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A MASS POINT MOVING ON A NONSMOOTH MANIFOLD IN $\mathbb{R}^n$ [1]

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We consider a mass point moving on a $n-1$ dimensional manifold with a wedge. To avoid a product of distributions in the equations of motion we regularize the wedge by smooth functions with a small parameter ϵ. Then the question arises whether there exists a limit for the velocity vector after passing the smoothed wedge. In this paper we will give a class of regularizations for which the limit exists and is independent of the special choice of the regularization itself. Furthermore, we give estimates for the quality of approximation depending on the parameter ϵ

1. Introduction

The description of a rigid body motion under a constraint by the position of its center of mass in general results in a nonsmooth admissible manifold. Especially we are interested in modelling a train on a rail (cf Frischmuth et al. (1994)). In the case of rail-wheel contact the center of mass of the wheel is passing a wedge when the contact point is jumping from the thread to the flange. By a wedge we denote in this context a $n-2$ dimensional submanifold where the gradient of the admissible manifold has jump discontinuities. A 2D-cut through the admissible manifold for realistic rail and wheel profiles is plotted below for illustration.

Having in mind the system of Lagrange's equations for this motion we immediately see that it can not be fulfilled in the classical sense when the mass point is crossing the wedge. This analytical difficulty also afflicts the

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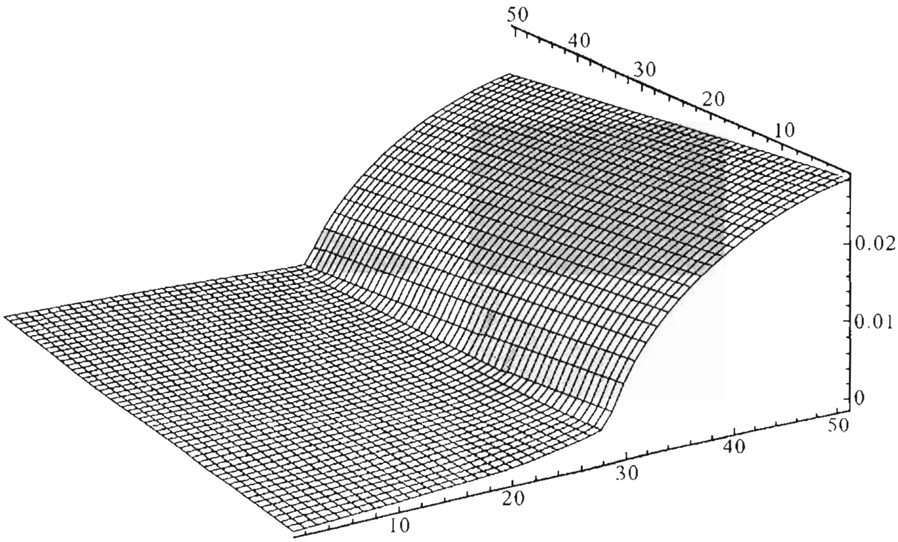

Fig. 1. Admissible manifold

numerical treatment of the problem forcing a small integration step size and therefore takes much more computing times.

The aim of this paper is to find a class of smooth regularizations of the wedge depending on a small parameter ϵ in such a way that as $\epsilon \to 0$ the wedge is reestablished and that the corresponding solutions to Lagrange's equations converge to a unique limit. To this end we use a small parameter technique which has approved to be a powerful tool to handle, e.g., self similar structures (cf Sanchez-Palencia (1980)) and irregular boundaries (cf Maz'ya and Hänler (1993); Frischmuth et al. (to appear)). In the latter context this technique is also referred to as the *homogenization method*. For a variety of applications of the method see Nayfeh (1973).

In the next section we will specify our model assumptions. The regularization procedure and the resulting limit problem as $\epsilon \to 0$ will be treated in Section 3. In Section 4 we will show with the help of error estimates that there exists a right limit of the velocity vector $v^+(0) = f(v^-(0))$ which is uniquely determined by the left limit for a certain class of regularizations. We show that f depends on local geometry data of the wedge only.

2. Model equations

We consider as a model of a wedge a constraint of the following type

$$x_n = \varphi(x') = a(x') + b(x')|c(x') - c(x'_0)| , \tag{2.1}$$

with $x = (x', x_n) \in \mathbb{R}^n$, $a, b, c \in C^2(\mathbb{R}^{n-1})$ (cf Frischmuth et al. (1994)).

Further we assume that the mass point hits the wedge at the point $x'_0 \in \mathbb{R}^{n-1}$ with $\nabla c(x'_0) \neq 0$.

Remark 1. *We want to emphasize that the above smoothness assumptions on a, b, c are the only essential suppositions we put on the wedge. The non-vanishing gradient for $c(x')$ at x'_0 ensures that there is a real wedge at this point excluding that way the trivial case.*

We introduce a new coordinate system in $\mathbb{R}^{n-1}$ originated at x'_0 in such a way that e_1 points in the direction of $\nabla c(x'_0)$ and that $\nabla a(x'_0)$ lies in the plane $e_1 e_2$. In a δ-neibourghood of 0 the constraint can be described in the form

$$\gamma(q) = \alpha_0 + \alpha_1 q_1 + \alpha_2 q_2 + \beta_0 |q_1| + \mathcal{O}(\delta^2) , \tag{2.2}$$

where

$$\alpha_0 = a(x'_0) \qquad \alpha_1 = \frac{\partial a}{\partial q_1}(x'_0)$$

$$\alpha_2 = \frac{\partial a}{\partial q_2}(x'_0) \qquad \beta_0 = \frac{b(x'_0)}{||\nabla c(x'_0)||}$$

For the motion of the mass point we have the system of Lagrange's equations

$$\frac{d}{dt} L_{\dot q} - L_q = Q \tag{2.3}$$

with the notations

- T – kinetic energy, $T(q, \dot q) = \frac{1}{2} m \{\dot q^2 + [\nabla \gamma(q) \dot q]^2\}$
- U – potential energy, $U(q) = mg\gamma(q)$
- L – Lagrange function, $L(q, \dot q) = T(q, \dot q) - U(q)$
- v – velocity vector, $v = [\dot q, \dot{\gamma(q)}]^\top$
- R – friction forces, $R = -\mu m \kappa |v| v$
- Q – generalized forces, $Q = -\mu m \kappa |v| A \dot q$

and

$$A = I + [\nabla \gamma(q)][\nabla \gamma(q)]^\top .$$

Then the system (2.3) takes the form

$$A\ddot{q} + \dot{q}^{\top} H(\gamma)\dot{q}\nabla\gamma(q) + g\nabla\gamma(q) = -\mu\kappa|v|A\dot{q} \ , \tag{2.4}$$

with $H(\gamma)$ denoting the Hessian matrix of γ. It is immediately seen from the form of γ that $H(\gamma)$ includes a δ-distribution and $\nabla\gamma(q)$ includes a Heaviside-type function for $q_1 = 0$. So a product of distribution is involved in the middle term on the left-hand side of (2.4) (cf Schwartz (1950); Colombeau and Roux (1988); Biagioni (1988); Berg and Frischmuth (1994)).

3. Regularization

First of all we notice that a change in $\gamma(q)$ by a constant has no influence on Eq (2.4). So for the sake we will put $\alpha_0 = 0$ from now on. Since we are interested in regularization near $q_1 = 0$ it is enough to consider the following linearized version of the constraint γ for any fixed $\delta > 0$

$$\gamma^{\delta}(q) = \Big[1 - \chi\Big(\frac{q_1}{\delta}\Big)\Big]\gamma(q) + \chi\Big(\frac{q_1}{\delta}\Big)(\alpha_1 q_1 + \alpha_2 q_2 + \beta_0|q_1|) \ . \tag{3.1}$$

where $\chi \in C^{\infty}(\mathbb{R})$ is a cut off function with

$$\chi(x) = \begin{cases} 1 & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| > 2 \end{cases}$$

Without loss of generality we assume that $\beta_0 > 0$. Let $\varphi \in C^2(\mathbb{R})$ such that

$$\varphi_\epsilon(x) = \epsilon\varphi\Big(\frac{x}{\epsilon}\Big)$$

be subject to the following conditions

$$\begin{array}{lll} \varphi_\epsilon(x) \to |x| & \forall x \in \mathbb{R} & \wedge \quad \epsilon \to 0 \\ \varphi'_\epsilon(x) \to \operatorname{sgn}(x) & \forall x \neq 0 & \wedge \quad \epsilon \to 0 \\ 0 \leq \varphi''_\epsilon(x) \to 0 & \forall x \neq 0 & \wedge \quad \epsilon \to 0 \end{array} \tag{3.2}$$

Remark 2. *As a standard example for φ we may consider the function*

$$\varphi(x) = \frac{2}{\pi}\int_0^x \arctan(s)\, ds \tag{3.3}$$

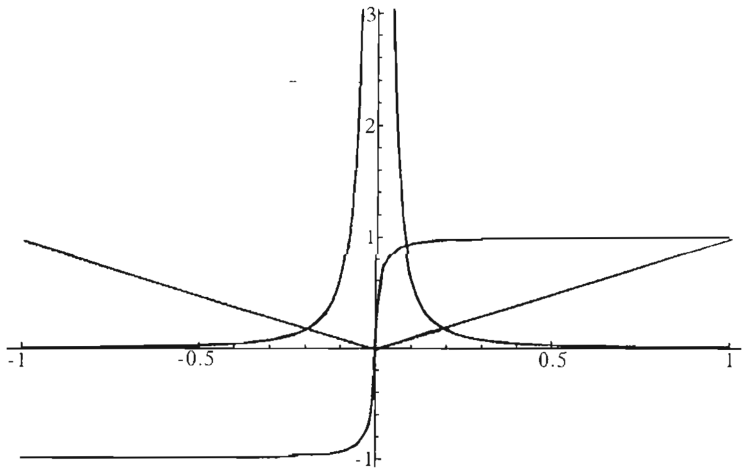

Fig. 2. $\varphi_\epsilon, \varphi'_\epsilon, \varphi''_\epsilon$ for $\epsilon = 0.01$

For the following confine ourselves to a δ-neighbourhood of $q_1 = 0$. Here we replace the modulus by φ_ϵ, i.e., we consider a constraint of the form

$$\gamma_\epsilon^\delta(q) = \alpha_1 q_1 + \alpha_2 q_2 + \beta_0 \epsilon \varphi\Big(\frac{q_1}{\epsilon}\Big) \tag{3.4}$$

To get rid of the ϵ we introduce new coordinates and a new time variable as follows

$$\tau = \frac{t - t_0}{\epsilon} \qquad\qquad \xi = \frac{q}{\epsilon}$$

Denoting by $\dot{\xi}$ the derivative with respect to τ we observe that

$$\dot{\xi} = \dot{q} \qquad\qquad \ddot{\xi} = \epsilon \ddot{q}$$

The constraint (3.4) now reads

$$\gamma(\xi) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \beta_0 \varphi(\xi_1) \tag{3.5}$$

The dependence on ϵ is now included in the ξ-variable and δ is fixed so we omitted the indices of γ above and in the following to keep the notations simpler. From the form of γ, see Eq (3.5), we immediately obtain that the Hessian matrix $H(\gamma)$ has just one entry, i.e.

$$H\Big(\gamma(\xi)\Big) = \frac{1}{\epsilon} \beta_0 \varphi''(\xi_1) e_1 e_1^T$$

Using the abbreviations

$$n(\xi_1) = |\nabla\gamma(\xi)| \qquad\qquad \widetilde{\kappa} = \epsilon\kappa|v| \,,$$

and upon the fact that

$$A^{-1} = I - \frac{1}{1+n^2}\nabla\gamma)(\nabla\gamma)^{\top} \,.$$

we can rewrite the system (2.4) in the new coordinates as follows

$$\ddot{\xi} + \beta_0\varphi''(\xi_1)\frac{1}{1+n^2(\xi_1)}\dot{\xi}_1^2\nabla\gamma(\xi) + \epsilon g\frac{1}{1+n^2(\xi_1)}\nabla\gamma(\xi) = -\mu\widetilde{\kappa}\dot{\xi} \tag{3.6}$$

By standard calculations we find

$$\widetilde{\kappa}^2 = \frac{\left[\ddot{\xi}^2 + \left(\mathcal{N}(\xi)\right)^2\right]\left[\dot{\xi}^2 + (\nabla\gamma,\dot{\xi}_1)^2\right] - \left[(\ddot{\xi},\dot{\xi}) + \mathcal{N}(\xi)(\nabla\gamma,\dot{\xi})\right]^2}{\left[\dot{\xi}^2 + (\nabla\gamma,\dot{\xi})^2\right]^2} \,, \tag{3.7}$$

where

$$\mathcal{N}(\xi) = \beta_0\varphi''(\xi_1)\dot{\xi}_1^2 + (\nabla\gamma.\ddot{\xi})$$

As usual in small parameter methods we introduce the formal expansion

$$\xi(\tau) = \sum_{n=0}^{\infty}\epsilon^n\xi^{(n)}(\tau) \tag{3.8}$$

Comparing the terms with respect to powers of ϵ we can get the resulting equations for each $\xi^{(n)}(\tau)$, $n = 1, \ldots$. For our aim it is enough to establish the equation in the zero order approximator $\xi^{(0)}$

$$\ddot{\xi} + \beta_0\varphi''(\xi_1)\frac{1}{1+n^2(\xi_1)}\dot{\xi}_1^2\nabla\gamma(\xi) = -\mu\widetilde{\kappa}\dot{\xi} \tag{3.9}$$

We will show in Section 4 that the difference $\xi - \xi^{(0)}$ is $\mathcal{O}(\epsilon)$ as $\epsilon \to 0$.

For the case $\mu = 0$ solution to the system (3.9) is obtained immediately

$$\dot{\xi} = \nu(\xi_1) = \begin{bmatrix} C_1\cos\psi \\ C_2 - C_1\dfrac{\alpha_2}{\sqrt{1+\alpha_2^2}}\sin\psi \\ C_3 \\ \vdots \\ C_{n-1} \end{bmatrix} \tag{3.10}$$

where $\psi(\xi_1)$ denotes the angle

$$\psi(\xi_1) = \arctan \frac{\alpha_1 + \beta_0\varphi'(\xi_1)}{\sqrt{1+\alpha_2^2}} \,. \tag{3.11}$$

and the constants C_i are determined by the initial velocity $v_{-\delta}(q)$ at $q_1 = -\delta$.

For the case with friction, i.e., $\mu > 0$ we presume a solution of the form

$$\dot{\xi} = e^{-\mu\rho(\xi_1)}\nu(\xi) \,.$$

where ν is the solution in the frictionless case. Substituting into Eq (3.7) we get

$$\begin{aligned} \tilde{\kappa}^2 &= \frac{\exp[-2\mu\rho(\xi_1)]}{\left[\nu^2 + (\nabla\gamma,\nu)^2\right]^2}\Big\{\nu'^2\nu_1^2 + \Big(\beta_0\varphi''(\xi_1)\nu_1^2 + \nu_1(\nabla\gamma,\nu')\Big)^2 + \\ &- \Big[\nu_1(\nu',\nu) + \Big(\beta_0\varphi''(\xi_1)\nu_1^2 + \nu_1(\nabla\gamma,\nu')\Big)(\nabla\gamma,\nu)\Big]^2\Big\} = \\ &= \left(\frac{\beta_0\varphi''(\xi_1)}{1+n^2}\nu_1^2\right)^2 \frac{1+n^2}{\nu^2 + (\nabla\gamma,\nu)^2} \end{aligned} \tag{3.12}$$

From Eq (3.9) we obtain the following equation in ρ

$$\begin{aligned} \rho' &= \frac{\beta_0|\varphi''(\xi_1)|}{\sqrt{1+n^2}}\nu_1 \frac{1}{\sqrt{\nu^2 + (\nabla\gamma,\nu)^2}} = \\ &= \frac{C_1}{|v^-|}\sqrt{\frac{1+n^2}{1+\alpha_2^2}}\psi' \cos\psi = \frac{C_1}{|v^-|}\psi' \end{aligned} \tag{3.13}$$

Here we used again the ψ angle introduced in Eq (3.10) and the fact that $\varphi''(\xi_1) \geq 0$, $\xi_1 \in \mathbb{R}$. Further we notice that the speed for $\mu = 0$

$$|v^-| = \sqrt{\nu^2 + (\nabla\gamma,\nu)^2} = \sqrt{C_1^2 + (1+\alpha_2^2)C_2^2 + C_3^2 + \ldots + C_{n-1}^2}$$

is independent of ξ_1 and therefore equal to the initial speed $|v_{-\delta}|$ at $q_1 = -\delta$. This states simply the fact that in the frictionless case the speed remains unchanged after passing the wedge. Let us denote by

$$\psi^{\pm} = \arctan \frac{\alpha_1 \pm \beta_0}{\sqrt{1+\alpha_2^2}}$$

the limit angle for $\xi_1 \to \pm\infty$. With the help of Eq (3.13) we find the solution to Eq (3.9)

$$\dot{\xi} = e^{-\mu \frac{C_1}{|v^-|}(\psi - \psi^-)} \nu(\xi_1) \tag{3.14}$$

Now we are able to impose the limit on the zero order approximator $\dot{\xi}^+$ as $\xi_1 \to \infty$

$$\dot{\xi}^+ = \exp\Big(-\mu \frac{C_1}{|v^-|}(\psi^+ - \psi^-)\Big) \begin{bmatrix} C_1 \cos \psi^+ \\ C_2 - \frac{\alpha_2}{\sqrt{1+\alpha_2^2}} C_1 \sin \psi^+ \\ \tilde{C} \end{bmatrix} \tag{3.15}$$

Here $\tilde{C}$ denotes an $n-3$ dimensional vector of constants, see Eq (3.10). For the initial velocity

$$\dot{\xi}^- = \dot{q}^-$$

we calculate straightforward the constants

$$C_1 = \frac{\dot{q}_1^-}{\cos \psi^-} \qquad C_2 = \dot{q}_2^- + \frac{\alpha_2}{\sqrt{1+\alpha_2^2}} \tan(\psi^-) \dot{q}_1^-$$
$$C_i = \dot{q}_i^- \qquad \forall\, i \in \{3, \dots, n-1\} \tag{3.16}$$

Combining Eqs (3.15) and (3.16) we obtain for the components of the velocity of the mass point after passing the wedge

$$\begin{bmatrix} \dot{q}_1^+ \\ \dot{q}_2^+ \\ \dot{q}_i^+ \end{bmatrix} = \exp\Big(-\frac{\mu \dot{q}_1^- (\psi^+ - \psi^-)}{|v^-| \cos \psi^-}\Big) \begin{bmatrix} \frac{\cos \psi^+}{\cos \psi^-} \dot{q}_1^- \\ \dot{q}_2^- - \frac{\alpha_2}{\sqrt{1+\alpha_2^2}} \frac{\sin \psi^+ - \sin \psi^-}{\cos \psi^-} \dot{q}_1^- \\ \dot{q}_i^- \end{bmatrix} \tag{3.17}$$

Remark 3. *It is easily seen from Eq (3.17) that the velocity vector after passing the wedge is independent of the special choice of the regularizing function φ. We only consider the fact that φ' is monotonic increasing and has the above required approximation properties.*

Further we want to emphasize that a change of velocity while passing the wedge happens only in the two directions e_1, e_2 which are determined by the gradients of c and a at x_0'.

In the following Section we will give error estimates depending on ϵ and δ.

4. Error estimates

First of all we want to define the class of regularizing functions in a more straighforward way. Let $\varphi \in C^2(\mathbb{R})$ be subject to the following conditions

(i)	$\lvert\varphi(x) - \lvert x\rvert\,\rvert \le C\lvert x\rvert^{1-\sigma}$	$\forall x \in \mathbb{R}$
(ii)	$\lvert\varphi'(x) - \mathrm{sgn}(x)\rvert \le C\lvert x\rvert^{-\sigma}$	$\forall x \neq 0$
(iii)	$0 \le \varphi''(x) \le C\lvert x\rvert^{-1-\sigma}$	$\forall x \neq 0$

for fixed $\sigma > 0$. Further we denote by $\dot{\xi} = \nu(\xi_1)$ the solution to Eq (3.6) with the initial values $\nu(-\delta/\epsilon) = \dot{q}^-$ and by $\dot{\xi}^{(0)} = \nu^{(0)}(\xi_1)$ the solution to Eq (3.9) obtained in Section 2 with the same initial data.

We will estimate the error produced by our regularization technique in two steps. In the first lemma we will find bounds on the difference between the exact solution to Eq (3.6) and its zero order approximator. In the second one we estimate the error produced by the cut off function.

Lemma 1. *For any $\varphi(x)$ subject to the above conditions and any $\epsilon > 0$ it holds*

$$|\nu(\xi_1) - \nu^{(0)}(\xi_1)| \le C|\epsilon\xi_1 + \delta|\,, \qquad -\frac{\delta}{\epsilon} \le \xi_1 \le \frac{\delta}{\epsilon} \tag{4.1}$$

with a constant $C = C(\varphi)$ if δ is sufficiently small.

• **Proof:** First we remark that the solution $\nu(\xi_1)$ to Eq (3.6) remains constant in the components $3, \ldots, n-1$. Thus it is enough to estimate ν_1 and ν_2. As before we start with the frictionless case, i.e., $\mu = 0$. We find that ν_1 is a solution to the Bernoulli equation

$$\nu_1' + \beta_0\varphi''(\xi_1)\frac{1}{1+n^2}(\alpha_1 + \beta_0\varphi')\nu_1 = -\epsilon g\frac{1}{1+n^2}(\alpha_1 + \beta_0\varphi')\nu_1^{-1} \tag{4.2}$$

Straightforward we compute the solution

$$\nu_1(\xi_1) = \sqrt{C\cos^2(\psi) + \frac{\epsilon g}{1+\alpha_2^2}[\alpha_1\xi_1 + \beta_0\varphi(\xi_1)]} \tag{4.3}$$

Using the fact that both solutions have the same initial condition at $\xi_1 = -\delta/\epsilon$ we obtain

$$|\nu_1(\xi_1) - \nu_1^{(0)}(\xi_1)| \le C|\epsilon\xi_1 + \delta|\,, \qquad -\frac{\delta}{\epsilon} \le \xi_1 \le \frac{\delta}{\epsilon} \tag{4.4}$$

The component ν_2 is subject to the linear equation

$$\nu_2' + \beta_0\varphi''(\xi_1)\frac{1}{1+n^2}(\alpha_1+\beta_0\varphi')\nu_1 = -\epsilon g\frac{1}{1+n^2}\alpha_2\nu_1^{-1} \tag{4.5}$$

Then from Eq (4.4) it follows immediately

$$|\nu_2(\xi_1) - \nu_2^{(0)}(\xi_1)| \leq C|\epsilon\xi_1 + \delta|\,, \qquad -\frac{\delta}{\epsilon} \leq \xi_1 \leq \frac{\delta}{\epsilon}\,, \tag{4.6}$$

if δ is sufficiently small. So with the case of no friction we are done.

If $\mu > 0$ we again presume a solution of the form $\dot{\xi} = \exp[-\mu\rho(\xi_1)]\nu(\xi)$. With the help of formula (3.13) and the above estimates we get

$$|\rho'^2(\xi_1) - \rho^{(0)\prime 2}(\xi_1)| \leq C|\epsilon\xi_1 + \delta|\,, \qquad -\frac{\delta}{\epsilon} \leq \xi_1 \leq \frac{\delta}{\epsilon}$$

Since Eqs (3.13), (3.14) and (4.3) imply that ρ' and $\rho^{(0)\prime}$ both are positive we have

$$|\rho'(\xi_1) - \rho^{(0)\prime}(\xi_1)| \leq C\sqrt{|\epsilon\xi_1 + \delta|}$$

and therefore

$$|\rho(\xi_1) - \rho^{(0)}(\xi_1)| \leq C|\epsilon\xi_1 + \delta|^{\frac{3}{2}}\,, \qquad -\frac{\delta}{\epsilon} \leq \xi_1 \leq \frac{\delta}{\epsilon} \tag{4.7}$$

A combination of Eqs (4.4), (4.6) and (4.7) yields

$$|e^{-\mu\rho(\xi_1)}\nu(\xi) - e^{-\mu\rho^{(0)}(\xi_1)}\nu^{(0)}(\xi)| \leq C|\epsilon\xi_1 + \delta|\,, \qquad -\frac{\delta}{\epsilon} \leq \xi_1 \leq \frac{\delta}{\epsilon} \tag{4.8}$$

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In order to get the proposed assertion concerning the velocities on the original curve before and after the wedge it remains to estimate the error produced by the cut off function. We get:

Lemma 2. *Let $q(t)$ be the solution to Eq (2.4) for the constraint γ_ϵ^δ. Then it holds for sufficiently small ϵ*

$$|\dot{q}_{\pm 2\delta} - \dot{q}_{\pm\delta}| \leq C\delta \tag{4.9}$$

with a constant $C = C(\chi)$.

• **Proof:** It is well known that for any constraint $\gamma \in C^2(\mathbb{R}^{n-1})$ there exists a C^2-solution $q(t)$ to Eq (2.4) for any given initial values. Furthermore the second time derivative $\ddot{q}(t)$ is bounded by a constant depending only on the initial values and on $\|\gamma\|_{C^2}$. Hence to get the above assertion it is enough to show that for a sufficiently small ϵ

$$\|\gamma_\epsilon^\delta(q)\|_{C^2} \le C\,, \qquad \delta \le q_1 \le 2\delta\,,$$

with a constant C independent of δ. We get

$$H\left(\gamma_\epsilon^\delta(q)\right) = [1-\chi(q_1)]H\left(\gamma(q)\right) + \chi(q_1)H\left(\gamma_\epsilon(q)\right) + \tag{4.10}$$
$$+\chi'(q_1)\Big\{e_1\nabla[\gamma_\epsilon(q)-\gamma(q)]^{\mathsf T} + \nabla[\gamma_\epsilon(q)-\gamma(q)]e_1^{\mathsf T}\Big\} + \chi''(q_1)[\gamma_\epsilon(q)-\gamma(q)]$$

The first term on the right-hand side is globally bounded by virtue of our assumptions about γ and χ. For the second term we find

$$\|H\left(\gamma_\epsilon(q)\right)\| = \left|\frac{\beta_0}{\epsilon}\varphi''\Big(\frac{q_1}{\epsilon}\Big)\right| < C\epsilon^\sigma\delta^{-1-\sigma}\,. \qquad \delta < q_1 < 2\delta\,.$$

i.e., it is bounded by a constant independent of δ for any $\epsilon < \delta^{1+1/\sigma}$. Now we consider the third term of Eq (4.10). We have

$$|\chi'(q_1)| \le C\delta^{-1}$$
$$\|\nabla(\gamma_\epsilon-\gamma_\delta)\| \le C\left|\varphi'\Big(\frac{q_1}{\epsilon}\Big)-1\right| \le C\Big(\frac{\epsilon}{\delta}\Big)^\sigma \le C\delta$$

for $\epsilon < \delta^{1+1/\sigma}$. On the other hand we have

$$\|\nabla(\gamma-\gamma_\delta)\| \le C\delta\,, \qquad \delta < q_1 < 2\delta\,,$$

since $\gamma_\delta = \alpha_1 q_1 + \alpha_2 q_2 + \beta_0|q_1|$ is the second order Taylor approximation of γ. A combination of the last three inequalities yields the boundedness of the third term of Eq (4.10). It remains to estimate the last term. We notice that

$$|\chi''(q_1)| \le C\delta^{-2}$$
$$|\gamma_\epsilon-\gamma_\delta| \le \beta_0\epsilon\left|\left|\frac{q_1}{\epsilon}\right| - \varphi\Big(\frac{q_1}{\epsilon}\Big)\right| \le C\delta^{1-\sigma}\epsilon^\sigma \le C\delta^2$$

for $\delta < q_1 < 2\delta$ and $\epsilon < \delta^{1+1/\sigma}$. As before we obtain from the Taylor series

$$|\gamma-\gamma_\delta| \le C\delta^2$$

Again a combination of these three inequalities immediately yields the boundedness of the last term on the right-hand side of Eq (4.10). Now we have that all second order derivatives of γ_ϵ^δ are bounded by a constant independent of δ whenever $\delta < q_1 < 2\delta$ and $\epsilon < \delta^{1+1/\sigma}$. It is easy to show that the same holds true for the gradient and the function itself.

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Combining both lemmas and using standard inequalities we immediately obtain:

Theorem 1. *Let the wedge be defined as in Eq (2.1), with* $a, b, c \in C^2(\mathbb{R}^{n-1})$ *and* $\nabla c(x_0') \neq 0$. *Then we have:*

In the above defined class of regularizing functions φ *there exists a unique limit* $\dot{q}^+$ *of the velocity after the wedge subject to formula (3.17), where* $\dot{q}^-$ *is the left-hand limit of the velocity before the wedge.*

In this way we are able to predict the velocity after the wedge knowing the velocity before. So for numerical aspects it is possible to stop the integration just before the wedge and to restart a new initial value problem just behindit.

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Punkt materialny poruszający się na niegładkiej rozmaitości w $\mathbb{R}^n$

Streszczenie

Rozpatrzmy punkt materialny poruszający się po $(n-1)$-wymiarowej rozmaitości z klinem. Aby uniknąć mnożenia dystrybucji w równaniach ruchu, regularyzujemy klin przy pomocy gladkich funkcji zależnych od małego parametru ϵ. Powstaje wówczas pytanie: czy istnieje graniczna wartość wektora prędkości przy przejściu przez wygladzony klin. W artykule podajemy klasę regularyzacji, dla których granica istnieje i jest niezależna od szczególnego wyboru samej regularyzacji. Ponadto podajemy ograniczenia dla jakości aproksymacji zależnych od parametru ϵ.

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