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NEW QUASI-NATURAL ARTIFICIAL VISCOSITY MODELS FOR COMPRESSIBLE FLUID FLOW, WITH AN IMPROVED ENTROPY PRODUCTION MECHANISM

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The paper presents formulation of artificial viscosity models based on the principle of entropy production for viscous compressible flows. An application of the derived models implemented in an h adaptive finite element code to a benchmark ramp test problem is included as a practical test of their performance.

Key words: artificial viscosity, compressible fluid flow, entropy production

1. Mathematical models of compressible fluid flow and artificial viscosity

We consider three mathematical models of compressible fluid flow. The first one, the Euler equations for inviscid gases, can be expressed as the system of conservation laws

$$U(x,t)_{,t} + f_i^E(U)_{,i} = 0 \tag{1.1}$$

where

$\boldsymbol{U}$ – vector of conservation variables

$\boldsymbol{f}_i^E$ – Eulerian fluxes.

For the complete listing of notation see Appendix A.

In the second model, the Navier-Stokes equations, new viscous and heat fluxes appear that correspond to heat conduction and viscous forces

$$U(x,t)_{,t} + f_i^E(U)_{,i} = f_i^\mu(U,\nabla U)_{,i} + f_i^\kappa(U,\nabla U)_{,i} \tag{1.2}$$

To close the above system additional experimental relations are supplied, namely Fourier's law, relating the heat flux q and the temperature gradient $T_{,i}$

$$q_i = -\kappa T_{,i}$$

Sutherland's law, expressing the viscosity coefficient μ in terms of temperature

$$\mu = \frac{1.45T^{\frac{3}{2}}}{T+110}\cdot 10^{-6}$$

the Prandtl number relationship, establishing the almost constant (in the presented considerations it is assumed exactly constant) ratio between the coefficient κ/c_V, appearing in heat flux terms in the Navier-Stokes equations, and the viscosity coefficient μ

$$\mathrm{Pr} = \frac{\mu\gamma c_V}{\kappa} = 0.72$$

and the Stokes hypothesis, relating two viscosity coefficients μ and λ

$$\lambda = -\frac{2}{3}\mu$$

In both models we assume that gas is calorically perfect, i.e. that

$$\gamma = \text{const} \qquad e = e_I + e_K = \frac{1}{2}u_l u_l + \frac{p}{(\gamma-1)\rho}$$

where

- e – total specific (per unit mass) energy
- e_I – specific internal energy
- e_K – specific kinetic energy.

Obviously, the Navier-Stokes equations are the more accurate of the two descriptions above. Nevertheless, in many cases one can still obtain good approximations of real flows using simpler (at least in terms of the number of physical processes considered) model of the Euler equations. The correspondence between the flows described by the two models is based only on heuristic and phenomenological observations and there is no mathematical evidence or estimate whether the solutions to the Euler equations can approximate, and

if yes then to which extent, the solutions to the Navier-Stokes equations. Moreover, it is well known that existence of boundary layers, the phenomenon neglected by the Euler model, may constitute the most important factor of a flow, changing essentially its character.

The Navier-Stokes equations can be expressed in a quasilinear form when natural viscosity matrices $\mathbf{K}^{\mu}_{ij}$ ($f^{\mu}_i = \mathbf{K}^{\mu}_{ij}(U)U_{,j}$) and matrices $\mathbf{K}^{\kappa}_{ij}$ ($f^{\kappa}_i = \mathbf{K}^{\kappa}_{ij}(U)U_{,j}$) are introduced

$$U(x,t)_{,t} + f^E_i(U)_{,i} = \left[\left(\mathbf{K}^{\mu}_{ij}(U) + \mathbf{K}^{\kappa}_{ij}(U)\right)U_{,j}\right]_{,i} \tag{1.3}$$

The third model of compressible fluid flow, regularized Euler equations, can be obtained by replacing the matrices $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$ in Eq (1.3) by some new matrices $\mathbf{K}^{AV}_{ij}$ which represent the model of artificial viscosity

$$U(x,t)_{,t} + f^E_i(U)_{,i} = \left[\mathbf{K}^{AV}_{ij}(U,\nabla U)U_{,j}\right]_{,i} \tag{1.4}$$

The reason for introducing the regularized Euler equations is the necessity for changing some properties of solutions to the system of equations describing the fluid flow. The Euler equations form a hyperbolic system of equations which in general can posess discontinuous solutions (in which case the equations have to be understood in a distributional sense). When one tries to solve the Euler equations using some straightforward approximation method (like central differences or the Galerkin finite elements) the numerical solutions become unstable. The situation is not much better in the case of the Navier-Stokes equations. The Navier-Stokes equations form a mixed hyperbolic-parabolic system of equations (the terms with matrices $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$ are all of the second order but the first rows in matrices $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$ contain only zeros) solutions of which are continuous[1]. However the physical and numerical experiments (as well as some mathematical reasoning in simplified situations) show that for real gases or their approximate models there still exist regions of very rapid changes in flow properties (shocks) with the thickness of approximately several mean free paths of gas particles ($\approx 10^{-7}$ m). This means that the accurate resolution of shocks in numerical simulations of flows around or inside macroscopic scale objects would still require very, very small grid sizes, usually making such numerical calculations impossible for today's computers. Additionaly, the accurate resolution of shocks is sometimes

[1]The mathematical theory of the Euler and the Navier-Stokes equations is still by no means complete. To procede further, we optimistically assume that the results which today are proved only for special cases will be confirmed in future for the general situation of arbitrary flows.

not necessary and we can achieve the aims of numerical computations using less accurate approximate models. In this way we arrive at regularizations to the Euler equations. The aim is to provide an approximate model of fluid flow in the same way as the Euler equations do, but without introducing the discontinuous solutions and, instead, having solutions where shocks are of a thickness possible to be resolved during numerical simulations.

The arguments above are presented to justify our approach to the regularized Euler equations: despite their name (adopted here in accordance with previous works on the subject) we will treat them as an approximate model of the Navier-Stokes equations, rather than of the Euler equations. In such a case several indications can be drawn for their design (which basically means the design of an artificial viscosity model). First, since the reason for their existence is to allow an approximation method to be able to resolve all flow features present in the fluid flow model, the model should somehow depend on the approximation method. In practice, these means introducing some parameters of the approximation method into the artificial viscosity terms (usually this parameter is a characteristic linear local mesh size, denoted by h). Furthermore, the method should be consistent, i.e. with the increasing accuracy of the method the model should become closer to the Navier-Stokes equations. Hence, artificial viscosity terms should vanish with the increased accuracy of computations and, at a certain point (the question at which should also be specified by the regularization procedure) the original Navier-Stokes second order terms (with matrices $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$) should be recovered. Introduction of the mesh parameter h as a multiplicative factor in the artificial viscosity terms allows for satisfaction of the first part of this requirement, i.e. it makes the artificial viscosity terms vanish with h decreasing to zero.

When proving the correctness of an artificial viscosity model the following procedure is often adopted: one assumes that the Euler equations provide the ultimate fluid flow model and tries to prove that with the increasing accuracy of computations and artificial viscosity terms tending to zero approximate solutions to the regularized Euler equations converge to the solutions to the Euler equations. Contrary to that, we consider artificial viscosity as an approximation to natural viscosity and heat conduction and, instead of proving convergence to the Euler model, we try to establish some, important in our view, links between the model of artificial viscosity and the natural viscosity and the heat conduction terms in the Navier-Stokes equations.

2. The second law of thermodynamics and entropy production

Systems of equations (1.1) and (1.2) express the principles of conservation of mass, momentum and energy for the corresponding fluid flow models. Compressible gases have also to satisfy the principle of entropy production i.e. the second law of thermodynamics, which, in the formulation of Clausius and Duhem, looks as follows

$$\eta_{,t} + f^{\eta}_{i,i} + \left(\frac{q_i}{e_I}\right)_{,i} \geq 0 \tag{2.1}$$

where $\eta(U) = \rho \ln(p\rho^{-\gamma})$ is the entropy density and $f^{\eta}_i(U) = \eta u_i$ are the corresponding entropy fluxes.

The second law of thermodynamics has to be added to the Euler equations as some extra condition. This is because the Euler equations can possess solutions which contradict the principle of entropy production and thus have no physical meaning. The role of condition (2.1) (with zero heat flux, since we neglect it in inviscid gas flow) is to rule out all unphysical solutions leaving the only one (as it is believed) that is physical.

The situation is different for viscous gases. The second law of thermodynamics is already built into the more comprehensive Navier-Stokes equations. To see this we multiply Eqs (1.2) by the gradient vector $\eta_{,U}$ and use the, so-called, compatibility condition for the entropy fluxes, $\eta_{,U} f^{E}_{i,U} = f^{\eta}_{i,U}$. After some transformations, we obtain the equation expressing the local entropy production for viscous gases

$$\eta_{,t} + f^{\eta}_{i,i} + \left(\frac{q_i}{e_I}\right)_{,i} = \frac{\mu}{e_I}\left(2D_{ij}D_{ij} - \frac{2}{3}u_{i,i}u_{j,j}\right) + \frac{c_V}{\kappa}\frac{1}{e_I^2}q_i q_i \geq 0 \tag{2.2}$$

We believe that the mechanism of entropy production constitutes an important part of the mathematical model describing the fluid flow. We have already used the principle of entropy production to prove stability properties of certain numerical methods (cf Banaś and Demkowicz (1996)). Now we postulate that the model of artificial viscosity should also provide a proper mechanism of entropy production, a mechanism that resembles the exact one presented in Eq (2.2).

3. Quasi-natural artificial viscosity

When trying to design a model of artificial viscosity with correct entropy production, the first idea which comes to mind is simply to replace the viscosity

coefficient (and related to it, by the Prandtl number relation, the heat flux coefficient) with some properly chosen function, such that the correct relations of an artificial viscosity model with both the physical description of the process and the approximate model are maintained (cf Dulikravich et al. (1989), Rachowicz (1993)). Below we will exploit the same ideas as in Dulikravich et al. (1989), with the difference in the choice of, as is there called, an artificial dissipation sensor (or indicator), which will be presented in the next section. We observe that if we substitute into the matrices $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$ the relations

$$\frac{\kappa}{c_V} = \frac{\mu\gamma}{\mathrm{Pr}} \qquad \lambda = -\frac{2}{3}\mu \tag{3.1}$$

we can find in both $\mathbf{K}^{\mu}_{ij}$ and $\mathbf{K}^{\kappa}_{ij}$ a common fraction μ/ρ which can be factored out of the sum of the matrices defining thus the following set of matrices $\overline{\mathbf{K}}_{ij}$

$$\overline{\mathbf{K}}_{ij} = \frac{\rho}{\mu}\mathbf{K}^{\mu}_{ij} + \frac{\gamma\rho}{\mathrm{Pr}}\frac{c_V}{\kappa}\mathbf{K}^{\kappa}_{ij}$$

such that

$$\frac{\mu}{\rho}\overline{\mathbf{K}}_{ij} = \mathbf{K}^{\mu}_{ij} + \mathbf{K}^{\kappa}_{ij}$$

In order to obtain a physically correct model of artificial viscosity we have to estimate now the dependance of the natural viscosity coefficient μ on the state variables. We observe that Sutherland's formula can be, when we assume that the temperature changes are of moderate range, approximated by a linear rule which leads to a linear dependence of μ upon the ratio p/ρ. This leads to the first of our new artificial viscosity models, which we choose to call "the quasi-natural artificial viscosity" (QN)

$$\mathbf{K}^{QN}_{ij} = \epsilon h \frac{p}{\rho^2}\overline{\mathbf{K}}_{ij}$$

where ϵ is a parameter and the mesh parameter h has been included as a multiplicative factor in accordance with our previous considerations.

To find another physically "almost" consistent model of artificial viscosity we proceed by analysing the simplest model of artificial viscosity with matrices $\mathbf{K}^{AV}_{ij}$ having only one variable parameter ϵ: $\mathbf{K}^{D}_{ij} = \epsilon\delta_{ij}\mathbf{I}$ (D due to the diagonal structure of matrices $\mathbf{K}^{D}_{ij}$). We compute the entropy production for this model in the same way as for the Navier-Stokes equations, substituting additionally $(\eta_{,U})_{,i} = \boldsymbol{U}^{\mathsf{T}}_{,i}\eta_{,UU}$ to get

$$\eta_{,t} + f^{\eta}_{i,i} - \left(\eta_{,U}\epsilon\delta_{ij}\boldsymbol{U}_{,j}\right)_{,i} = \boldsymbol{U}^{\mathsf{T}}_{,i}(-\eta_{,UU})\epsilon\delta_{ij}\boldsymbol{U}_{,j} \geq 0 \tag{3.2}$$

The last inequality results from the fact that the matrix $-\eta_{,UU}$ is symmetric and positive definite. When we compare the corresponding terms in Eqs (3.2) and (2.2)

$$\left(\frac{q_i}{e_I}\right)_{,i} = -\left(\frac{\kappa}{c_V}\frac{1}{\rho e_I}[e_K - e_I, -u_k, 1]U_{,i}\right)_{,i}$$

with

$$-\left(\eta_{,U}\epsilon\delta_{ij}U_{,j}\right)_{,i} = -\left(\frac{\epsilon}{e_I}[e_K - \gamma e_I + \ln(p\rho^{-\gamma})e_I, -u_k, 1]U_{,i}\right)_{,i}$$

and

$$\frac{c_V}{\kappa}\frac{1}{e_I^2}q_i q_i = \frac{\kappa}{c_V}\frac{1}{\rho^2 e_I^2}U_{,i}^{\mathsf{T}}\begin{bmatrix} (e_K - e_I)^2 & -u_j(e_K - e_I) & e_K - e_I \\ -u_i(e_K - e_I) & u_i u_j & -u_i \\ e_K - e_I & -u_j & 1 \end{bmatrix}U_{,i}$$

with

$$U_{,i}^{\mathsf{T}}(-\eta_{,UU})\epsilon\delta_{ij}U_{,j} = \frac{\epsilon}{\rho e_I^2}U_{,i}^{\mathsf{T}}\begin{bmatrix} \gamma e_I^2 + e_K^2 & -u_j e_K & e_K - e_I \\ -u_i e_K & e_I\delta_{ij} + u_i u_j & -u_i \\ e_K - e_I & -u_j & 1 \end{bmatrix}U_{,i}$$

we observe that the entropy production due to the heat flux q_n in the Navier-Stokes model is very similar to the entropy production due to the artificial dissipation in the diagonal model. The D model of artificial viscosity provides only an approximate mechanism for entropy production, however now the artificial viscosity terms make the system of equations elliptic (all equations have some form of dumping) which should make the method more robust.

4. Dependence of the coefficients in the artificial dissipation models on the flow field

Models of artificial dissipation usually consist of two parts: one that contains information on how the artificial viscosity terms are oriented in space and how they change depending on the equation they are applied to (in our representation this information will be carried by the structure of matrices $\mathbf{K}_{ij}^{AV}$), and the second part usually in the form of some variable parameter in front of all artificial viscosity terms. The role of this parameter (an artificial viscosity sensor, Dulikravich and Dorney (1989)) is to indicate the points at which we should apply artificial viscosity and to determine the actual amount of artificial viscosity to be applied.

In our QN model this coefficient is derived from physical considerations: we demand that it depends upon the flow variables in the same way as natural viscosity and heat conduction coefficients do. However this results in a situation where in the regions of smooth although not constant flow, the terms of QN artificial viscosity are of several orders of magnitude bigger than the corresponding terms in the Navier-Stokes equations. Hence in these regions, where the original flow is practically inviscid and where we expect that accurate solutions can be obtained without artificial viscosity at all, the QN artificial viscosity introduces substantial dumping and the quality of approximation deteriorates. So there arises the question what else can be put in front of matrices $\overline{\mathsf{K}}_{ij}$ instead of the factor p/ρ^2?

The answer consistent with physical reasonings we try to accept in this paper could be as follows. We have to apply artificial viscosity near shocks where the standard approximation methods become unstable. Shocks are the places where the right-hand side of Eq (1.3) is big, and so is the left hand side of the equations. Hence, it is reasonable to base the indicator part of artificial viscosity on the residual of the Euler equations (cf Hansbo and Johnson (1991))

$$\boldsymbol{r}_E := U(x,t)_{,t} + \boldsymbol{f}_i^E(U)_{,i}$$

In other words, we should just apply the artificial viscosity terms at the places where departure of approximate solutions from the Euler model is the biggest, requiring the change of the model of a fluid flow. Once again we would like to stress that this reasoning turns out to be natural and straightforward when we understand our simulations as an approximation of the Navier-Stokes equations.

When based on $\boldsymbol{r}_E$ the quasi-natural viscosity takes on the final form

$$\mathsf{K}_{ij}^{RQN} = Ch^2\|\boldsymbol{r}_E\|\overline{\mathsf{K}}_{ij}$$

where $\|\boldsymbol{r}_E\|$ is some vector norm of $\boldsymbol{r}_E$ and R in RQN indicates the fact that we have used the residual of the Euler equations. We implemented the RQN model of artificial viscosity with the norm $\|\boldsymbol{r}_E\|_\eta = \sqrt{-\boldsymbol{r}_E\eta_{,UU}\boldsymbol{r}_E}$ which naturally arises in the context of compressible fluid flows. Another h in front of the matrices $\overline{\mathsf{K}}_{ij}$ appears in order to maintain the original relation between the artificial viscosity and the mesh size (the artificial viscosity should vanish with increasing density of the mesh). Since $\boldsymbol{r}_E$ depends on the gradient of solution in a quasi linear way, the product $h\boldsymbol{r}_E$ is of the same order during mesh refinements so the whole artificial viscosity terms behave asymptotically in the same way as in the QN model.

The residual $\boldsymbol{r}_E$ can be used as an artificial viscosity indicator also in a slightly different form. If we multiply it by the gradient vector $\eta_{,U}$ we get the local departure from the inviscid entropy balance at a point

$$\eta_{,U} \cdot \boldsymbol{r}_E = \eta_{,t} + f^{\eta}_{i,i}$$

Now the absolute value $|\eta_{,U} \cdot \boldsymbol{r}_E|$ can be used as an indicator where we should use artificial viscosity with the clear physical interpretation as an indicator of places where the entropy balance should be restored. A new model in which this form of $\boldsymbol{r}_E$ is used

$$\mathsf{K}^{EQN}_{ij} = Ch^2|\eta_{,U} \cdot \boldsymbol{r}_E|\overline{\mathsf{K}}_{ij}$$

has a letter E to indicate the entropy balance considerations.

Both the abovementioned artificial viscosity coefficients or indicators can also be used with the diagonal model yielding the next two models of artificial dissipation

$$\mathsf{K}^{RD}_{ij} = Ch^2\|\boldsymbol{r}_E\|\delta_{ij}\mathsf{I}$$
$$\mathsf{K}^{ED}_{ij} = Ch^2|\eta_{,U} \cdot \boldsymbol{r}_E|\delta_{ij}\mathsf{I}$$

5. Performance of the proposed artificial viscosity models

We implemented the proposed artificial viscosity models in our 2D h-adaptive finite element code for compressible fluid flow problems (cf Banaś and Demkowicz (1994)) based on the second order Taylor-Galerkin time marching algorithm. For space discretization we employed an automatic h-adaptivity with an error indicator developed by Erikson and Johnson (1993), once again based on the residual of the Euler equations.

The solution to a well known benchmark problem of 2D compressible gas dynamics, the so-called ramp problem (cf Woodward and Colella (1984)) was chosen as a test for the models of artificial viscosity. A shock, of a strength Mach 10, is travelling along and perpendicular to the wall, represented by a sloped part of the boundary (see Fig.1 and Fig.2). At time $t = 0$ it meets a corner of a wall with which it makes an angle of 60 degrees. A complicated structure of double Mach reflection of the shock at the wall developes with a jet of a dense fluid along the wall just behind the shock. The problem has simple conditions (the bottom part is just a reflecting wall while the motion of the

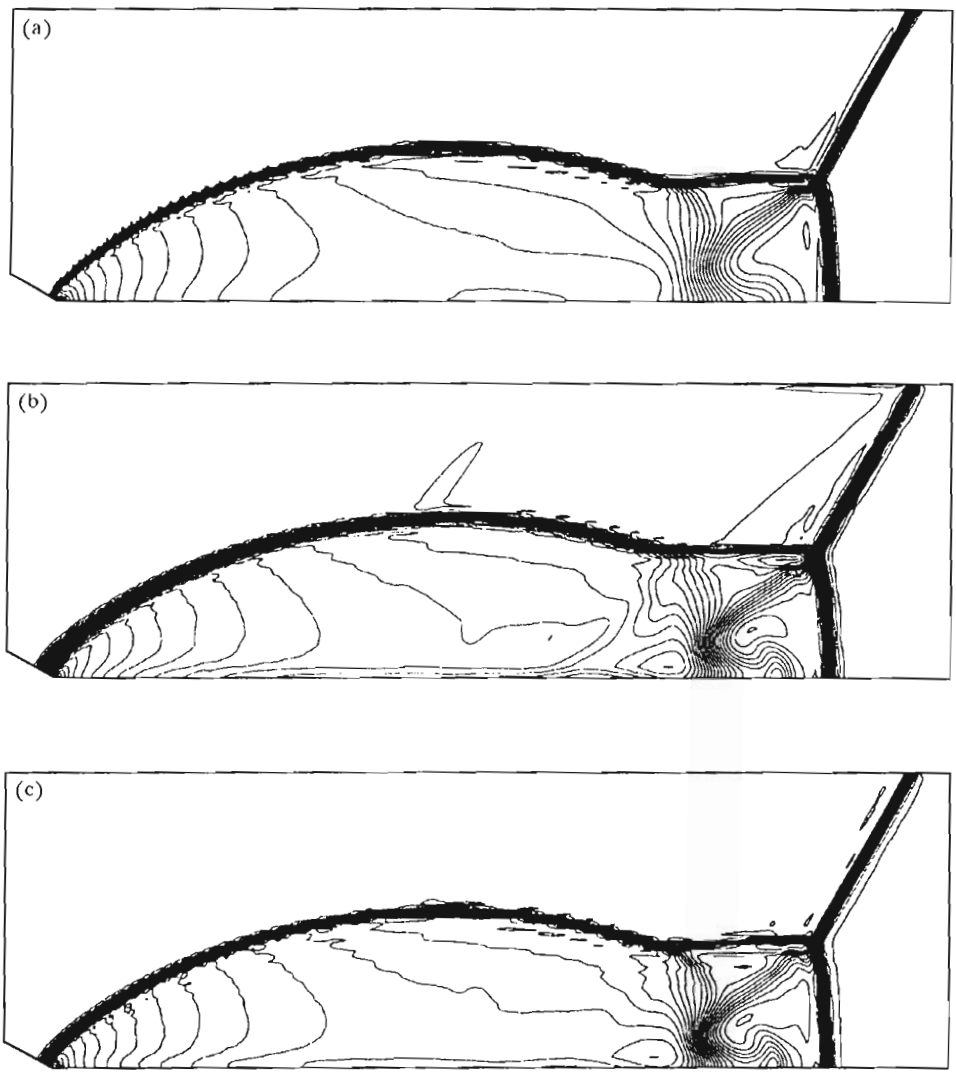

Fig. 1. The ramp problem – density contours at the time $t = 0.2$s: (a) $\Delta\rho = 0.4$ – QN model, $C = 3$; (b) $\Delta\rho = 0.4$ – ED model, $C = 1.5$; (c) $\Delta\rho = 0.4$ – RQN model, $C = 8$. The smallest grid size $\Delta x = 1/98$, $\Delta y = 1/56$, three levels of refinement

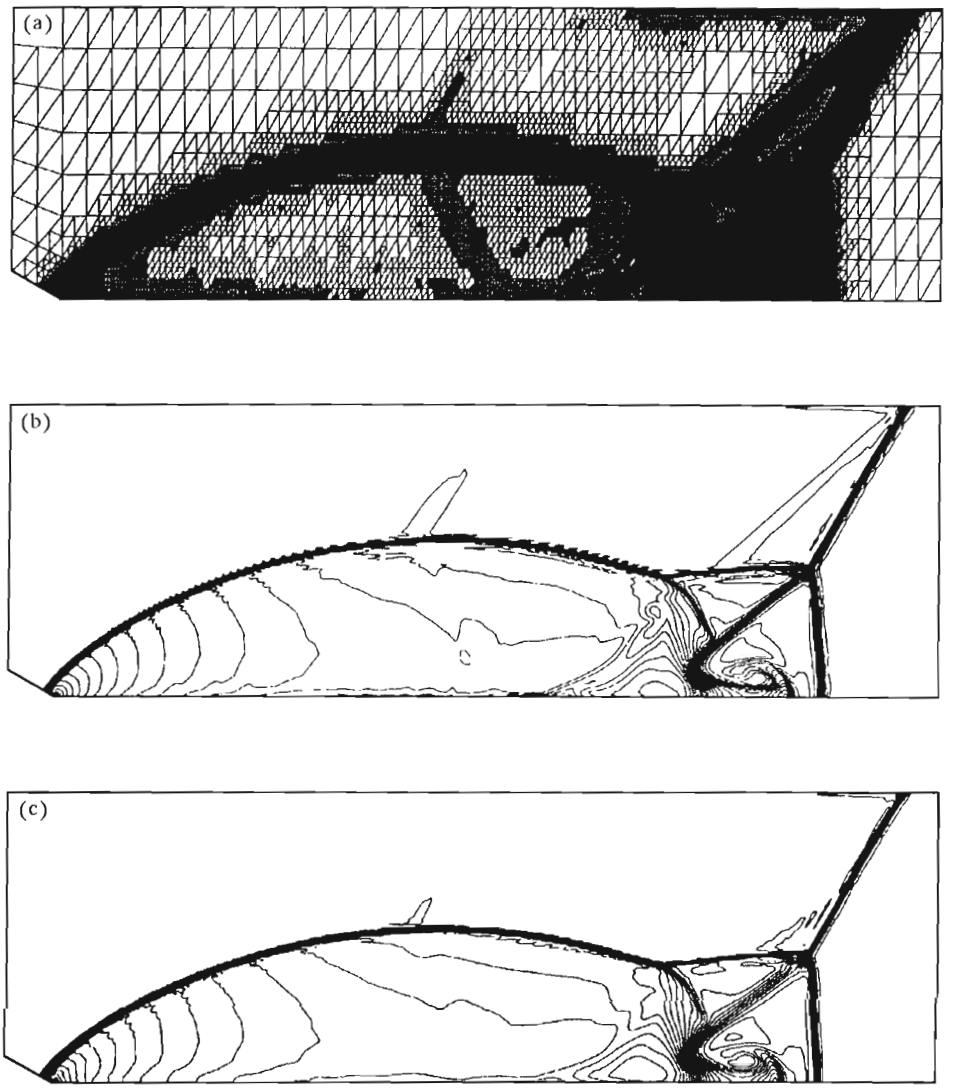

Fig. 2. The ramp problem – an example h-adaptive finite element mesh (a) and two density contours at the time $t = 0.2$s: (b) $\Delta\rho = 0.4$ – RD model, $C = 4$; (c) $\Delta\rho = 0.4$ – EQN model, $C = 4$. The smallest grid size $\Delta x = 1/196$, $\Delta y = 1/112$, four levels of refinement

shock along the top boundary is directly enforced by the Dirichlet boundary conditions) so the performance of the computer code depends solely on the algorithm.

We present a series of results, for different artificial viscosity models and two different maximal refinement levels of h adaptive meshes. For both cases of maximal refinement levels (three and four) the limiting value for refinement indicator was chosen small enough to ensure that the most complicated features of the flow are solved on portions of the mesh with a uniform element size. In Fig.1 the results are presented for three levels of refinement allowed, with the smallest grid size $\Delta x = 1/98$ and $\Delta y = 1/56$. In Fig.2, for four levels of refinement, the smallest grid size was $\Delta x = 1/196$ and $\Delta y = 1/112$. In all cases the Courant-Friedrichs-Levy (CFL) number was kept constant during the simulations and equal to one (the element size used to compute the CFL number was taken as $h = \sqrt{\Delta x \Delta y/2}$).

As one could expect the first AV model, QN, the only one which has not been based on the residual of the Euler equations, gave the poorest results with overly smeared thick shocks. The other models gave comparable results, each having its advantages and disadvantages. Although all the models require much more computational tests in order to evaluate their quality, certain observations can already be made. There have been no important differences between the methods based on the norm of $\boldsymbol{r}_E$ and the local entropy production $|\eta_{,U} \cdot \boldsymbol{r}_E|$. The methods based on the quasi-natural model resulted in the solutions with thicker shocks and small oscillations after the stationary shocks. The methods based on the diagonal model had a worse resolution of contact discontinuities and produced overshots close to shocks, often leading to the appearence of negative pressure.

6. Conclusions

We presented a derivation of several new and a justification for several existing artificial viscosity models based on the idea of approximating in the numerical solutions the entropy production mechanism present in the Navier-Stokes model. The models proved to give accurate and stable results for a difficult transient benchmark problem of 2D gas dynamics when applied along with the second order Taylor-Galerkin finite element scheme. We believe they can be also effectively used with different time and space integration algorithms.

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Appendix A. Notation

$\boldsymbol{x}$	–	point inside Ω_C
t	–	time
Ω_C	–	computational domain, $\Omega_C \subset \mathbb{R}^n$, $n = 1, 2, 3$
$\boldsymbol{n}$	–	outward unit vector normal to the boundary $\partial\Omega_C$
ρ	–	density
u_i	–	ith component of velocity

e	–	total specific energy, $e = e_K + e_I$
e_K	–	specific kinetic energy, $e_K = \frac{1}{2}u_l u_l$
e_I	–	specific internal energy, $e_I = \frac{p}{(\gamma-1)\rho}$
p	–	pressure, $p = (\gamma - 1)(\rho e - \frac{1}{2}\rho u_i u_i)$
γ	–	ratio of specific heats, $\gamma = c_V/c_p$, c_V at a constant volume, c_p at a constant pressure
$\boldsymbol{U}$	–	vector of conservation variables, $U = [\rho, \rho u_k, \rho e]^\top$
$\boldsymbol{\sigma}$	–	tensor of viscous stresses, $\sigma_{ij} = 2\mu D_{ij} + \lambda\delta_{ij}D_{kk}$
$\mathbf{D}$	–	rate of deformation tensor, $D_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
μ, λ	–	coefficients of viscosity, ($\lambda = -\frac{2}{3}\mu$ by the Stokes hypothesis)
$\boldsymbol{q}$	–	heat flux vector, $q_i = -\kappa T_{,i}$, κ – coefficient of thermal conductivity
T	–	temperature, $T = \frac{1}{c_V}e_I$
$\eta(U)$	–	nondimensional entropy density, $\eta(U) = \rho\ln(p\rho^{-\gamma})$
$f_i^\eta(U)$	–	fluxes of nondimensional entropy, $f_i^\eta(U) = \eta u_i$
$\eta_{,U}, \eta_{,UU}$	–	gradient and hessian of nondimensional entropy with respect to conservation variables ($-\eta_{,UU}$ is symmetric and positive definite)

$$\eta_{,U} = \frac{1}{e_I}\Big[e_K - \gamma e_I + \ln(p\rho^{-\gamma})e_I, -u_k, 1\Big]$$

$$\eta_{,UU} = -\frac{1}{\rho e_I^2}\begin{bmatrix} \gamma e_I^2 + e_K^2 & -u_j e_K & e_K - e_I \\ -u_i e_K & e_I\delta_{ij} + u_i u_j & -u_i \\ e_K - e_I & -u_j & 1 \end{bmatrix}$$

$\boldsymbol{f}_i^E$	–	Eulerian fluxes, $\boldsymbol{f}_i^E = [\rho u_i, \rho u_i u_j + p\delta_{ij}, (\rho e + p)u_i]^\top$
$(\boldsymbol{f}_i)_{,U}$	–	Jacobians of Eulerian fluxes
$\boldsymbol{f}_i^\mu$	–	viscous fluxes, $\boldsymbol{f}_i^\mu = [0, \sigma_{ik}, \sigma_{ij}u_j]^\top = \mathbf{K}_{ij}^\mu(U)U_{,j}$
$\mathbf{K}_{ij}^\mu(U)$	–	natural viscosity matrices

$$\mathbf{K}_{ij}^\mu = \frac{1}{\rho}\begin{bmatrix} 0 & 0 & 0 \\ -\mu u_l\delta_{ij} - \mu u_i\delta_{jl} - \lambda u_j\delta_{il} & \mu\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \lambda\delta_{jk}\delta_{il} & 0 \\ -(\lambda+\mu)u_i u_j - 2\mu e_K\delta_{ij} & \mu u_k\delta_{ij} + \mu u_j\delta_{ik} + \lambda u_i\delta_{jk} & 0 \end{bmatrix}$$

$\boldsymbol{f}_i^{\kappa}$ – vectors containing components of heat flux $\boldsymbol{q}$,

$$\boldsymbol{f}_i^{\kappa} = \left[0, 0, -q_i\right]^{\mathrm{T}} = \mathbf{K}_{ij}^{\kappa}(U)U_{,j}$$

$$\mathbf{K}_{ij}^{\kappa} = \delta_{ij}\frac{\kappa}{c_V}\frac{1}{\rho}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_K - e_I & -u_k & 1 \end{bmatrix}$$

$\mathbf{K}_{ij}^{AV}(U, \nabla U)$	–	matrix functions corresponding to a particular model of artificial viscosity
$\mathbf{K}_{ij}^{TG}$	–	matrix functions of second order Taylor-Galerkin (Lax-Wendroff) approximation method, $\mathbf{K}_{ij}^{TG} = \frac{\Delta t}{2}(\boldsymbol{f}_i)_{,U}(\boldsymbol{f}_j)_{,U}$
Δt	–	*time step length*, $\Delta t = t^{n+1} - t^n$
h	–	characteristic linear local mesh (element) size
C	–	constant
$\mathbf{I}$	–	identity matrix
δ_{ij}	–	Kronecker delta
$(\cdot)_{i,j,k,l}$	–	subscripts corresponding to space dimensions, $i, j, k = 1, \ldots, n$
$(\cdot)_{\infty}$	–	subscript for free stream values
$(\cdot)_n$	–	subscript denoting normal component of a vector
$(\cdot)^n$	–	superscript for values at time t^n
$(\cdot)_{,t}, (\cdot)_{,i}$	–	partial derivatives, $\partial/\partial t$, $\partial/\partial x_i$

summation convention is always implied by repeated indices

in all vectors and matrices the middle rows and columns should be expanded according to the number of space dimensions n

Nowe quasi-naturalne modele sztucznej lepkości dla symulacji przepływów gazów ściśliwych, z poprawionym mechanizmem produkcji entropii

Streszczenie

Zaprezentowano wyprowadzenie nowych modeli sztucznej lepkości dla numerycznej symulacji przepływów gazów ściśliwych w oparciu o zasadę produkcji entropii dla

przepływów lepkich zawartą w równaniach Naviera-Stokesa, Modele, po implementacji w programie adaptacyjnej metody elementów skończonych, zostały przetestowane na przykładzie symulacji interakcji fali uderzeniowej z klinem. Uzyskane wyniki potwierdzają przydatność wypracowanych modeli do symulacji przepływów z silnymi falami uderzeniowymi.

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