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LIMIT LOAD OF ELASTOMER ELEMENTS

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The internal structure of elastomers allows for great elastic deformations to take place. Breakdown of such hyperelastic materials or the transition into plastic stress state is discussed. The elastic potential with the material constants determined experimentally is derived, and critical conditions based on the Huber-Mises and Tresca material failure theories are specified. As an example, the stresses and deformations in some of machine elements loaded in a combined way are determined.

Key words: hyperelasticity, elastic potential, critical conditions, limit load

1. Introduction

Structural materials sensitive to strain rate, such as polymers and elastomers, differ much from metals in their mechanical and rheological properties. The deformation process in these materials depends directly on the atomic and molecular structure. The mechanisms of deformation result from the nature of the material. The structure of the material is spatially disordered and it consists of transversal and longitudinal chains of particles. Between the bonded chains there are the forces of internal friction and the intermolecular forces (van der Waals), Dudziak and Anisimowicz (1990).

Main bonds are the chemical ones, which affect the magnitudes of elastic strains ε_H and creep strains ε_N (cf Dudziak, 1990), Fig.1 and Fig.2.

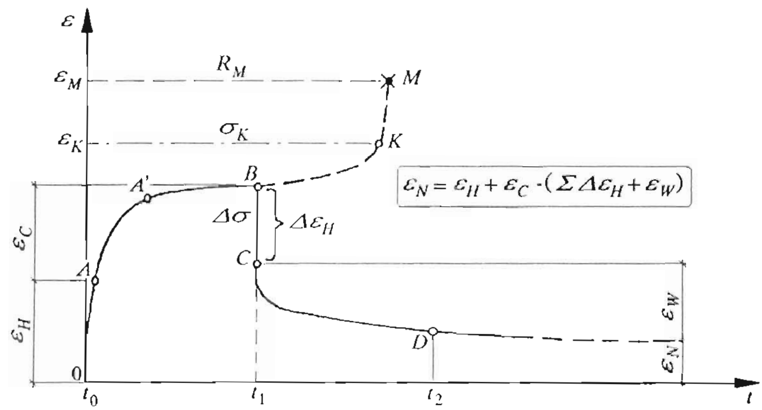

Fig. 1. Tensile test of elastomers

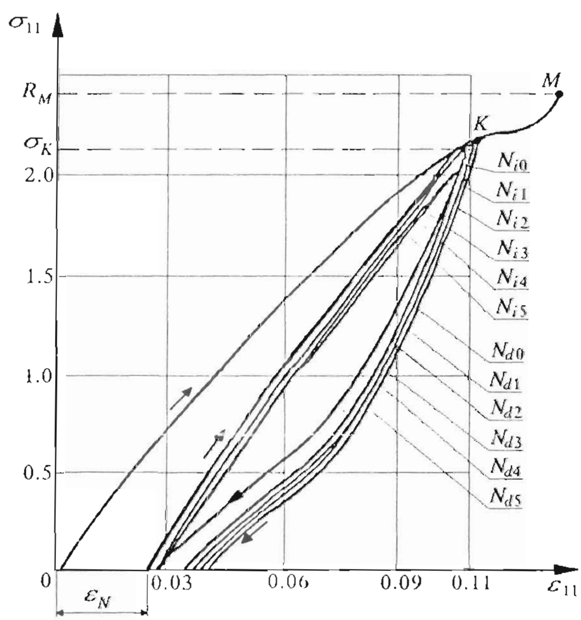

Fig. 2. Tensile characteristic (stress-strain curve) of elastomers;
$N_{i0}, N_{i1}, ..., N_{i5}$ – cycles of load increase, $N_{d0}, N_{d1}, ..., N_{d5}$ – cycles of load decrease

In principle, the bonds of van der Waals forces influence the deformation mechanism of creep strains ε_N and also the mechanical and rheological properties such as; for example, tensile strength, different elasticity moduli for loading and unloading to the point K, see Fig.2.

The ultimate tensile strength R_M of a material depends mainly on intermolecular (interbonds) forces. If the external load exceeds the intermolecular forces of the bonds (point K), Fig.1 and Fig.2, the rapid elongation of the material can be observed and then it breakes (point M). The point K is called the critical point of the loading and the corresponding load is called the critical loading. The area of effective use of the loading acting upon the structure made of such elements extends up to that point. The experiments have proved that the critical point K is contained within the range of $0.95R_M \div 0.98R_M$, Fig.1 and Fig.2.

The use of critical loads in the design process of structural elements made of elastomers enables more complete use of their strength, in accordance with the physical nature of the material. However, one may find serious difficulties in determination of the critical state of stress or deformation in the case of structural compound loadings. In the paper, the use of Huber-Mises or Tresca material failure theories enables formulation of a general form of the limit conditions. The elastomer is treated here as a hyperelastic material (in Green's formulation, Green and Zerna (1968), it is known to be nonlinear elastic) so, we need to know the function of elastic potential for the description of this material.

2. Equations of hyperelasticity; limit stress

The elastic behaviour of elastomers is well defined by the constitutive equations of hyperelasticity (cf Green and Zerna, 1968) formulated on the base of strain potential W. The equations of motion of such medium can be written in the form

$$x^i = x^i(X^\alpha) \tag{2.1}$$

where X^i are initial coordinates (state B^0) with metric tensor $G_{\alpha\beta}$, and x^i are the current coordinates (state B) with the metric tensor g_{ij}. The Cauchy-Green strain tensor and strain invariants are as follows

$$B^{ij} = F^i_\alpha F^i_\alpha G^{\alpha\beta} = I_1 g^{ij} - g^{ir} g^{is} G_{rs} \tag{2.2}$$

$$I_1 = G_{rs}g^{rs}$$
$$I_2 = \frac{1}{2}(I_1^2 - g^{rn}g^{sn}G_{rs}G_{mn}) \tag{2.3}$$
$$I_3 = g^{rm}G_{ms}$$

wheras $I_3 = 1$ for elastomers (incompressibility).

If the material is homogeneous and isotropic within the elastic range, the elastic potential is a function of strain invariants, i.e. $W = W(I_1, I_2, I_3)$, and the stress state is expressed by the following relations, (see Green and Zerna, 1968; Wesołowski and Woźniak, 1970)

$$\tau^{ij} = \phi g^{ij} + \psi B^{ij} + pG^{ij} \tag{2.4}$$

where

$$\phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1} \qquad \psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2} \qquad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}$$

In the case of axial tension the critical stress state is achieved for $\sigma = \sigma_K$. To determine the limit state in three-axial tension we will use the Huber-Mises and Tresca material failure theories. The material reaches a critical state when physical components of the stress tensor satisfy the following general condition (see Mielniczuk and Sawczuk, 1972 and 1975)

$$f(\sigma_1^1, \sigma_2^2, \sigma_3^3) = 0 \tag{2.5}$$

The physical stress components have the same numerical values as the mixed components of the stress tensor, i.e.

$$\tau_j^i = \tau^{ir}G_{rj} = \sigma_j^i$$

Taking into consideration the fact that the deviatoric part of stress tensor is responsible for the critical state (destruction) of elastomers, i.e.

$$t^{ij} = \tau^{ij} - \frac{1}{3}\tau_k^k G^{ij} = \phi\Big(g^{ij} - \frac{1}{3}I_1 G^{ij}\Big) + \psi\Big(B^{ij} - \frac{2}{3}I_2 G^{ij}\Big)$$

the Huber-Mises and Tresca limit conditions are as follows

$$(t_1^1)^2 + (t_2^2)^2 + (t_3^3)^2 = \frac{2}{3}\sigma_K^2 \tag{2.6}$$

and

$$\max\Big\{|t_1^1 - t_2^2|, |t_2^2 - t_3^3|, |t_3^3 - t_1^1|\Big\} = \sigma_K \tag{2.7}$$

Hereinafter the limit strain states are specified for the two boundary problems. It enables one to determine the limit loads.

3. Deformation of a spherical shell

A thick-walled spherical shell is loaded by the external pressure P_e, and internal pressure P_i. The material of the shell is assumed to be incompressible $I_3 = 1$. The internal and external radii are r_1 and r_2 for the initial state B^0, and R_1 and R_2 for the deformed state B, respectively (Fig.3). The critical strain is expressed as follows

$$Q(R) = \frac{r}{R} \tag{3.1}$$

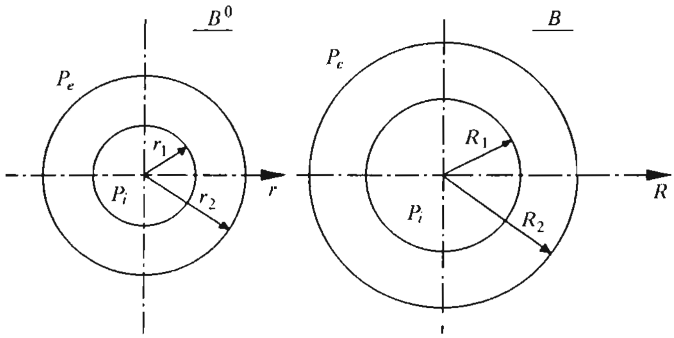

Fig. 3. Deformation of a thick-walled shell

The invariants of the strain state are of the form

$$I_1 = Q^4 + \frac{2}{Q^2} \qquad I_2 = 2Q^2 + \frac{1}{Q^4} \qquad I_3 = 1 \tag{3.2}$$

and the components of the stress state, according to Eq (2.4), are

$$\begin{aligned} t_1^1 &= \frac{2}{3}\Big[\phi\Big(\frac{1}{Q^2} - Q^4\Big) + \psi\Big(\frac{1}{Q^4} - Q^2\Big)\Big] \\ t_2^2 &= t_3^3 = -\frac{1}{2}t_1^1 \end{aligned} \tag{3.3}$$

The limit condition is the same for both material failure theories, and it is of the form

$$\phi\Big(Q^4 - \frac{1}{Q^2}\Big) + \psi\Big(Q^2 - \frac{1}{Q^4}\Big) = \pm\sigma_K \tag{3.4}$$

If the elastic potential (functions ϕ and ψ) is known then the critical strain and critical load of a shell can be determined from the above criterion.

Mechanical properties of some elastomers (e.g. rubber) are acceptably represented by the potential function which depends on the first strain tensor invariant I_1. We have $W = C(I_1 - 3)$ for the so called neoHOOKEan, where C is a material constant (cf Wesołowski and Woźniak, 1970).

Thus, the limit condition (3.4) has the following more simple form

$$Q^6 - \frac{\sigma_K}{2C} Q^2 - 1 = 0 \tag{3.5}$$

This equation has two real roots, which, otherwise, represent the solution to the boundary value problem, therefore

$$Q_{1,2} = \pm\sqrt{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \Big(\frac{\sigma_K}{6C}\Big)^3}} + \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} + \Big(\frac{\sigma_K}{6C}\Big)^3}}} \tag{3.6}$$

The limit stress state can be determined from Eq (3.1) for $Q_{1,2}$.

4. Critical bending of a strip

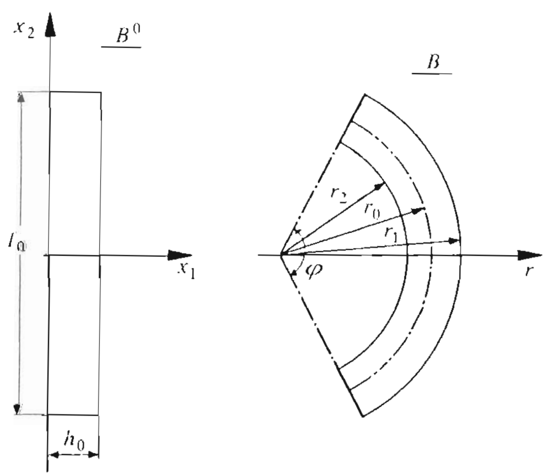

Fig. 4. Cylindrical bending of the strip

A plate strip made of elastomer is shown in Fig.4. The strain tensor invariants are of the form

$$I_1 = A^2 r^2 + \frac{1}{A^2 r^2} + 1 \qquad I_2 = I_1 \qquad I_3 = 1 \tag{4.1}$$

where

$$A = \frac{2h_0}{r_1^2 - r_2^2}$$

The physical components of the stress tensor deviator are as follows (cf Dudziak and Mielniczuk, 1990)

$$3t_1^1 = \phi\Big(\frac{2}{A^2r^2} - A^2r^2 - 1\Big) + \psi\Big(\frac{1}{A^2r^2} - 2A^2r^2 + 1\Big)$$

$$3t_2^2 = \phi\Big(2A^2r^2 - \frac{1}{A^2r^2} - 1\Big) + \psi\Big(A^2r^2 - \frac{2}{A^2r^2} + 1\Big)$$

$$3t_3^3 = \phi\Big(2 - A^2r^2 - \frac{1}{A^2r^2}\Big) + \psi\Big(A^2r^2 + \frac{1}{A^2r^2} - 2\Big)$$

and the limit conditions according to the Huber-Mises and Tresca material failure theories are expressed in the following way

$$\begin{aligned}(\phi^2 + \psi^2)\Big(A^4r^4 + \frac{1}{A^4r^4} - A^2r^2 - \frac{1}{A^2r^2}\Big) + \\ +\phi\psi\Big(A^4r^4 + \frac{1}{A^4r^4} + 2A^2r^2 + \frac{2}{A^2r^2} - 6\Big) = \sigma_K^2\end{aligned} \tag{4.2}$$

and

$$(\phi + \psi)\Big(A^2r^2 - \frac{1}{A^2r^2}\Big) = \pm\sigma_K \tag{4.3}$$

From the distribution of stresses it can be seen that the extreme fibres, situated at $r = r_1$ and $r = r_2$, are subject to the highest stresses. Assuming the same model of the material as above the Huber-Mises condition takes the form

$$A^4r_1^4 + \frac{1}{A^2r_1^4} - A^2r_1^2 = \frac{\sigma_K^2}{4C}$$

Substituting for $x = r_1 - r_2^2/r_1^2$ we obtain the equation

$$\frac{16h_0^4}{x^4} + \frac{x^4}{16h_0^4} - \frac{4h_0^2}{x^2} - \frac{x^2}{4h_0^2} - \frac{\sigma_K^2}{4C} = 0 \tag{4.4}$$

from which the solution curves have been determined for the bands with different thicknesses (Fig.5a), and for the band of constant thickness with different values of the axial limit stress (Fig.5b).

For the Tresca criterion we obtain the following simple formula

$$A^2r_1^2 - \frac{1}{A^2r_1^2} = \frac{\sigma_K}{2C}$$

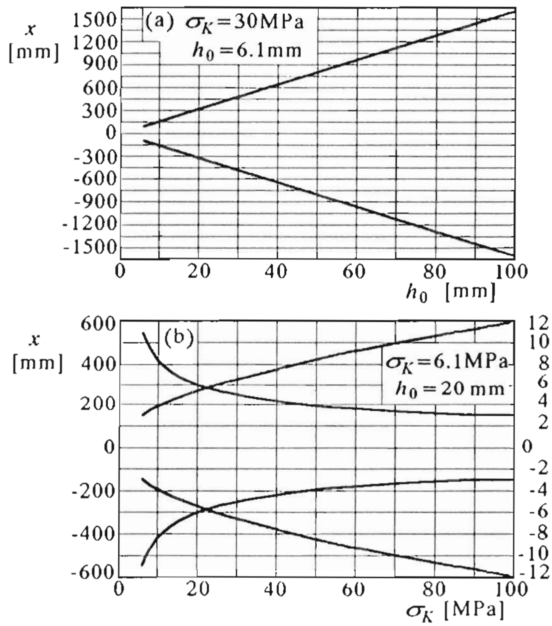

Fig. 5. Solution to the bending problem according to the Huber-Mises material failure theory

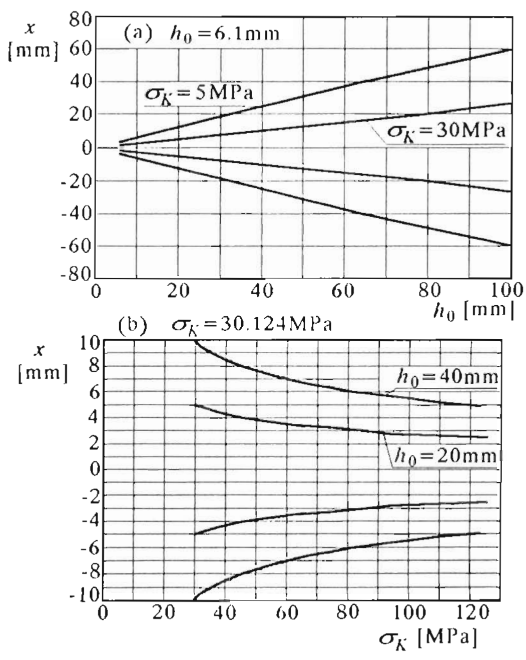

Fig. 6. Solution to the bending problem according to the Tresca material failure theory

The same substitution as above yields

$$\frac{1}{4h_0^2}x^4 \pm \frac{\sigma_K}{2C}x^2 = 4h_0^2 \tag{4.5}$$

Fig.6 illustrate the solution contour lines obtained from the above equations for variable band thicknesses and different limit stresses.

Knowing the solutions of Eqs (3.9) and (3.10) we determine the limit stresses in a bent band loaded by the limit bending moment.

Concluding we must say that the determined boundary conditions are important from engineering point of view. The first boundary condition can be used to determine the limit pressures for thick-walled rubber coats (tyres, balls). The second one can be used to determine the limit loads for rubber belts.

References

1. DUDZIAK M., 1990, *Problems of Internal Fricton and Dissipation of Energy in Rubber V-Belts,* Technical University of Poznań, Nr 229, 1-171
2. DUDZIAK M., ANISIMOWICZ M., 1990, Relaxation of V-Belts Stresses, *Res Mechanika,* **29**, 1-12
3. DUDZIAK M., MIELNICZUK J., 1990, Deformierung des Gummiflachriemens bei komplexem Belastungszustand, *Ing. Archiv,* **60**, 157-164
4. GREEN A.E., ZERNA W., 1968, *Theoretical Elasticity,* Clarendon Press, Oxford
5. MIELNICZUK J., SAWCZUK A., 1972, A Note on Finite Elastic-Plastic Bending, *AMS,* **24**, 1083-1088
6. MIELNICZUK J., SAWCZUK A., 1975, On Yielding of Hyperelastic Solids, *ZAMM,* **55**, 41-46
7. WESOŁOWSKI Z., WOŹNIAK C., 1970, *Elements of Nonlinear Theory of Elasticity,* PWN, Warsaw

Obciążenia graniczne elementów z elastomerów

Streszczenie

Wewnętrzna struktura elastomerów, odmienna od struktury metali, wpływa na przebieg procesu odkształcania i powstawanie dużych deformacji sprężystych. Rozważa się problem zniszczenia lub uplastycznienia takiego materiału jako materiału hipersprężystego. Sformułowano warunki graniczne w oparciu o hipotezy Hubera-Missesa i Treski wykorzystując postać potencjału sprężystego określoną doświadczalnie. Wyznaczono odkształcenia i w efekcie naprężenia dla elementów maszyn poddanych złożonym obciążeniom.

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