \frac{\widehat{\theta} - \beta}{\sigma_E} \tag{4.8}$$

where

$$\widehat{\theta} = \theta_0 + \int_{\theta_0}^{\theta} \frac{\sigma_E}{\tau(\theta)}\, d\theta \tag{4.9}$$

If $\tau = \text{const} = \sigma$ the original model is reobtained. Otherwise we impose the condition $\tau(\theta_0) = \sigma_E$ which allows us to determine ψ_{20}. In this case the present model equations coincide with the original ones in the neighborhood of θ_0 up to the second order terms in $\theta - \theta_0$.

We perform now numerical experiments and study the influence of both parameters θ_0 (resp. C) and ψ_{20} on the behaviour of the numerical solutions to $(3.4)_2$ and $(3.4)_3$. Fig.1 presents a typical pulse. We assumed here a homogeneous and relaxed state as the initial condition, a Dirichlet condition $\theta = \theta_0$ on the left boundary $x = 0$ and a Neumann condition $q = 0$ at the right boundary $x = l$. For details of the set-up of heat pulse experiments (geometry of the specimen, pulse duration and height), cf Frischmuth and Cimmelli (1995).

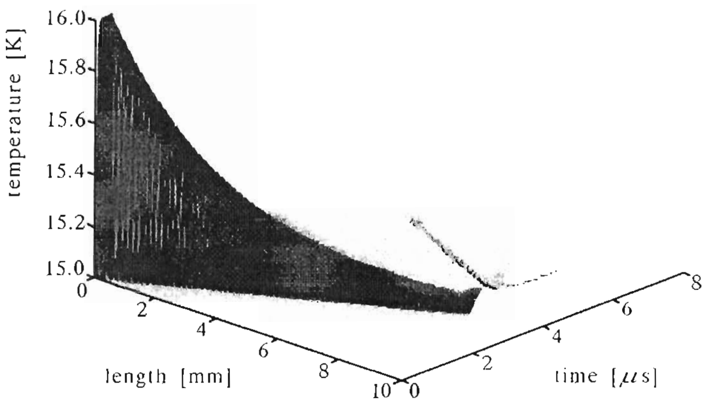

Fig. 1. Numerical solution for $\theta_0 = 15$

It turns out that the decay of the amplitude depends very strongly on σ while the influence of θ_0 and ψ_{20} is negligible. Most essential is the dependency of the arrival times on the initial value of the temperature. Fig.2 presents the temperature increment on the right end of the specimen as a function of initial temperature and time.

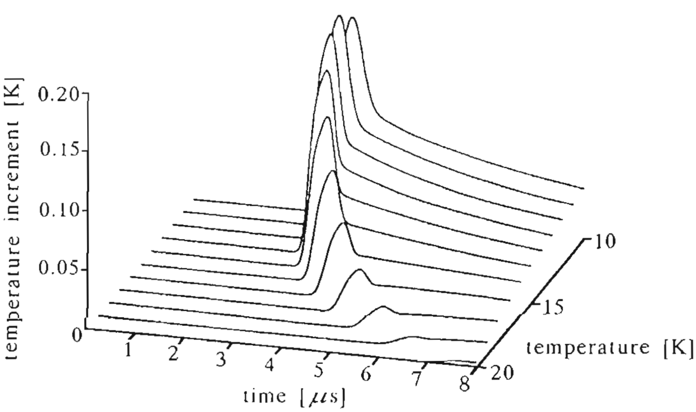

Fig. 2. Temperature increment at right end of the specimen

Comparison with the experimental appearance times (leading edge as well as maximum) shows an acceptable agreement. However, it is obvious that the maximal pulse amplitude occurs at a slightly lower temperature than in the experiments.

5. Concluding remarks

We have constructed a thermodynamically admissible semi-empirical heat conduction model which reproduces the wave speeds observed in second sound experiments. We removed two drawbacks of the previous variant which was used in the first numerical experiments. The first concerns available data of the specific heat and heat conductivity, in particular the latter curve exhibits a peak at the critical temperature. The second drawback concerns the variable relaxation time, meeting the requirements of the second law, while at the same time giving the right wave speed in a certain range of temperatures. The modelling of the pulse amplitudes, improvement of the approximation of the wave speed as well as the consideration of coupling with elastic waves is still under way.

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Hiperboliczne przewodnictwo ciepła ze zmiennym czasem relaksacji

Streszczenie

Jest wiele współzawodniczących modeli rozprzestrzeniania się impulsów ciepła. Model przewodnictwa ciepła, wykorzystujący koncepcję temperatury semi-empirycznej, jest zbudowany w ramach podejścia z wewnętrznymi zmiennymi stanu. Celem niniejszego artykułu jest dobór funkcji materiałowych w taki sposób, aby uzyskać ilościową zgodność z danymi doświadczalnymi z pełnej dziedziny dopuszczalnych temperatur. Przedstawiono nowe przybliżenia analityczne danych doświadczalnych i przedyskutowano nierówności Clausiusa-Duhema. Nowy model weryfikuje się numerycznie.

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