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LAGRANGE'S EQUATIONS FOR TWO-DIMENSIONAL MOTION OF A CYLINDER IN PERFECT FLUID

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Lagrange's formulation of system of particles mechanics is based on Lagrange's description of motion. When a rigid body moves in a velocity field of fluid it is convenient to use Lagrange's description of the body motion and the Euler one for the velocity field of fluid. In such a description a Lagrangian equal to the kinetic energy minus the potential one does not lead to correct differential equations of motion. In our case the Lagrangian is defined as a function that leads to the correct differential equations for the problem. The problem is discussed for the case of a circular cylinder moving in a space and half-space of fluid. First of all, the resultant hydrodynamic forces acting on the cylinder are calculated on the basis of pressures and the differential equations for the problem are established. The Lagrangian is given for a general case of motion in space. In the case of half-space with a perfect bottom, it is assumed that the radius of the cylinder is small and an approximate description is introduced. The variational formulations based on Hamillton's principle are useful in numerical solutions.

Key words: perfect fluid, motion of body, Lagrangian

1. Introduction

The plane motion of a cylinder with radius a in inviscid and incompressible fluid in motion is considered. This means that viscous effects like separation of the boundary layer and shedding of vortices are not considered. These effects are discussed e.g. by Sarpkaya (1976) and Sumer and Fredsoe (1988). The aim of the paper is to reduce the problem of interaction between the body and fluid to the description of cylinder motion and thus to a set of two ordinary differential equations and to find a Lagrangian that leads to these equations.

It is natural to consider the motion of the cylinder in Lagrange's description and the velocity field of fluid in the Euler one. Lagrange's system of particles dynamics is in Lagrange's description. Thus, in a general case, a function that represents the instantaneous kinetic energy of the body and fluid minus the potential energy considered as a Lagrangian does not lead to correct differential equations of motion. Luke (1961) used Euler's description and proposed a Lagrangian for the case of water waves in a layer of fluid. The proposed Lagrangian leads by Hamilton's principle of least action and standard procedures to the known differential equations of the problem.

In our case the resultant hydrodynamic forces are calculated on the basis of pressures in the fluid and the differential equations for the problem are established. The standard two-dimensional formulation of solutions of potential theory in complex variables is used following Lamb (1957). Firstly, the problem of motion in the space of fluid is considered. This problem is simple in calculations and gives a good insight into the analysis and physical meaning.

Similar problem of motion in the upper half-space with a perfect rigid boundary on the x axis is considered. This problem is the basis for the consideration of the motion of a cylinder placed near a bottom in a wave field. The problem of motion in a half-space was discussed by von Mueller (1929), Carpenter (1958), Yamamoto et al. (1974), Wilde (1993), and Wilde et al. (1993). In the present paper a rigorous formulation is established for the case of the external velocity corresponding to a field homogeneous in space and variable in time. Such a case is the standard basis for the formulations used in engineering applications.

Yamamoto et al. (1974) considered, in some cases, the influence of a non-homgeneous velocity field. In this paper the differential equations for this problem are derived in a formal way. The Lagrangian obtained for the homogeneous case is generalised and then Lagrange's equations are derived to establish the differential equations for the problem. The formulation is approximate because the boundary conditions on the cylinder surface are not satisfied in an exact way.

The final equations may be supplemented by additional terms introducing the neglected viscous effects. The standard additional terms usually correspond to the Morison formula (cf Morison et al. (1950)).

2. Velocity fields

Let us assume that the external velocity field of fluid has a potential

$\Phi(x, y, t)$ that satisfies the Laplace equation and that the complex potential $W(z, t)$, where $z = x + \mathrm{i}y$ and t denotes time is an analytic function in the space for each value of the parameter t. Thus the complex potential may be represented in the fixed Cartesian co-ordinate system by the following series

$$W_f = \sum_{n=0}^{\infty} g_n(t) z^n \tag{2.1}$$

The velocities u_x and u_y are related to the complex potential by

$$u_x + \mathrm{i}u_y = \overline{\frac{\partial W(z,t)}{\partial z}} \tag{2.2}$$

where the bar over a symbol denotes the complex conjugate.

Let us denote by

$$z_0(t) = x_0(t) + \mathrm{i}y_0(t) \tag{2.3}$$

the complex number that describes the instantaneous position of the centre of the cylinder. The complex potential (2.1) may be expanded into a Taylor series in the neighborhood of the point $z_0(t)$. It follows

$$W_f = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n W(z,t)}{\partial z^n}\Big|_{z=z_0(t)} [z - z_0(t)]^n = \sum_{n=0}^{\infty} b_n[z_0(t), t](z^*)^n \tag{2.4}$$

where

$$b_n[z_0(t), t] = \sum_{m=n}^{\infty} g_m(t) \binom{m}{n} z_0^{m-n}(t) \qquad z^* = z - z_0(t)$$

Eq (2.4) is written in a moving Cartesian co-ordinate system.

In applications it is often enough to consider a few terms of the expansion. If in Eq (2.1) the coefficients $g_n(t)$ are equal to zero for $n > 2$, then

$$\begin{aligned}
W_f &= b_0[z_0(t), t] + b_1[z_0(t), t][z - z_0(t)] + b_2[z_0(t), t][z - z_0(t)]^2 \\
b_0[z_0(t), t] &= g_0(t) + g_1(t) z_0(t) + g_2(t) z_0^2(t) \\
b_1[z_0(t), t] &= g_1(t) + 2g_2(t) z_0^2(t) \\
b_2[z_0(t), t] &= g_2(t)
\end{aligned}$$

and $b_n[z_0(t), t] = 0$ for $n > 2$. If we are interested in the approximation of velocity field in a neighbourhood of a point $z_0(t)$ at a given time and a fixed radius a_f then when

$$|b_n[z_0(t), t]| a_f^n \ll |b_1[z_0(t), t]| a_f \qquad n > 2$$

we may take $b_n[z_0(t), t] = 0$ for $n > 2$.

It should be stressed that, according to Eq (2.4), b_n is a function of $z_0(t)$ and t. It is easy to calculate that

$$\begin{aligned} &\frac{\partial}{\partial z_0} b_n(z_0, t) = (n+1) b_{n+1}(z_0, t) \\ &\frac{d}{dt} b_n(z_0, t) = (n+1) b_{n+1}(z_0, t)\dot{z}_0(t) + \frac{\partial}{\partial z_0} b_n(z_0, t) \end{aligned} \tag{2.5}$$

According to Eq (2.2) in view of Eq (2.4) the complex velocity at the time t and point $z_0(t)$ is

$$u_x(x_0, y_0, t) + \mathrm{i} u_y(x_0, y_0, t) = \overline{b_1(z_0, t)} \tag{2.6}$$

The corresponding complex acceleration as a material time derivative is

$$\begin{aligned} a_x(x_0, y_0, t) + \mathrm{i} a_y(x_0, y_0, t) &= \frac{D}{Dt}\Big[\frac{\partial}{\partial x}\Phi(x, y, t) + \mathrm{i}\Phi(x, y, t)\Big]_{z=z_0} = \\ &= \Big[\frac{\partial}{\partial t}\overline{W_f} + \frac{\partial}{\partial z} W_f \frac{\partial^2}{\partial z^2}\overline{W_f}\Big]_{z=z_0} = \frac{\partial}{\partial t}\overline{b_1(z_0, t)} + 2 b_1(z_0, t)\overline{b_2(z_0, t)} \end{aligned} \tag{2.7}$$

where the symbol D/Dt denotes the material time derivative.

Let us introduce the cylindrical coordinates r and α, the origin of which lies at the centre of the cylinder. The normal unit vector to the surface is

$$\boldsymbol{n} = \boldsymbol{e}_x \cos\alpha + \boldsymbol{e}_y \sin\alpha \tag{2.8}$$

where $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the Cartesian base vectors. The normal component of fluid velocity at the cylinder surface is

$$u_n = \boldsymbol{u}\boldsymbol{n} = u_x \cos\alpha + u_y \sin\alpha \tag{2.9}$$

In complex variable notations it can be written

$$z^* = r e^{\mathrm{i}\alpha} \qquad\qquad u_n = \mathrm{Re}\Big[\frac{\partial W}{\partial z} e^{\mathrm{i}\alpha}\Big]_{r=a} \tag{2.10}$$

where a is the cylinder radius.

Thus the normal velocity on the surface of the cylinder due to the external velocity field is

$$u_n = \mathrm{Re}\Big[\sum_{m=1}^{\infty} m b_m a^{m-1} e^{\mathrm{i}m\alpha}\Big] \tag{2.11}$$

To cancel this component, an additional velocity field outside the cylinder is assumed in the form

$$W_r(z^*,t) = \sum_{m=1}^{\infty} f_s(t)[z^*(t)]^{-s} \tag{2.12}$$

where the time t is a parameter.

The normal velocity on the cylinder surface due to this velocity field is

$$u_n = \mathrm{Re}\sum_{s=1}^{\infty}(-s)f_s(t)a^{-s-1}\mathrm{e}^{-\mathrm{i}s\alpha} \tag{2.13}$$

It follows from the boundary condition on the surface of the cylinder for the instantaneous motion

$$f_s(t) = a^{2s}\overline{b_s(t)} \tag{2.14}$$

The cylinder moves as a rigid body without rotations. Thus the velocities of all points are equal to the velocity of the centre denoted by $\dot{z}_0(t)$. The corresponding complex potential due to the motion of the cylinder for $r \geq a$ is

$$W_c = -a^2\dot{z}_0(t)[z^*(t)]^{-1} \tag{2.15}$$

where $\dot{z}_0(t) = \dot{x}_0(t) + \mathrm{i}\dot{y}_0(t)$.

Finally the total complex potential of the complete instantaneous velocity field of fluid is

$$W = W_f + W_r + W_c \tag{2.16}$$

3. Pressures and resultant forces acting on the cylinder

The pressures p in the fluid at the point (x,y) are given by the formula

$$p = -\rho g y - \rho\dot{\phi} - \frac{1}{2}\rho(u_x^2 + u_y^2) \tag{3.1}$$

where

ρ – density of the fluid

g – acceleration of the gravitational field

u_x, u_y – components of the velocity field that is given as the gradient of the potential ϕ

and the dot over a symbol denotes the time derivative.

The resultant hydrodynamic force (the hydrostatic pressure is neglected) is

$$\boldsymbol{F} = -\oint p\Big|_{r=a} \boldsymbol{n}\, ds \tag{3.2}$$

where $ds = a d\alpha$ is the arc element and $\boldsymbol{n}$ is the unit outward normal vector. In complex number notations it follows

$$F_x + \mathrm{i}F_y = \frac{1}{2}\rho \oint \left(\dot{W} + \overline{\dot{W}}\right) \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha + \frac{1}{2}\rho \oint \frac{\partial W}{\partial z} \overline{\frac{\partial W}{\partial z}} \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha \tag{3.3}$$

where the values under the integral are taken as in (3.2) for $r = a$.

Now to calculate the integrals one has to substitute the expressions (2.16), (2.4), (2.12), (2.14) and (2.15) into (3.3). The final results are

$$\begin{aligned}
&\frac{1}{2}\rho \oint \left(\dot{W}_f + \overline{\dot{W}_f}\right) \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = \pi\rho a^2 \overline{\dot{b}_1} - 2\pi\rho a^2 \overline{b_2 \dot{z}_0(t)} \\
&\frac{1}{2}\rho \oint \left(\dot{W}_r + \overline{\dot{W}_r}\right) \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = \pi\rho a^2 \overline{\dot{b}_1} \\
&\frac{1}{2}\rho \oint \left(\dot{W}_c + \overline{\dot{W}_c}\right) \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = -\pi\rho a^2 \ddot{z}_0(t) \\
&\frac{1}{2}\rho \oint \frac{\partial W_f}{\partial z} \overline{\frac{\partial W_f}{\partial z}} \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = \pi\rho \sum_{n=1}^{\infty} n(n+1) a^{2n} b_n \overline{\dot{b}_{n+1}} \\
&\frac{1}{2}\rho \oint \frac{\partial W_r}{\partial z} \overline{\frac{\partial W_r}{\partial z}} \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = \pi\rho \sum_{n=1}^{\infty} n(n+1) a^{2n} b_n \overline{\dot{b}_{n+1}} \\
&\frac{1}{2}\rho \oint \frac{\partial W_c}{\partial z} \overline{\frac{\partial W_c}{\partial z}} \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = 0 \\
&\frac{1}{2}\rho \oint \left[\frac{\partial W_f}{\partial z} \overline{\frac{\partial W_r}{\partial z}} + \frac{\partial W_r}{\partial z} \overline{\frac{\partial W_f}{\partial z}}\right] \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = 0 \\
&\frac{1}{2}\rho \oint \left[\frac{\partial W_f}{\partial z} \overline{\frac{\partial W_c}{\partial z}} + \frac{\partial W_c}{\partial z} \overline{\frac{\partial W_f}{\partial z}}\right] \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = 0 \\
&\frac{1}{2}\rho \oint \left[\frac{\partial W_c}{\partial z} \overline{\frac{\partial W_r}{\partial z}} + \frac{\partial W_r}{\partial z} \overline{\frac{\partial W_c}{\partial z}}\right] \mathrm{e}^{\mathrm{i}\alpha} a\, d\alpha = -2\pi\rho a^2 \overline{b_2 \dot{z}_0(t)}
\end{aligned} \tag{3.4}$$

The total time derivative has to be calculated according to the second relation in Eqs (2.5). The final formula for the force exerted by the fluid on the cylinder is

$$F_x + \mathrm{i}F_y = -\pi\rho a^2 \ddot{z}_0(t) + 2\pi\rho a^2 \overline{\frac{\partial b_1(z_0, t)}{\partial t}} + 2\pi\rho \sum_{n=1}^{\infty} n(n+1) a^{2n} b_n(z_0, t) \overline{b_{n+1}(z_0, t)} \tag{3.5}$$

In view of Eq (2.7) this expression may be rewritten in the following form

$$F_x + \mathrm{i}F_y = -\pi\rho a^2\ddot{z}_0(t) + 2\pi\rho a^2\Big[a_x(z_0,t) + \mathrm{i}a_y(z_0,t)\Big] + + 2\pi\rho\sum_{n=1}^{\infty} n(n+1)a^{2n}b_n(z_0,t)\overline{b_{n+1}(z_0,t)} \tag{3.6}$$

If the external velocity field corresponds to the case $g_n(t) = 0$ for $n > 2$ then the term with the summation symbol vanishes and a very simple expression results that may be written in the following form

$$F_x + \mathrm{i}F_y = -\pi\rho a^2\ddot{z}_0(t) + 2\pi a^2\rho\frac{D}{Dt}\Big[\Phi_{,x}(x,y,t) + \mathrm{i}\Phi_{,y}(x,y,t)\Big]_{z=z_0} \tag{3.7}$$

This expression may be used if the radius is small compared with a characteristic length, for example the length of the surface water wave. It should be noted that the material time derivative has to be taken.

4. Lagrange's equations of motion

The Lagrangian of the system is

$$L = T - V \tag{4.1}$$

where T is the kinetic energy of the system and V is the potential energy that depends upon the displacements of the cylinder and results from the action of elastic supports.

Let us assume that the coefficients $g_s(t)$ in the formula for the complex potential of the external velocity field are not changed due to the motion of the cylinder. In such a case the generalised coordinates of the system q_α are x_0 and y_0.

Lagrange's differential equations of motion are

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{q}_\alpha}\Big) - \frac{\partial L}{\partial q_\alpha} = F_\alpha \tag{4.2}$$

where F_α are the generalised hydrodynamic forces

$$L = \frac{1}{2}\pi a^2\rho_c\Big[\dot{x}_0^2(t) - \dot{y}_0^2(t)\Big] - \frac{1}{2}\Big\{k_x x_0^2(t) + k_y[y_0(t) - y_p]^2\Big\} \tag{4.3}$$

where

ρ_c – density of the cylinder
k_x, k_y – elastic spring constants
y_p – defines the position of the centre at rest.

In the considered case system of differential equations (4.2) corresponds to two equations, the first one is obtained by differentiation with respect to $\dot{x}_0$ and x_0, and the second one to $\dot{y}_0$ and y_0, respectively.

The hydrodynamic forces are given by Eq (3.6) and in a simplified form by Eq (3.7). The first term on the right side may be shifted to the left side. In a simplified case the final differential equations are

$$\begin{aligned} \pi a^2(\rho_c + \rho)\ddot{x}_0(t) + k_x x_0(t) &= 2\pi a^2 \rho \frac{D}{Dt}\Big[\Phi_{,x}(x,y,t)\Big]_{x=x_0,y=y_0} \\ \pi a^2(\rho_c + \rho)\ddot{y}_0(t) + k_x x_0(t) &= 2\pi a^2 \rho \frac{D}{Dt}\Big[\Phi_{,y}(x,y,t)\Big]_{x=x_0,y=y_0} \end{aligned} \tag{4.4}$$

If there are no springs and the density of the cylinder is equal to the density of the fluid then the differential equations describe the phenomenon of identical accelerations of the cylinder and fluid particle at the instanteneous position of the cylinder centre (in the absence of the cylinder). It is true if the radius is small and the omitted terms in (3.1) are negligible.

It should be noted that ρ in the first term represents the added mass of fluid. The equations result from the description within the theory of perfect incompressible fluids, thus they do not describe the boundary layer and the detachment of vortices. To obtain a better description one has to take into account the real added mass represented by an experimental coefficient C_m. The differential equations of the problem become

$$\begin{aligned} \pi a^2(\rho_c + \rho C_m)\ddot{x}_0(t) + k_x x_0(t) &= 2\pi a^2 \rho (1 + C_m)\frac{D}{Dt}\Big[\Phi_{,x}(x,y,t)\Big]_{x=x_0,y=y_0} \\ \pi a^2(\rho_c + \rho C_m)\ddot{y}_0(t) + k_x x_0(t) &= 2\pi a^2 \rho (1 + C_m)\frac{D}{Dt}\Big[\Phi_{,y}(x,y,t)\Big]_{x=x_0,y=y_0} \end{aligned} \tag{4.5}$$

To obtain a good agreement with the experiments usally a drag force term is added, for example of the type given by Morison et al. (1950), but this problem will not be discussed in this paper.

There is no difficulty to take into account the full expression for the resultant hydrodynamic force (3.6), acting on the cylinder.

The motion of the cylinder is described in Langrange's description, while the velocity field of fluid is in the Euler one. The formulation given by Eqs (4.2) and (4.3) is standard.

Let us look for the Lagrangian L^* that leads directly to the equations of motion from Hamilton's principle of least action that states a conservative mechanical system moves from time t_1 to time t_2 in such a way that

$$J = \int_{t_1}^{t_2} L^* \, dt \tag{4.6}$$

called the action integral, has an extreme value. Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_\alpha}\right) - \frac{\partial L^*}{\partial q_\alpha} = 0 \tag{4.7}$$

should lead to the differential equations of motion obtained previously by considering the resultant forces due to fluid pressure. Luke (1961) proposed a Lagrangian of this type for the wave motion. It should be stressed that in his case as well in our case the formulae for the Lagrangian are not derived in a mathematical way but stated ad hoc.

It is easy to verify that in our case the Lagrangian is

$$\begin{aligned} L^* &= \frac{1}{2}(\rho + \rho_c)\pi a^2 \dot{z}\overline{\dot{z}} - \rho\pi a^2(b_1\dot{z}_0 + \overline{b_1\dot{z}}) + \\ &+ \rho\pi \sum_{n=1}^{\infty} n a^{2n} b_n \overline{b_n} - \frac{1}{2}k_x x_0 - \frac{1}{2}k_y(y_0 - y_p) \end{aligned} \tag{4.8}$$

Let us consider the instanteneous kinetic energy of the external velocity field that corresponds to the area of the cylinder cross section

$$T_K = \frac{1}{2}\rho \iint_{A_c} \frac{\partial W_f}{\partial z} \overline{\frac{\partial W_f}{\partial z}} \, dA \tag{4.9}$$

Simple integration leads to the expression

$$T_K = \frac{1}{2}\rho\pi \sum_{n=1}^{\infty} n a^2 b_n \overline{b_n} \tag{4.10}$$

that appears in the Lagrangian (4.8).

Simple transformations lead to the following more familiar form of the Lagrangian in real variables

$$\begin{aligned} L^* &= \frac{1}{2}(\rho+\rho_c)\pi a^2[\dot{x}_0^2(t)+\dot{y}_0^2(t)] + \\ &- 2\rho\pi a^2\Big[\Phi_{,x}(x,y,t)\dot{x}_0(t)+\Phi_{,y}(x,y,t)\dot{y}_0(t)\Big] + \\ &+ \rho\iint\limits_{A_c}\Big[\Phi_{,x}^2(x,y,t)+\Phi_{,y}^2(x,y,t)\Big]\,dA - \frac{1}{2}k_x x_0^2 - \frac{1}{2}k_y(y_0-y_p)^2 \end{aligned} \tag{4.11}$$

where the subscript after the comma indicates differentiation and Φ is the potential function, equal to the real part of W.

If variability of the external velocity field is small within the area of the cylinder cross section the integral may be approximated by

$$\iint\limits_{A_c}\Big[\Phi_{,x}^2(x,y,t)+\Phi_{,y}^2(x,y,t)\Big]\,dA \approx \rho\pi a^2\Big[\Phi_{,x}^2(x_0,y_0,t)+\Phi_{,y}^2(x_0,y_0,t)\Big] \tag{4.12}$$

Within this approximation the resulting Lagrange's equations are exactly the same as exactly differential equations (4.4). Eq (4.12) may be used for the justification of the introduced simplification.

The variational formulation is very useful when numerical methods are used to obtain solutions. In such a case the differentials are replaced by finite differences and analytical integration is replaced by numerical methods. The final result is a set of algebraic equations containing unknown displacements at chosen times.The variational formulation is constructed within the framework of theory of conservative mechanical systems. In the variational formulation only the first derivatives appear and the numerical approximations of the second derivatives in the final differential equations are obtained from variational calculus. When finite differences are introduced into the differential equations the approximations are done in an independent way that may lead to terms which introduce energy into the system.

5. The case of a surface wave field

Let us consider a layer of fluid of depth h with gravitational surface waves. In the classical wave theory the fluid is inviscid and incompressible and the flow has a potential that satisfies the Laplace equation. For the simplest case of linear harmonic progressive waves the complex potential is

$$W_f = \frac{H\omega}{2k}\frac{1}{\sinh(kh)}[\sin(kz)\cos(\omega t) - \cos(kz)\sin(\omega t)] \tag{5.1}$$

where

H	–	wave height
ω	–	angular wave frequency, $\omega = 2\pi/T$
T	–	wave period
k	–	wave number, $k = 2\pi/L$
L	–	wave length

and $z = x + iy$, the x axis coincides with line of the bottom, $z = 0 + iy_p$ describes the initial position of the centre of the cylinder at rest. The wave number and angular frequency are related by the dispersion equation

$$\frac{\omega^2 h}{g} = kh \tanh(kh) \tag{5.2}$$

where g is the acceleration of gravitational field.

The first terms of the power series expansion in the neighbourhood of the origin of the fixed coordinate system are

$$W_f = \frac{H\omega}{2k}\frac{1}{\sinh(kh)}\Big[-\sin(\omega t) + (kz)\cos(\omega t) + \frac{1}{2}(kz)^2\sin(\omega t) + ...\Big] \tag{5.3}$$

This expression corresponds to the power series given by Eq (2.1). It should be noted that all the coefficients $g_n(t)$ are real functions of time. From definition of the wave number k it follows

$$kz = \frac{2\pi z}{L} \tag{5.4}$$

If the cylinder radius a compared with the wave length is very small, the cylinder at the initial position is close to the bottom and stays during the motion in the neighbourhood of the initial position, then the absolute value of kz is a very small number. For the roughest approximation

$$W_f = u_x(t)z \tag{5.5}$$

and thus the velocity field may be considered as homogeneous in space and harmonic in time and corresponds to a rigid body motion of the fluid parallel to the x axis.

Let us discuss the physical meaning of the coefficients in the case when we assume that the coefficients are zero for n greater than two. It follows from Eq (2.2) that the instantaneous components of velocity at the position of the cylinder centre are

$$\begin{aligned} u_x(x_0, y_0, t) &= g_1(t) + 2g_2(t)x_0(t) \\ u_y(x_0, y_0, t) &= -2g_2(t)y_0(t) \end{aligned} \tag{5.6}$$

To calculate the accelerations one has to calculate the material time derivative. It follows

$$
\begin{aligned}
a_x(x_0, y_0, t) &= \dot{g}_1(t) + 2\dot{g}_2(t)x_0(t) + 2g_2(t)[g_1(t) + g_2(t)x_0(t)] \\
a_y(x_0, y_0, t) &= -2\dot{g}_2(t)y_0(t) + 4g_2^2(t)
\end{aligned}
\tag{5.7}
$$

It should be noted that the components of velocity and acceleration are not independent and are fixed by the values of two real coefficients. Our velocity field is not arbitrary. The potential has to satisfy the Laplace equation and the boundary conditions on the bottom.

The potential function in the moving coordinate system with the help of Eqs (2.4) is

$$
\begin{aligned}
W_f &= [g_0(t) + g_1(t)z_0(t) + g_2(t)] + \\
&+ [g_1(t) + 2g_2(t)z_0(t)][z - z_0(t)] + g_2(t)[z - z_0(t)]^2 + \dots
\end{aligned}
\tag{5.8}
$$

The trigonometric functions of complex arguments have convergent power series expansions for all z and thus the representation in the form (2.1) is justified. It should be, however, noted that Eq (5.1) holds only for points in the fluid. Thus the potential may be used only in the case when the distance of the cylinder to the free water surface is sufficiently long. When the cylinder is close to the free surface the additional velocity field defined by the boundary condition on the cylinder surface has an influence on the boundary conditions on the free surface and results in additional progressive surface waves. In such a case it we can not assum that the external velocity field is independent of the motion of the cylinder as it is done in the present paper.

Let us consider the simplest case of fluid motion given by the potential (5.5). The standard method of analysis of cylinder motion close to the bottom is to consider the motion of two cylinders in the space, the second cylinder corresponds to the mirror reflection of the first one in the x axis and moving with the velocities of centre equal to $\overline{\dot{z}_0(t)}$. The complex potential that satisfies the boundary conditions on the surfaces of cylinder and the x axis (cf Wilde (1995)) is

$$
W = u_x(t) - a^2 \sum_{j=0}^{\infty} \mu_j \Big\{ \frac{\dot{z}_0(t) - u_x(t)}{z - z_0(t) + \mathrm{i}aq_j} + \frac{\overline{z_0(t)} - u_x(t)}{z - z_0(t) + \mathrm{i}[2y_0(t) - aq_j]} \Big\}
\tag{5.9}
$$

where $q_0 = 0$, $q_{j+1} = 1/(2y_0 - aq_j)$ and $\mu_0 = 1$, $\mu_{j+1} = \mu_j q_j^2$.

The resultant force acting on the cylinder may be calculated from the hydrodynamic pressures on the cylinder as it was done in Section 4. Another standard approach is to consider the multiply connected region bounded by the curve S_0 fixed in space, called the control surface, given by the relations

$$\begin{array}{lll} x = R\cos\beta & y = R\sin\beta & 0 < \beta < \pi \\ -R < x < R & y = 0 & \end{array} \tag{5.10}$$

and a moving curve S corresponding to the surface of the cylinder.

Consideration of the resulting momentum of the fluid in the region and the resultant forces on the surfaces S_0 and S leads with the help of integral transformations to the following formula for the force acting on the cylinder

$$\boldsymbol{F}(t) = \frac{d}{dt}\oint\limits_{S} \rho\Phi\boldsymbol{n}\, ds + \rho\oint\limits_{S_0}\Big[\frac{1}{2}(\Phi_{,x}^2 + \Phi_{,y}^2)\boldsymbol{n}_0 - \Phi_{,n_0}\mathrm{grad}\Phi\Big]\, ds_0 \tag{5.11}$$

where

$\boldsymbol{n}$ – unit normal to the cylinder

$\boldsymbol{n}_0$ – unit normal to the control surface

$\Phi_{,n_0}$ – velocity component normal to the control surface.

In complex number notation the vector $\boldsymbol{n}$ corresponds to $e^{i\alpha}a\, d\alpha$ in a moving coordinate system fixed at the centre of the cylinder. The function Φ is the real part of the complex potential W. Thus

$$\Phi = \frac{1}{2}(W + \overline{W}) \tag{5.12}$$

where $z - z_0(t) = ae^{i\alpha}$.

In complex number notation the vector $\boldsymbol{n}_0\, ds_0$ corresponds to $e^{i\beta}R\, d\beta$. The integration is in the fixed in space coordinate system and

$$\Phi_{,x}^2 + \Phi_{,y}^2 = \frac{\partial W}{\partial z}\overline{\frac{\partial W}{\partial z}} \tag{5.13}$$

where $z = Re^{i\beta}$.

Finally, the hydrodynamic force on the cylinder is expressed in terms of a complex function. The real part corresponds to the horizontal component and the imaginary part to the vertical one. The integration is a tedious exercise when applying the residue theorem. An essential part of the calculations is estimation of the contour integral values when R goes to infinity. In the calculations of the second integral on the control surface the following identity given by Wilde (1995) was used

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{4\mu_j\mu_n}{\left[2\frac{y_0}{a}-q_j-q_n\right]^3}=-\frac{1}{2}C'_m(v) \tag{5.14}$$

where $v = y_0/a$ and the derivative is taken with respect to the dimensionless variable v and

$$C_m(v)=1+2\sum_{n=1}^{\infty}\mu_n$$

After integration the final formulae for the components of the hydrodynamic force on the cylinder are

$$\begin{aligned} F_x &= \pi\rho a^2\dot{u}_x(t)-\pi\rho a^2C_m(v)[\ddot{x}_0(t)-\dot{u}_x(t)]+\\ &-\pi\rho aC'_m(v)[\dot{x}_0(t)-u_x(t)]\dot{y}_0(t)\\ F_y &= -\pi\rho a^2C_m(v)\ddot{y}_0(t)-\frac{1}{2}\pi\rho aC'_m(v)\Big\{-[\dot{x}_0(t)-u_x(t)]^2+\dot{y}_0^2(t)\Big\} \end{aligned} \tag{5.15}$$

The final differential equations of the problem are

$$\begin{aligned} &\pi a^2[\rho_c+\rho C_m(v)]\ddot{x}_0-\pi a^2\rho[1+C_m(v)]\dot{u}_x+\\ &+\pi a\rho C'_m(v)[\dot{x}_0-u_x]\dot{y}_0+k_xx_0=0\\ &\pi a^2[\rho_c+\rho C_m(v)]\ddot{y}_0+\frac{1}{2}\pi a\rho C'_m(v)\Big[-(\dot{x}_0-u_x)^2+\dot{y}_0^2\Big]+k_y(y_0-y_p)=0 \end{aligned} \tag{5.16}$$

The first terms in both equations show that $C_m(v)$ is the coefficient of added mass of fluid which depends upon the distance to the bottom. The differential equations are conjugate and nonlinear. If there are no springs and the density of the cylinder is equal to the density of the fluid then $\dot{x}_0(t) = u_x(t)$, $\dot{y}_0(t) = 0$ is a solution. The cylinder moves like the particle of fluid corresponding to the cylinder centre. If there is no external velocity field the differential equations reduce to those given by Wilde (1995) for free vibrations. If the cylinder does not move then

$$\begin{aligned} k_xx_0(t) &= \pi a^2\rho[1+C_m(v)]\dot{u}_x(t)\\ k_yy_0(t) &= \frac{1}{2}\pi a\rho C'_m(v)u_x^2(t) \end{aligned} \tag{5.17}$$

If the spring constants tend to infinity then the displacements approaches zero, but their product is equal to the forces in the springs. The obtained relations are equal to the standard ones.

The Lagrangian that leads directly to the differential equation has the form

$$\begin{aligned} L^* &= \frac{1}{2}\pi a^2\rho[1+C_m(v)]\Big\{[\dot{x}_0 - u_x(t)]^2 + \dot{y}_0^2\Big\} + \\ &+ \frac{1}{2}\pi a^2(\rho_c - \rho)[\dot{x}_0^2 + \dot{y}_0^2] - V[x_0, y_0] \end{aligned} \tag{5.18}$$

where V is the potential energy of the springs

$$V = \frac{1}{2}\Big[k_x x_0^2 + k_0(y_0 - y_p)^2\Big]$$

Now let us discuss a more general case when the external velocity field is given by a potential that has terms up to $n = 2$ in power series expansion. The external potential has a term $g_2[z - z_0(t)]^2$. It is easy to calculate the corresponding additional potential to satisfy the boundary conditions (cf Eqs (2.12) and (2.14)) for the case of a full space. For the half-space it is necessary to construct a solution that satisfies the boundary conditions on the x axis. Then the potential (5.9) has to be supplemented by this solution. The same procedure may be applied to obtain the resultant forces on the cylinder.

Let us look at the approximate formulation. We generalise the Lagrangian (satisfy the boundary condition only for the first singularity). Let us take the Lagrangian

$$\begin{aligned} L^* &= \frac{1}{2}\pi a^2\rho[1+C_m(v)]\Big\{[\dot{x}_0 - g_1(t) - 2g_2(t)x_0]^2 + [\dot{y}_0 + 2g_2(t)y_0]^2\Big\} + \\ &+ \frac{1}{2}\pi a^2(\rho_c - \rho)[\dot{x}_0^2 + \dot{y}_0^2] - V[x_0, y_0] \end{aligned} \tag{5.19}$$

The generalisation is done with the help of Eqs (5.6). The standard form of Lagrange's equations is used to obtain the differential equations for the problem

$$\begin{aligned} &\pi a^2[\rho_c + \rho C_m(v)]\ddot{x}_0 - \pi a^2\rho[1+C_m(v)]a_x(x_0, y_0, t) + \\ &+\pi a\rho C_m'(v)[\dot{x}_0 - u_x(x_0, y_0, t)]\dot{y}_0 + k_x x_0 = 0 \\ &\pi a^2[\rho_c + \rho C_m(v)]\ddot{y}_0 - \pi a^2\rho[1+C_m(v)]a_y(x_0, y_0, t) + \\ &+\frac{1}{2}\pi a\rho C_m'(v)\Big[-(\dot{x}_0 - u_x)^2 + \dot{y}_0^2 - u_y^2\Big] + k_y(y_0 - y_p) = 0 \end{aligned} \tag{5.20}$$

In the final form Eqs (5.6) and (5.7) were used. It should be noted that accelerations defined as material time derivatives are introduced by the variational calculus. It should be stressed that the relations are derived for the potential flow and thus the velocities and accelerations at the cylinder centre can not be taken in an arbitrary way.

6. Conclusions

The differential equations of plane motion of a circular cylinder in a space of fluid with a given velocity field are derived within the framework of Lagrange's formulation of mechanics. The expression for the Lagrangian is derived, based on the comparison to the solution obtained by the analysis of hydrodynamic forces due to pressures in the fluid on the cylinder surface. The Lagrangian is not a difference between the instantaneous kinetic energy of the fluid and cylinder, and the potential energy of the springs. Only in the absence of the external velocity field, i.e. for free vibrations such a statement is true.

It is shown that in the case when the external velocity field changes very little within the cross-section of the cylinder the final equations are very simple. It is necessary only to take the material time derivative of the external velocity field given in Euler's description at the instantaneous position of the centre.

The case of a cylinder moving in a half-space of fluid is considered when the external velocity field is homogeneous in space and variable in time with only one component parallel to the ideal bottom. The Lagrangian is given and the differential equations are derived.

The formulations may be generalised to the case of a non-homogeneous velocity field in an approximate way. For a rigorous solution it is necessary to introduce higher order singularities to satisfy the boundary conditions on the surface of the cylinder. In the case when the cylinder radius is very small compared with the length of the water wave the approximate formulation is justified to get an insight into the behaviour. It should be noted, however, that the boundary conditions on the surface of the cylinder are not satisfied and one may expect discrepancies between the theoretical solution and the experimental values. The expressions reduce to the standard forms for the cases when the cylinder does not move and as a second one there is no external velocity field.

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Równania Lagrange'a dla dwuwymiarowego ruchu cylindra w cieczy doskonałej

Streszczenie

Teoria Lagrange'a dynamiki układu punktów materialnych jest sformułowana w układzie śledzącym ruch poszczególnych punktów. W przypadku ruchu ciała sztywnego w cieczy wygodnie jest określać ruch ciała w opisie Lagrange'a, natomiast pole prędkości w cieczy w opisie Eulera. W takim opisie Lagrangian równy różnicy energii kinetycznej i energii potencjalnej nie prowadzi do prawidłowych równań różniczkowych ruchu. W zagadnieniu rozważanym w pracy określa się Lagrangian jako funkcję, która prowadzi do równań Lagrange'a identycznych z otrzymanymi bezpośrednio z równań Newtona. Zagadnienie jest badane dla przypadku ruch cylindra w przestrzeni i półprzestrzeni cieczy. Najpierw określa się wypadkowe siły hydrodynamiczne działające na cylinder na podstawie analizy ciśnień w cieczy i wyznacza się równania różniczkowe ruchu. Podano wyrażenie na Lagrangian dla przypadku ruchu w cieczy o polu prędkości określonym poprzez funkcję analityczną. W przypadku półprzestrzeni z idealnym dnem zakłada się, że średnica cylindra jest mała i wprowadza się opis przybliżony. Sformułowania wariacyjne bazujące na zasadzie Hamiltona są dogodne w przypadku stosowania metod numerycznych dla uzyskania efektywnych rozwiązań.

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