MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 35, 1997

FREE VIBRATIONS OF A LONGITUDINALLY LOADED COLUMN WITH ROTARY INERTIA ELEMENTS

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The influence of the rotary inertia and axial load on the natural frequency of non-linear two-member column is studied. The perturbation method is used for solving the problem. Numerical results concerning both the vibration frequency and modes are compared with those from experiment. For equal ratios of the mass per unit length to flexural rigidity for both members the rotary inertia does not affect the natural frequency. The critical load caused the divergence instability of the system is also found.

Key words: natural transverse vibrations, divergence instability, compound column

1. Introduction

The problem of the transverse vibration of beams with attached inertia elements has been discussed by several authors. Srinath and Das (1967) studied the vibrations of a simply supported beam, carrying a mass and having rotational inertia at a certain general point along the span. Natural frequencies and modes for beams having a continuous mass distribution as well as a finite number of concentrated masses with rotary inertia were determined by Jacquot and Gibson (1972). Both the free and forced vibrations of a uniform beam elastically restrained against rotation at one end and against translation at the other end, and carrying a lumped mass with rotary inertia and external loading at an arbitrary intermediate point were analysed by Hamdan and Jubran (1991). The effects of location and magnitude of the lumped mass and rotary inertia on eigenfrequency parameters and resonance response of the beam-mass system on the base of parametric studies were examined in that work. Hamdan and Abdel Latif (1994) performed a numerical convergence study of three discretization methods; i.e., Rayleigh-Ritz, Galerkin and finite element, as applied to the analysis of free bending linear vibration of a beam carrying inertia elements at intermediate points. A cantilever beam carrying a lumped mass with rotary inertia at an arbitrary intermediate point and another at the beam tip is used as a case of study. The beam mode shapes tended to change rapidly as the end or intermediate inertia become large. Increasing the end and intermediate rotational inertia resulted in a decrease in the investigated first five frequency parameters. The deterministic and random vibration response analysis of a uniform, mass-loaded, hysteretically damped beam, the left end of which is attached by both translational and rotational springs and the right end of which is free and carrying a heavy tip mass was presented by Chang (1995). Geometrically non-linear structures were studied. Przybylski et al. (1996) demonstrated the influence of prestress, axial force as well as distribution of both the axial and flexural rigidities on the natural frequency of non-linear two-member frame on the basis of both numerical and experimental results. Tomski et al. (1994) found the effect of both the prestressing force and concentrated mass on the natural frequency of a compound beam. The same system was analysed by Tylikowski (1991) who applied dynamic approach to establishing two different kinds of stability. Tomski and Kukla (1990) observed the effects of initial imperfection and amplitude on the free vibrations of a system composed of a beam and a spring. All the results obtained in the above mentioned works prove the necessity of investigation of the rotary inertia effect on natural vibrations of geometrically non-linear beams.

The present contribution aims at examination of the effect of both the rotary inertia of eccentrically mounted mass as well as the compressive force on the natural frequencies of two-member geometrically non-linear columns.

The same column, scheme of which is shown in Fig.1, has been investigated theoretically and experimentally. The construction consists of three rods. The two identical rods (1) are made of duralumin (forming one rod) and reveal the bending stiffness E_1I_1, while the central rod (2) of the bending stiffness E_2I_2 is made of steel. Free ends of the rods are connected by means of the rigid element (3), in which the rods (1) and (2) are nutted against translation (4).

The block (3) is fixed to ball bearings mounted on the frame (5), what results in the fixed pivot support conditions. Other ends of the rods (1) and (2)

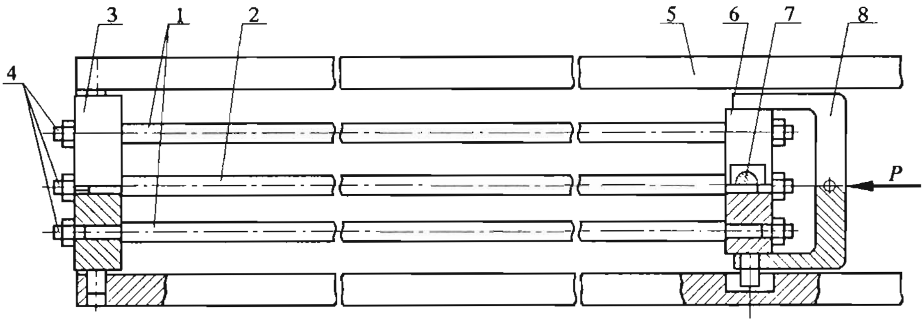

Fig. 1. Design of two-member column

are joined by means of the rigid block (6) and nutted. The block is restrained against rotation in the frame (5) by means of ball bearings, resulting in the pinned support conditions. There is the rigid rod (7) carrying a lumped mass fixed to the block (6), (see also Fig.2), as well as a lug (8) ensuring the column to be longitudinally loaded.

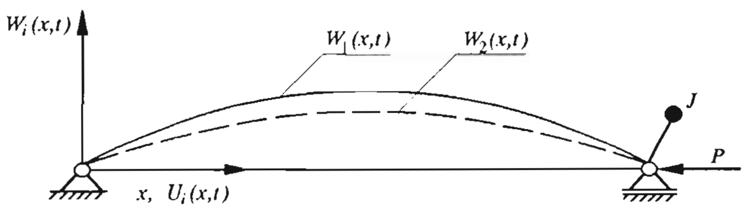

Fig. 2. Theoretical model of the investigated system

2. Boundary value problem

The scheme of deformed axes of the column under investigation is given in Fig.2.

The equations representing transverse vibration of the column ith rod loaded by the longitudinal force are as follows

$$\frac{\partial^4 w_i(\xi,\tau)}{\partial \xi^4} + \lambda_i \frac{\partial^2 w_i(\xi,\tau)}{\partial \xi^2} + \varpi_{ni}^2 \frac{\partial^2 w_i(\xi,\tau)}{\partial \tau^2} = 0 \tag{2.1}$$

where

$$w_i(\xi,\tau) = \frac{W_i(x,t)}{l} \qquad \lambda_i = \frac{S_i l^2}{E_i I_i} \qquad \varpi^2_{ni} = \frac{\rho_i A_i \Omega^2_n l^4}{E_i I_i} \qquad i = 1,2$$

denote non-dimensional transverse displacements, load parameters and non-dimensional natural frequencies, respectively

$$\xi = \frac{x}{l} \qquad \tau = \Omega_n t$$

and Ω_n stands for the nth natural frequency, $E_i I_i$ denotes bending stiffness of the ith rod and $\rho_i A_i$ stands for a mass per unit length of the ith rod.

The formulae for the longitudinal displacements of ith rod under the boundary conditions $U_i(0,\tau) = U_2(0,\tau) = 0$ have the following form

$$u_i(\xi,\tau) = -\frac{\lambda_i}{\varphi_i}\xi - \frac{1}{2}\int\limits_0^\xi \Big[\frac{\partial w_i(\zeta,\tau)}{\partial \zeta}\Big]^2 \, d\zeta \tag{2.2}$$

where

$$\varphi_i = \frac{A_i l^2}{I_i} \qquad u_i(\xi,\tau) = \frac{U_i(x,\tau)}{l}$$

The boundary value problem formulated above can be solved employing the small parameter method, i.e., expanding the relevant quantities into the exponential series with respect to the amplitude parameter ε ($\varepsilon \ll 0$) (cf Evansen, 1968)

$$\begin{aligned} w_i(\xi,\tau) &= \sum_{j=1}^{N} \varepsilon^{2j-1} w_{i2j-1}(\xi,\tau) + 0(\varepsilon^{N+1}) \\ \lambda_i &= \lambda_{i0} + \sum_{j=1}^{N} \varepsilon^{2j}\lambda_{i2j}(\tau) + 0(\varepsilon^{N+1}) \\ \varpi^2_{ni} &= \omega^2_{ni}\Big(1 + \sum_{j=1}^{N}\varepsilon^{2j}\nu_{2j}\Big) + 0(\varepsilon^{N+1}) \qquad i = 1,2 \end{aligned} \tag{2.3}$$

where ν_{2j} stands for the frequency correction coefficient, and

$$\begin{aligned} & w_{i1}(\xi,\tau) = w^{(1)}_{i1}(\xi)\cos\tau \qquad w_{i3}(\xi,\tau) = w^{(1)}_{i3}(\xi)\cos\tau + w^{(3)}_{i3}(\xi)\cos 3\tau \\ & w_{i5}(\xi,\tau) = w^{(1)}_{i5}(\xi)\cos\tau + w^{(3)}_{i5}(\xi)\cos 3\tau + w^{(5)}_{i5}(\xi)\cos 5\tau \\ & \lambda_{i2}(\tau) = \lambda^{(0)}_{i2}\cos 2\tau \qquad \lambda_{i4} = \lambda^{(0)}_{i4} + \lambda^{(2)}_{14}\cos 2\tau + \lambda^{(4)}_{i4}\cos 4\tau \end{aligned} \tag{2.4}$$

Substituting Eqs (2.3) into the equations of motion (2.1) supplied with the formulae for longitudinal displacements (2.2), upon equating to zero the terms of respective ε exponents, one obtains the following equations of motion and longitudinal displacements

$$
\begin{aligned}
0(\varepsilon^0): &\quad u_{i0}(\xi) = -\frac{\lambda_{i0}}{\varphi_i}\xi \\
0(\varepsilon^1): &\quad w_{i1}^{IV}(\xi,\tau) + \lambda_{i0} w_{i1}^{II}(\xi,\tau) + \omega_{ni}^2 \ddot{w}_{i1}(\xi,\tau) = 0 \\
0(\varepsilon^2): &\quad u_{i2}(\xi,\tau) = -\frac{\lambda_{i2}}{\lambda_i}\xi - \frac{1}{2}\int\limits_0^{\xi} \Big[w_{i1}^{I}(\zeta,\tau)\Big]^2\, d\zeta \\
0(\varepsilon^3): &\quad w_{i3}^{IV}(\xi,\tau) + \lambda_{i0} w_{i3}^{II}(\xi,\tau) + \omega_{ni}^2 \ddot{w}_{i3}(\xi,\tau) = \\
&= -\lambda_{i2}(\tau) w_{i1}^{II}(\xi,\tau) - \omega_{ni}^2 \ddot{w}_{i1}(\xi,\tau) = 0
\end{aligned}
\tag{2.5}
$$

Roman numerals and dots denote derivatives with respect to ξ and τ, respectively.

In view of Eqs (2.3) and (2.5) are to be solved under the following boundary conditions

$$
\begin{aligned}
& w_{11}(0,\tau) = w_{21}(0,\tau) = w_{11}(l,\tau) = 0 \\
& w_{11}^{I}(0,\tau) = w_{21}^{I}(0,\tau) \qquad\qquad w_{11}^{I}(l,\tau) = w_{21}^{I}(l,\tau) = 0 \\
& w_{11}^{II}(l,\tau) + \mu w_{21}^{II}(l,\tau) + \frac{\omega^2 J l}{E_1 I_1}\ddot{w}_{11}^{I}(l,\tau) = 0 \\
& w_{11}^{II}(0,\tau) + \mu w_{12}^{II}(0,\tau) = 0 \\
& \frac{\lambda_{10}}{\varphi_1} = \frac{\lambda_{20}}{\varphi_2} \qquad\qquad \lambda_{10} + \lambda_{20}\frac{1}{\mu} = p
\end{aligned}
\tag{2.6}
$$

where J is the lumped mass inertial moment about the axis crossing the symmetry axis of the column at the point of rod (7) attachment (Fig.1), and perpendicular to the vibration plane and

$$
p = \frac{Pl^2}{E_1 I_1} \qquad\qquad \mu = \frac{E_2 I_2}{E_1 I_1}
\tag{2.7}
$$

Consider free vibration of the system

$$
w_{ij}(\xi,\tau) = \mathrm{e}^{\mathrm{i}\tau} y_{ij}(\xi)
\tag{2.8}
$$

General solution to Eqs (2.5b), by virtue of Eq (2.8) has the following form

$$y_{i1}(\xi) = A_{i1}\cosh(\alpha_{i1}x) + B_{i1}\sinh(\alpha_{i1}x) + C_{i1}\cos(\beta_{i1}x) + D_{i1}\sin(\beta_{i1}x) \tag{2.9}$$

where

$$\alpha_{i1} = \sqrt{-\frac{1}{2}\lambda_{io} + \sqrt{\frac{1}{4}\lambda_{io}^2 + \omega_{ni}^2}} \qquad \beta_{i1} = \sqrt{\frac{1}{2}\lambda_{io} + \sqrt{\frac{1}{4}\lambda_{io}^2 + \omega_{ni}^2}} \tag{2.10}$$

From the results obtained by Przybylski et al. (1996) it follows that for the vibration amplitudes small enough, i.e., when the frequency correction coefficient is close to zero, this problem can be satisfactorily solved when taking only two terms (ε_0 and ε_1) from the expansions (2.3c) into account.

Since the solution to the boundary value problem given above is well-known, and can be obtained with the use of standard mathematics programs on PC-type computers, in Section 3 only the results of natural frequency calculations are given.

3. Natural frequencies – numerical results and experiments

The research (both calculations and experiments) was arranged in two series.

The first one aims at examination of the effect of the inertial moment J on the natural frequencies of the column for different values of ratios $\rho_1 A_1/(E_1 I_1)$ and $\rho_2 A_2/(E_2 I_2)$ at $P = 0$. The bending stiffnesses and masses per unit length of each rod are given in Table 1.

The second series concerns the influence exerted by the compressive force upon the natural frequency curve course as well as the determination of the divergence critical force magnitude for the geometrically non-linear column for the three selected values of inertial moment $J = 1.65 \cdot 10^{-4}$, 0.084 and 0.273 kgm^2, respectively.

Table 1. Physical data of columns

Col.	Rod 1 [mm]	Rod 2 [mm]	E_1I_1 [Nm^2]	E_2I_2 [Nm^2]	ρ_1A_1 [kg/m]	ρ_2A_2 [kg/m]	$\frac{\rho_1A_1}{E_1I_1}$ [s^2/m^4]	$\frac{\rho_2A_2}{E_2I_2}$ [s^2/m^4]
B_1	$2\cdot\emptyset 14$	$\emptyset 10$	254.95	95.72	0.866	0.619	0.003	0.006
B_2	$2\cdot\emptyset 14$	$\emptyset 14$	254.95	367.72	0.866	1.213	0.003	0.003
B_3	$2\cdot\emptyset 14$	$\emptyset 20$	254.95	1532.0	0.866	2.476	0.003	0.002

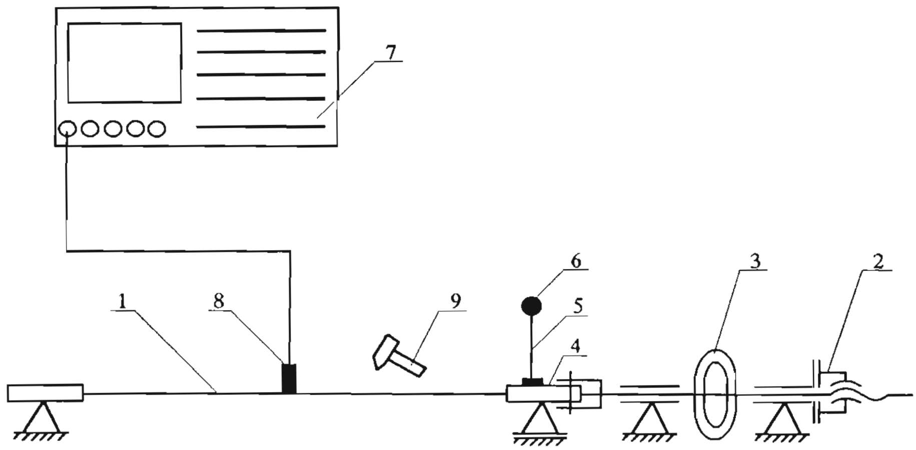

Fig. 3. Experimental set-up for measuring frequency

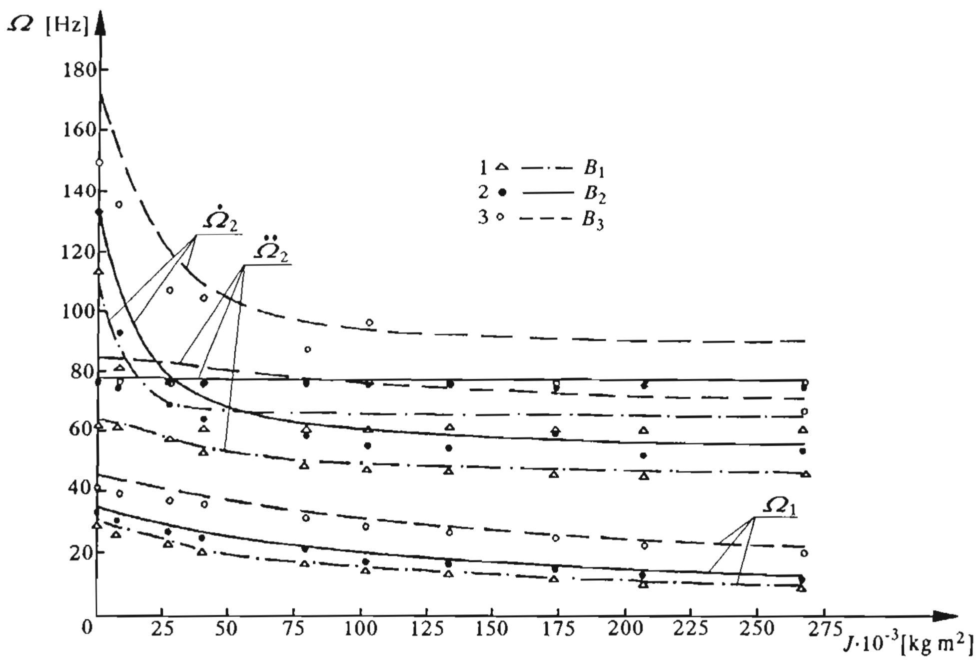

Fig. 4. Two first eigenfrequencies versus the moment of inertia J for columns of different physical properties

Experiments were conducted on the stand, scheme of which is shown in Fig.3. The compressive force P acting upon the column (1) appears due to the stretching system (3) supplied with the dynamometer (3). The current magnitude is read off on the strain gauge fixed to the dynamometer. The rigid rod (7) is mounted on the block enabling the lumped mass (6) to be attached.

Experimental investigations into natural frequencies have been conducted with the aid of the Bruel-Kjaer single-channel vibration analyser (5) of 2115 type combined with the accelerometer (8) of 4381 type. The system was excited by means of the impactor (9).

Table 2. Second mode shapes for columns with different moment of inertia J

		$J=1.625\cdot10^{-4}$ kg·m^2	$J=0.032$ kg·m^2	$J=0.084$ kg·m^2	$J=0.084$ kg·m^2
B_1	$\dot{\Omega}_2$ [Hz]	116.9	68.19	65.98	65.52
	$\ddot{\Omega}_2$ [Hz]	63.13	56.11	49.87	47.15
B_2	$\dot{\Omega}_2$ [Hz]	136.67	74.61	61.92	56.83
	$\ddot{\Omega}_2$ [Hz]	77.6	77.6	77.6	77.6
B_3	$\dot{\Omega}_2$ [Hz]	174.2	114.65	95.59	90.45
	$\ddot{\Omega}_2$ [Hz]	84.11	82.06	77.63	72.0

Fig.4 shows the results obtained when performing the first series of tests, with dots denoting the experimental results and solid lines representing the numerical ones, respectively. Fig.4 provides the opportunity to study changes of the natural frequency as a function of the inertial moment J. Increasing the end rotary inertia yields the decrease in frequency. It is worth noticing that there are two second frequency curves for each column marked by $\overset{*}{\Omega}_2$

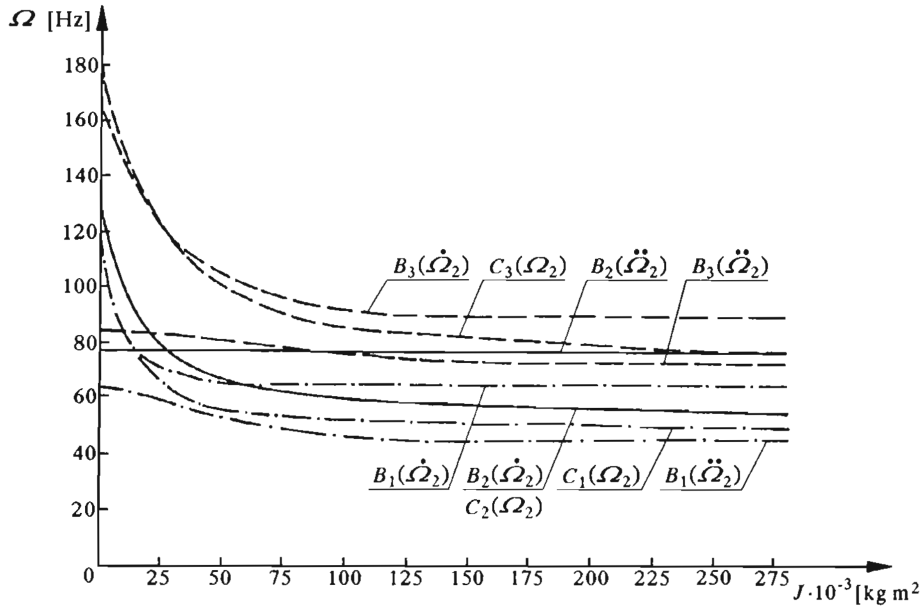

Fig. 5. Comparison of eigenfrequencies for a single and two-member columns

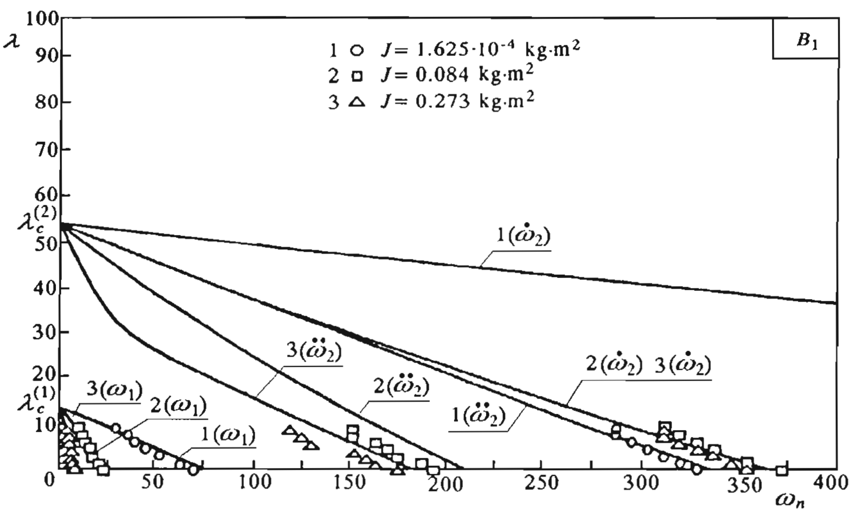

Fig. 6. Eigencurves for columns B_1 with a lumped mass of different rotary inertia J

and $\overset{**}{\Omega}_2$. The curve $\overset{**}{\Omega}_2$ for the column B_2 $\left(\frac{\rho_1 A_1}{E_1 I_1} = \frac{\rho_2 A_2}{E_2 I_2}\right)$ is parallel to the horizontal – an increase in J does not affect this frequency. Explanation of such a phenomenon is easy when studying the vibration modes. In Table 2 the second modes are shown for four values of J. Each mode has two nodes with the exception of modes for the column B_2 and frequency $\overset{**}{\Omega}_2$. The characteristic shape of those modes (zero deflection angles at both ends of the column) makes clear that an arbitrary value of J cannot influence the natural vibration when the column vibrates in that mode.

In the case when the free vibration amplitude takes very small values (the measured values were about $10\,\mu$m), it affects the natural frequencies of the geometrically non-linear column insignificantly (comp. Przybylski et al., 1996).

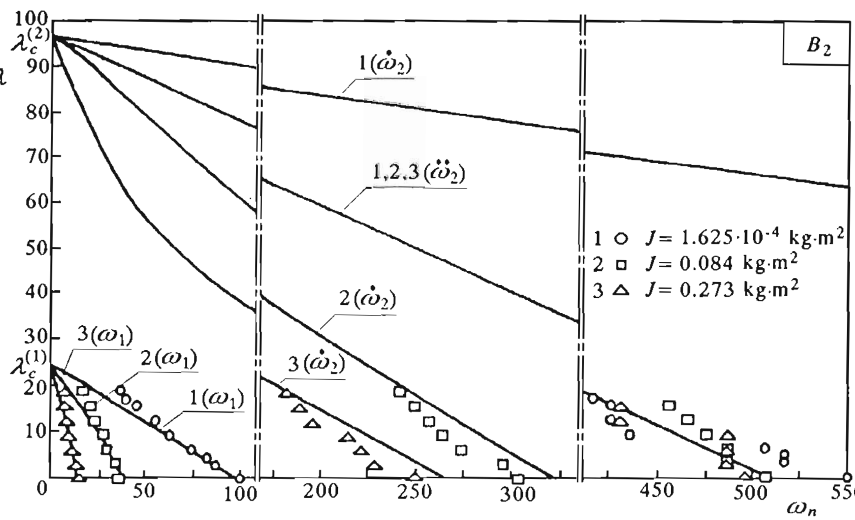

Fig. 7. Eigencurves for columns B_2 with a lumped mass of different rotary inertia J

Courses of the second vibration frequency curves are compared in Fig.5 with adequate curves for a single-member column of bending rigidity and mass per unit length equal to the corresponding sums for a double-member column, respectively, i.e. $EI = E_1I_1 + E_2I_2$, $\rho A = \rho_1 A_1 + \rho_2 A_2$. Curves for single columns are marked with $C_i(\Omega_2)$, and the C_i column has its EI and ρA as the column B_i $(i = 1,2,3)$. The curves $C_1(\Omega_2)$ and $C_3(\Omega_2)$ are placed between curves $B_1(\overset{*}{\Omega}_2)$ and $B_1(\overset{**}{\Omega}_2)$, and $B_3(\overset{*}{\Omega}_2)$ and $B_3(\overset{**}{\Omega}_2)$, respectively, whereas the curve $C_2(\Omega_2)$ overlaps the curve $B_2(\overset{*}{\Omega}_2)$.

The runs of dimensionless natural frequencies ω_n of the system versus the load parameter λ obtained numerically (solid lines) and experimentally (dots) were plotted in Fig.6 for the column B_1, and in Fig.7 for the column B_2 for three magnitudes of the inertial moment J. A good correlation between the corresponding results should be noted. Despite of a different course, the second eigenvalue curves for columns B_1 and B_2 for different J coincide at the point of coordinates $(0, \lambda_c^{(2)})$ where the column loses its stability through divergence (second critical load).

4. Conclusions

For compound geometrically non-linear columns, in contrast to the single column configurations, within the range of the first and second frequencies an additional natural frequency appears, involving an additional mode shape.

For two-member columns the natural frequency $\overset{**}{\Omega}_2$ is independent of the rotary inertia of a lumped mass fixed to one end of the column if $\frac{\rho_1 A_1}{E_1 I_1} = \frac{\rho_2 A_2}{E_2 I_2}$. It happens because the corresponding mode is characterised by zero deflection angles at the column ends, and in result the mass remains immovable during vibrations. For all cases when $\frac{\rho_1 A_1}{E_1 I_1} \neq \frac{\rho_2 A_2}{E_2 I_2}$ the increase in the rotary inertia causes the decrease in the natural frequencies.

The pinned compound column with rotary inertia elements attached to one end of the column, subject to the compressive force loses its stability through divergence: for $\Omega_1 \to 0$ the first critical force appears, while for $\overset{*}{\Omega}_2$ and $\overset{**}{\Omega}_2$, despite different courses of the natural frequency curves, they coincide at one point (for $\overset{*}{\Omega}_2 = \overset{**}{\Omega}_2 \to 0$), where only one value of the critical force appears.

Acknowledgment

This research has been supported by the State Committee for Scientific Research, Warsaw, Poland under grant BS-01-302/91.

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Drgania własne obciążonej wzdłużnie kolumny dwuprętowej z elementem inercyjnym

Streszczenie

W pracy zbadano wpływ siły wzdłużnej oraz inercji obrotowej masy skupionej zamocowanej mimośrodowo na końcu swobodnie podpartej geometrycznie nieliniowej kolumny dwuprętowej na drgania takiego układu. Do rozwiązania zagadnienia zastosowano metodę małego parametru. Wyniki otrzymane numerycznie były weryfikowane doświadczalnie na stanowisku. Stwierdzono, że przy równych relacjach masy na jednostkę długości do sztywności na zginanie dla obu prętów istnieje postać drgań układu, przy której zmiana momentu inercyjnego nie wpływa na częstość drgań. Wyznaczono także dwie pierwsze siły krytyczne określające niestatecznośc dywergencyjną układu.

Manuscript received October 31, 1996; accepted for print November 29, 1996