MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 35, 1997

GEOMETRICALLY NONLINEAR MODELS OF ELASTIC AND ELASTIC-PLASTIC BEAMS

ANNA SŁAWIANOWSKA

Institute of Fundamental Technological Research, Polish Academy of Sciences

Following the moderate rotation theory of shells, we develop a similar treatment of elastic and elastic-plastic beams, undergoing small deformations and moderate rotations. We assume that a beam can deform in a fixed vertical plane only, and that the applied load does not act on its vertical walls. No Bernoulli-type constraints are imposed.
The one-dimensional problem studied is described by a system of four ordinary differential equations, being the equilibrium equations (with boundary conditions) of the considered beams. These equations have been derived on the basis of variational techniques.

Key words: beams, geometrical nonlinearity

1. Introduction

The problem of geometrical nonlinearity in beam theory was investigated by many authors. Most of them seem to deal with elastic problems, e.g. [2,3,6,7,10]. Gałka et al. (1994) formulated the complementary energy principle for two models of geometrically nonlinear compressed elastic beams. Also Mikkola (1989) dealt with variational approach to elastic geometrically nonlinear beams and trusses.

The novelty of the beam theory examined by Reissner (1972) was an incorporation of the transverse shear deformation. A geometrically non-linear plane problem was considered. Shear deformation beam-bending theories as adequate models for anisotropic and composite beams were investigated by Rychter (1993). A refined Bernoulli-Euler type theory was treated as a special case.

Shield (1992) considered a small strain pure bending of a beam or wide strip using finite-deformation theory and neglecting some higher-order terms. The shell theory was used as a starting point.

Reffering to elastic-plastic structures like beams, plates and shells, one has to cite the book by Washizu (1975), where a variational approach to linear and geometrically nonlinear problems is consequently used.

Schmidt and Weichert (1989) explored the rate variational principle due to Neal (1972) for the case of quasi-static problems of elastic-plastic shells. On the basis Schmidt and Weichert (1989) we will carry out a variational study of elastic-plastic beams. It is interesting to note that Telega (1976) showed that Neal's principle can be included into the variational framework of potential operators.

Novozhilov and other authors, see Pietraszkiewicz (1980), pointed out that in the case of thin structures; such as: shells, plates and beams, rotations are often large or moderately large, although strains are small.

The aim of this paper is to formulate an approach to the geometrically nonlinear theory of elastic and elastic-plastic beams using consistent strain- and rotation-based order estimates. In the case of elastic beams the variational principle of virtual work will be used. The equilibrium equations and static boundary conditions will be deduced from this principle.

For the elastic-plastic beams the rate variational principle due to Neal (1972) will yield their rate equilibrium equations and static boundary conditions. The only effective dimension and the straight-line reference configuration of the middle axis make the beam problem easier than the adequate one for shells.

2. Geometry of the problem and essential assumptions

We will consider a homogeneous beam of the length l and a constant rectangular cross-section of height h and breadth b. A coordinate system in space is chosen in such a way that the middle axis of the beam in its reference configuration occupies the segment $(0, l)$ on the first coordinate axis and the vertices of the "left" end surface of the beam have coordinates $(0, -h/2, -b/2)$, $(0, -h/2, b/2)$, $(0, h/2, -b/2)$, $(0, h/2, b/2)$; see Fig.1. We use the same coordinate system to parameterize the reference position of material points and current spatial points as well. The material (Lagrangian) coordinates are denoted by Θ^i, $i = 1, 2, 3$ whereas the spatial (Eulerian) coordinates are denoted by x^i. The total boundary surface of the beam in the reference configuration will be denoted by $\mathcal{A}$ and its interior by $\mathcal{V}$. The boundary planes with $\Theta^3 = h/2$, $\Theta^3 = -h/2$, $\Theta^2 = b/2$, $\Theta^2 = -b/2$, $\Theta^1 = l$, $\Theta^1 = 0$, are denoted, respectively, by $\mathcal{A}_{3+}, \mathcal{A}_{3-}, \mathcal{A}_{2+}, \mathcal{A}_{2-}, \mathcal{A}_{1+}, \mathcal{A}_{1-}$. The beam axis is denoted by $\mathcal{M}$.

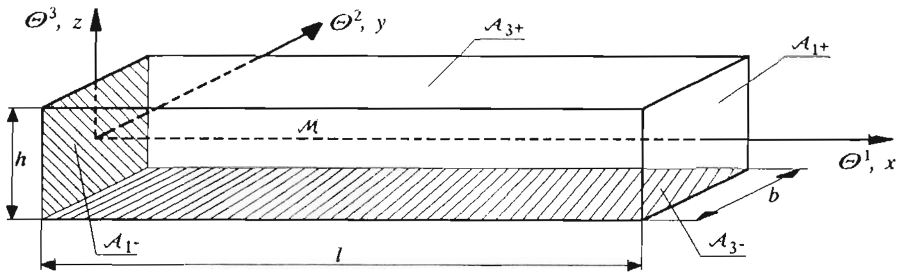

Fig. 1.

A simple geometry of the problem and the above choice of adapted coordinates imply that the corresponding field of local bases $\boldsymbol{g}_i$, $i = 1,2,3$ in the beam continuum is constant and coincides with the system of basic versors

$$\boldsymbol{g}_1 = [1,0,0] \qquad \boldsymbol{g}_2 = [0,1,0] \qquad \boldsymbol{g}_3 = [0,0,1] \tag{2.1}$$

Being orthonormal this system coincides with its own dual basis, $\boldsymbol{g}^i$, $i = 1,2,3$, because the metric components take on the standard Cartesian values $g_{ij} = \delta_{ij}$. Therefore the notation used by Schmidt and Weichert (1989) simplifies significantly; particularly, there is no need to introduce a separate symbol $\boldsymbol{a}_i$ for the midline values of $\boldsymbol{g}_i$.

Configurations of the beam will be described by a vector mapping $\boldsymbol{\Theta} \Rightarrow \boldsymbol{x}(\Theta)$, i.e., analytically, by the systems of functions $x^i(\Theta^1,\Theta^2,\Theta^3)$. The corresponding displacement vector is given by $\boldsymbol{V}(\Theta) = \boldsymbol{x}(\Theta) - \boldsymbol{\Theta}$, or $V^i = x^i(\Theta^j) - \Theta^i$.

We assume that the beam deforms only in the (Θ^1,Θ^3) coordinate plane. Thus, all the involved fields will not depend on Θ^2. Therefore, we may write

$$\boldsymbol{x}(\Theta^1,\Theta^2,\Theta^3) = \Big(\Theta^1 + u(\Theta^1,\Theta^3),\ \Theta^2,\ \Theta^3 + w(\Theta^1,\Theta^3)\Big) \tag{2.2}$$

and we see that

$$\boldsymbol{V}(\Theta^1,\Theta^2,\Theta^3) = \Big(u(\Theta^1,\Theta^3),\ 0,\ w(\Theta^1,\Theta^3)\Big) \tag{2.3}$$

Next simplifying assumption is that of flat and homogeneously deformable cross-sections; thus, displacements $u(\Theta^1,\Theta^3)$ and $w(\Theta^1,\Theta^3)$ have the following form

$$u(\Theta^1,\Theta^3) = \overset{0}{u}(\Theta^1) + \Theta^3\,\overset{1}{u}(\Theta^1)$$
$$w(\Theta^1,\Theta^3) = \overset{0}{w}(\Theta^1) + \Theta^3\,\overset{1}{w}(\Theta^1) \tag{2.4}$$

In the literature this condition is usually referred to as an assumption of flat cross-sections. We do not assume cross-sections to remain ortogonal to the middle axis during the deformation process, i.e., no restrictions of Love-Kirchhoff or Bernoulli (cf Schmidt and Weichert (1989)) type are imposed.

Our study is confined to the important in practice case of small deformations and moderate rotations.

We use the Cartesian orthonormal coordinates, thus we do not distinguish between contravariant and covariant objects. Let us first recall some essential kinematical relations.

The gradient of deformation is given by

$$X^i_j = \frac{\partial x^i}{\partial \Theta^j} = \delta^i_j + V^i{}_{,j} \tag{2.5}$$

The finite deformation tensor has the form

$$E_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i} + V_{k,i}V_{k,j}) \tag{2.6}$$

The infinitesimal deformation tensor η_{ij} and the rotation tensor Ω_{ij} are given by

$$\eta_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i}) \tag{2.7}$$

$$\Omega_{ij} = \frac{1}{2}(V_{i,j} - V_{j,i}) \tag{2.8}$$

Expressing E_{ij} by η_{ij} and Ω_{ij} one obtains

$$E_{ij} = \eta_{ij} + \frac{1}{2}\Omega_{ri}\Omega_{rj} + \frac{1}{2}(\eta_{ri}\Omega_{rj} + \eta_{rj}\Omega_{ri}) + \frac{1}{2}\eta_{ri}\eta_{rj} \tag{2.9}$$

Only for infinitesimal gradients $V_{i,j}$ the quantities η_{ij}, Ω_{ij} may be interpreted physically as rigorous measures of deformation and rotation respectively (cf Eringen (1962)). Nevertheless, they are always well – defined, and in a moderate range they provide a convenient approximation tool (cf Pietraszkiewicz (1980); Schmidt and Weichert (1989)), which consists in neglecting some, but not all, higher-order terms. We shall follow the approximation method used by Schmidt and Weichert (1989) for shells and adopt it to the theory of beams.

According to our simplifying assumptions, the beam deforms in the (Θ^1, Θ^3)-plane, thus

$$\Omega_{23} = 0 \qquad \Omega_{21} = 0 \tag{2.10}$$

and the only non-vanishing component of Ω is

$$\Omega := \Omega_{31} = -\Omega_{13} \tag{2.11}$$

We assume that the angle Ω (in radians) satisfies $|\Omega| < 1$ and $\Omega = O(\vartheta)$ is moderate, i.e., there is no assumption $|\Omega| \ll 1$. Instead, it is assumed that $\Omega^2 \ll 1$. Next, it is assumed that η_{ij} is small, i.e., of the order of $O(\vartheta^2)$. Then also the total tensor E_{ij} is of the order of $O(\vartheta^2)$.

Our approximate treatment of geometrically nonlinear beams consists in retaining in Eq (2.9) of all terms up to the order $O(\vartheta^3)$, and rejecting of all higher order terms. This approximation is physically justified and effective in practical problems.

By substituting Eqs (2.10) and (2.11) into (2.9) we obtain

$$\begin{aligned}
E_{11} &= \eta_{11} + \eta_{31}\Omega + \frac{1}{2}\Omega^2 + O(\vartheta^4)\\
E_{13} &= \eta_{13} + \frac{1}{2}(\eta_{33} - \eta_{11})\Omega + O(\vartheta^4) = E_{31}\\
E_{33} &= \eta_{33} - \eta_{31}\Omega + \frac{1}{2}\Omega^2 + O(\vartheta^4)
\end{aligned} \tag{2.12}$$

Obviously, the approximation assumptions have been taken into account (the accuracy order is $O(\vartheta^4)$).

Accepting kinematical assumption (2.4) we can express the quantities Ω and η_{ij} through the generalized displacement fields $\overset{0}{u}$, $\overset{1}{u}$, $\overset{0}{w}$, $\overset{1}{w}$

$$\Omega = \frac{1}{2}(\overset{0}{w}{}' - \overset{1}{u}) + \frac{1}{2}\,\overset{1}{w}{}'\Theta^3 \tag{2.13}$$

$$\begin{aligned}
\eta_{11} &= \overset{0}{u}{}' + \overset{1}{u}{}'\Theta^3\\
\eta_{13} &= \frac{1}{2}(\overset{1}{u} + \overset{0}{w}{}') + \frac{1}{2}\,\overset{1}{w}{}'\Theta^3\\
\eta_{33} &= \overset{1}{w}
\end{aligned} \tag{2.14}$$

where $(\cdot)' = \dfrac{d(\cdot)}{d\Theta^1}$.

We observe that no Bernoulli-type orthogonality assumption is used. Nevertheless, it may be interesting to see how such additional constraints could be taken into account within the framework of our approach. We have admitted configurations of the form

$$\boldsymbol{x}(\Theta^1,\Theta^2,\Theta^3) = \Big(\Theta^1 + \overset{0}{u}(\Theta^1) + \Theta^3\,\overset{1}{u}(\Theta^1), \Theta^2, \Theta^3 + \overset{0}{w}(\Theta^1) + \Theta^3\,\overset{1}{w}(\Theta^1)\Big) \tag{2.15}$$

The field of vectors tangent to the deformed midline is given by

$$\boldsymbol{t}_1(\Theta^1) = \frac{\partial \boldsymbol{x}}{\partial \Theta^1}(\Theta^1,0,0) = [1 + \overset{0}{u}{}'(\Theta^1),\ 0,\ \overset{0}{w}{}'(\Theta^1)] \tag{2.16}$$

The vectors tangent to the deformed Θ^3 and Θ^2-lines and attached at the deformed midline $\mathcal{M}$ are given by

$$\begin{aligned} \boldsymbol{t}_2(\Theta^1) &= \frac{\partial \boldsymbol{x}}{\partial \Theta^2}(\Theta^1,0,0) = [0,1,0] \\ \boldsymbol{t}_3(\Theta^1) &= \frac{\partial \boldsymbol{x}}{\partial \Theta^3}(\Theta^1,0,0) = [\overset{1}{u}(\Theta^1),\ 0,\ 1 + \overset{1}{w}(\Theta^1)] \end{aligned} \tag{2.17}$$

The Bernoulli constraints mean that $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ are orthogonal to $\boldsymbol{t}_1$ for any value of admissible Θ^1, thus

$$\boldsymbol{t}_1 \cdot \boldsymbol{t}_2 = 0 \qquad\qquad \boldsymbol{t}_1 \cdot \boldsymbol{t}_3 = 0 \tag{2.18}$$

It is clear that the first condition is fulfilled automatically, without imposing any constraints on the variables $\overset{0}{u}, \overset{1}{u}, \overset{0}{w}, \overset{1}{w}$. The second condition in Eq (2.18) gives

$$\overset{1}{u} = -\frac{\overset{0}{w}{}'(1 + \overset{1}{w})}{1 + \overset{0}{u}{}'} \tag{2.19}$$

Constraint (2.19) is of a differential form, and $\overset{1}{u}$ is expressed by means of derivatives of $\overset{0}{u}$ and $\overset{0}{w}$. This is the reason for which the Bernoulli beam is described by a higher-order differential equation.

However, we do not impose the Bernoulli constraints and the beam will be described by a system of four second-order ordinary differential equations.

3. Variational formulation of a theory of geometrically nonlinear beams. Elastic problem

Let us formulate general ideas of reducing of the problem of three-dimensional continuum to the effective one-dimensional system. We will follow the two-dimensional reduction carried out by Schmidt and Weichert (1989) for shells.

The theory of elastic-plastic problems is based on the rate variational formulation due to Neal (1972). The last formulation is relatively complicated, thus, it seems instructive to begin with a simpler and less complicated elastic problem.

The principle of virtual work will consequently be used to derive the governing equations. Let us mention, incidentally, that the incremental principle due to Neal was developed just on the basis of principle of virtual work. Let us remind the formulation of this principle. First, we recall the formulation of this principle for the three-dimensional problem.

We assume that in the reference configuration the body occupies a domain $\mathcal{V}$ with the boundary $\mathcal{A}$. On a part $\mathcal{A}_s \subset \mathcal{A}$ of this boundary external surface forces ${}^*t^i$ are prescribed, whereas on the remaining part of $\mathcal{A}$ i.e. on $\mathcal{A}_v$ a displacement field is prescribed. Let $\boldsymbol{F} = (F^i)$ denote the density of body forces (per unit mass), and ρ – the mass density in the reference configuration. Internal elastic forces are described by the second Piola-Kirchhoff stress tensor $\mathbf{S}$. The constitutive law $S_{ij} = \widehat{S}_{ij}(\mathbf{E})$ is known and will be used in the variational principle. There is a wide range of practically important geometrically nonlinear problems for which the linear constitutive relation

$$S_{ij} = C_{ijkl}E_{kl} \tag{3.1}$$

may be used. Then, in the special case of isotropic bodies we have

$$S_{ij} = \alpha E_{ij} + \beta E^k{}_k \delta_{ij} \tag{3.2}$$

The principle of virtual work has the form

$$\int\limits_{\mathcal{V}} \Big(S_{ij}\delta E_{ij}(V) - \rho F^i \delta V_i\Big)\, d\mathcal{V} - \int\limits_{\mathcal{A}_s} {}^*t^i \delta V_i d\mathcal{A} = 0 \tag{3.3}$$

where $\boldsymbol{V} = (V_i)$ is any kinematically admissible displacement field and $\boldsymbol{V} = \boldsymbol{V}_0$ on $\mathcal{A}_s$ is prescribed. When deriving equilibrium equations we must express the variations δE_{ij} by δV_i; one can easily show that

$$\delta E_{ij} = L_{ij}^{ab}(\delta V_a)_{,b} \tag{3.4}$$

where

$$L_{ij}^{ab} = \frac{1}{2}(\delta_i^a \delta_j^b + \delta_j^a \delta_i^b + \delta_i^b V^a{}_{,j} + \delta_j^b V^a{}_{,i}) \tag{3.5}$$

The variational procedure with boundary conditions imposed on kinematical fields

$$\delta V_i\Big|_{\mathcal{A}_v} = 0 \tag{3.6}$$

gives the equilibrium equations, whereas on the surface $\mathcal{A}_s$ we obtain the conditions of equilibrium balance between internal elastic and external prescribed forces.

One of the known advantages of using variational principles is, due to modern numerical techniques, that they enable one to find approximate solutions. Besides, passing on from the three-dimensional model (3D) to a one-dimensional (1D) beam problem, defined along the midline, is almost automatic.

Let us carry out the dimensional reduction by using Eq (3.3). To this end we must perform integration with respect to the variables Θ^2, Θ^3. Let us consider the separate terms appearing in Eq (3.3). The first term is

$$I_1 = \int_{\mathcal{V}} S^{ij}\delta E_{ij}\, d\mathcal{V} = \int_0^l d\Theta^1 \int_{-b/2}^{b/2} d\Theta^2 \int_{-h/2}^{h/2} S^{ij}\delta E_{ij}\, d\Theta^3 \tag{3.7}$$

Eqs (2.13), (2.14) enable one to write the expansion (see (2.12))

$$\delta E_{ij}(V) = \sum_n (\Theta^3)^n \delta \overset{n}{E}_{ij} \qquad\qquad n = 0,1 \tag{3.8}$$

We can express $\delta\overset{n}{E}_{ij}$ through variations $\delta\overset{0}{u}$, $\delta\overset{1}{u}$, $\delta\overset{0}{w}$, $\delta\overset{1}{w}$. The quantities E_{ij} and S_{ij} are independent of Θ^2 while $\overset{n}{E}_{ij}$ and $\delta\overset{n}{E}_{ij}$ are independent of Θ^3. Thus, performing integration in Eq (3.7) over $d\Theta^2$, $d\Theta^3$ we obtain

$$I_1 = \int_0^l d\Theta^1 \sum_n \overset{n}{\mathcal{L}}{}^{ij} \delta \overset{n}{E}_{ij} \qquad\qquad n = 0,1 \tag{3.9}$$

where

$$\overset{n}{\mathcal{L}}{}^{ij} = \int_{-b/2}^{b/2} d\Theta^2 \int_{-h/2}^{h/2} (\Theta^3)^n S^{ij}\, d\Theta^3 \tag{3.10}$$

is the nth moment of S^{ij} with respect to the variable Θ^3.

The next term of Eq (3.3) is

$$I_2 = -\int_{\mathcal{V}} \rho F^i \delta V_i\, d\mathcal{V} \tag{3.11}$$

where ρF^i – components of the vector of body force.

Assuming $F^1 = 0$, $F^2 = 0$, $F^3 = -g$ (gravitational origin of body forces), we obtain

$$I_2 = \int_0^l d\Theta^1 \sum_n \overset{n}{\mathcal{B}}\, \delta \overset{n}{V}_3(\Theta^1) \tag{3.12}$$

provided the mass density ρ is constant within the beam. The coefficients $\overset{n}{\mathcal{B}}$ in the above formula are given by

$$\overset{n}{\mathcal{B}} = \frac{\rho g b}{n+1}\left(\frac{h}{2}\right)^{n+1}\left(1+(-1)^n\right) \tag{3.13}$$

thus $\overset{n}{\mathcal{B}} = 0$ for $n = 1$ and $\overset{n}{\mathcal{B}} = \rho g b h$ for $n = 0$, and $\overset{n}{V}_3$ are expansion coefficients of V_3 at $(\Theta^3)^n$, thus, as it is seen from Eq (2.3)

$$\overset{0}{V}_3 = \overset{0}{w} \qquad \overset{1}{V}_3 = \overset{1}{w} \qquad \overset{n}{V}_3 = 0 \qquad \text{for } n > 1 \tag{3.14}$$

The third term of Eq (3.3) is

$$I_3 = -\int_{\mathcal{A}_s} {}^*t^k \delta V_k \, d\mathcal{A} \tag{3.15}$$

This quantity must be expressed as a sum of terms corresponding to the boundary planes, orthogonal to respective axes. For the faces perpendicular to the coordinate line Θ^3 we have

$$I_{33} = -b \int_0^l \sum_n \overset{n}{\mathcal{P}}{}^k \, \delta \overset{n}{V}_k(\Theta^1)\, d\Theta^1 \tag{3.16}$$

where

$$\overset{n}{\mathcal{P}}{}^k = \left[{}^*t^{3k}(\Theta^3)^n\right]_{-h/2}^{+h/2} \tag{3.17}$$

$$\begin{aligned} \overset{0}{V}_1 &= \overset{0}{u} & \overset{1}{V}_1 &= \overset{1}{u} \\ \overset{0}{V}_2 &= 0 & \overset{1}{V}_2 &= 0 \\ \overset{0}{V}_3 &= \overset{0}{w} & \overset{1}{V}_3 &= \overset{1}{w} \end{aligned} \tag{3.18}$$

The difference structure of (3.17) is due to the fact that the external normal versors of $\mathcal{A}_{3+}$ and $\mathcal{A}_{3-}$ are negative to each other.

The last contribution to I_3, denoted by I_{31}, is obtained of an integration over the surfaces $\mathcal{A}_{1-}$, $\mathcal{A}_{1+}$. This term occurs when the external loads are applied at the ends of beam. One can show that

$$I_{31} = \sum_n {}^*\!\underset{(0)}{\overset{n}{\mathcal{L}}}{}^k \, \delta \overset{n}{V}_k(0) - \sum_n {}^*\!\underset{(l)}{\overset{n}{\mathcal{L}}}{}^k \, \delta \overset{n}{V}_k(l) \tag{3.19}$$

where

$$\begin{aligned} {}^*\!\underset{(0)}{\overset{n}{\mathcal{L}}}{}^k &= b \int\limits_{-h/2}^{h/2} {}^*t^{1k}\Big|_{\Theta^1=0} (\Theta^3)^n \, d\Theta^3 \\ {}^*\!\underset{(l)}{\overset{n}{\mathcal{L}}}{}^k &= b \int\limits_{-h/2}^{h/2} {}^*t^{1k}\Big|_{\Theta^1=l} (\Theta^3)^n \, d\Theta^3 \end{aligned} \tag{3.20}$$

and, obviously, $\overset{n}{V}_k$ are given by Eqs (3.18).

When the external loads ${}^*t^{1k}$ are absent, ${}^*\underset{(0)}{V}{}^k(\Theta^3)$ or ${}^*\underset{(l)}{V}{}^k(\Theta^3)$ are prescribed on $\mathcal{A}_{1-}$, $\mathcal{A}_{1+}$, then the corresponding term drops out of Eq (3.19), and we put

$$\begin{aligned} V^k(0,\Theta^2,\Theta^3) &= {}^*\underset{(0)}{V}{}^k(\Theta^3) \\ V^k(l,\Theta^2,\Theta^3) &= {}^*\underset{(l)}{V}{}^k(\Theta^3) \end{aligned} \tag{3.21}$$

Now, we can come back to the variational principle (3.3)

$$I = I_1 + I_2 + I_{33} + I_{31} = 0 \tag{3.22}$$

where the separate contributions to I are given by Eqs (3.9), (3.12), (3.16), and (3.19) ÷ (3.21). Thus we have obtained a one-dimensional variational principle with the integration variable Θ^1 and kinematical variables $\overset{0}{u}$, $\overset{1}{u}$, $\overset{0}{w}$, $\overset{1}{w}$. This principle leads to a system of four ordinary second-order differential equations; if at $\Theta^1 = 0$ and/or $\Theta^1 = l$ external loads are prescribed, it contains also static boundary conditions at the beam ends.

4. Equilibrium equations of geometrically nonlinear elastic beams; plane deformation

Let us come back to Eq (3.22) which is our virtual work principle for geometrically nonlinear elastic beams. We remaind here that S^{ij} is the second Piola-Kirchhoff stress tensor, and E_{ij} is the Green strain tensor. We are dealing with beams at small strains but moderately large rotation of the cross-section points (see Eq (2.13)). The quantities $\overset{0}{u}$, $\overset{0}{w}$ denote an elongation and a deflection of the midline points of the beam.

The coordinate plane (Θ^1, Θ^3) is a deformation plane of the beam (see Fig.1).

Let us compare the kinematical hypotheses used in the present and Bernoulli's beam theories, respectively.

Present theory	Bernoulli's theory
$u = \overset{0}{u} + \Theta^3 \overset{1}{u}$	$u = \overset{0}{u} - \Theta^3 (\overset{0}{w})'$
$w = \overset{0}{w} + \Theta^3 \overset{1}{w}$	$w = \overset{0}{w}$
$v = 0$	$v = 0$

(4.1)

where $(\cdot)' = \frac{d(\cdot)}{d\Theta^1}$.

Assumptions about the order of magnitude of certain quantities (see Section 2) yield

$$\overset{1}{w} = O(\vartheta^2) \qquad (\overset{0}{w})' - \overset{1}{u} := \overset{1}{\varphi} = O(\vartheta)$$
$$(\overset{0}{u})' = O(\vartheta^2) \qquad (\overset{0}{w})' + \overset{1}{u} := \overset{2}{\varphi} = O(\vartheta^2) \tag{4.2}$$
$$(\overset{1}{w})'\Theta^3 \leq O(\vartheta^2)$$

These estimations will be applied below. Let us note that the following two substitutions

$$\overset{1}{\varphi} := (\overset{0}{w})' - \overset{1}{u} \qquad \overset{2}{\varphi} := (\overset{0}{w})' + \overset{1}{u} \tag{4.3}$$

have been introduced above.

Now, let us use an expansion of E_{ij} in the form (cf Schmidt and Reddy (1988))

$$
\begin{aligned}
E_{11}(\Theta^1,\Theta^3) &= \overset{0}{E}_{11}(\Theta^1) + \Theta^3\,\overset{1}{E}_{11}(\Theta^1) + O(\vartheta^4)\\
E_{13}(\Theta^1,\Theta^3) &= \overset{0}{E}_{13}(\Theta^1) + \Theta^3\,\overset{1}{E}_{13}(\Theta^1) + O(\vartheta^4)\\
E_{33}(\Theta^1,\Theta^3) &= \overset{0}{E}_{33}(\Theta^1) + O(\vartheta^4)
\end{aligned}
\tag{4.4}
$$

where, because of the order of magnitude, $\overset{n}{E}_{ij}= 0$ for $n \geq 2$, $i,j = 1,3$.

From Eqs (2.12), (2.13) taking into account Eqs (4.2) we have

$$
\begin{aligned}
\overset{0}{E}_{11} &= (\overset{0}{u})' + \frac{1}{4}\,\overset{1}{\varphi}\overset{2}{\varphi} + \frac{1}{8}(\overset{1}{\varphi})^2\\
\overset{1}{E}_{11} &= (\overset{1}{u})' + \frac{1}{2}\,\overset{1}{\varphi}\,(\overset{1}{w})'\\
\overset{0}{E}_{13} &= \frac{1}{2}\,\overset{2}{\varphi} - \frac{1}{4}\,\overset{1}{\varphi}\left((\overset{0}{u})' - \overset{1}{w}\right)\\
\overset{1}{E}_{13} &= \frac{1}{2}(\overset{1}{w})' - \frac{1}{4}\,\overset{1}{\varphi}\,(\overset{1}{u})'\\
\overset{0}{E}_{33} &= \overset{1}{w} - \frac{1}{4}\,\overset{1}{\varphi}\overset{2}{\varphi} + \frac{1}{8}(\overset{1}{\varphi})^2
\end{aligned}
\tag{4.5}
$$

It will be convenient to distinguish the following three integrals in the principle of virtual work

$$
\underbrace{\int_{\mathcal{V}} S^{ij}\delta E_{ij}(V)\,d\mathcal{V}}_{I_1} - \underbrace{\int_{\mathcal{V}} \rho F^i\delta V_i\,d\mathcal{V}}_{I_2} - \underbrace{\int_{\mathcal{A}_s} {}^*t^i\delta V_i\,d\mathcal{A}}_{I_3} = 0
\tag{4.6}
$$

The above variational principle becomes effective when δE_{ij}, $i,j = 1,3$ are expressed through δV_i and the constitutive law is assumed. The variational procedure with boundary conditions imposed on independent variables: $\delta V_i\Big|_{\mathcal{A}_v} = 0$, yields the equilibrium equations of the beam.

Now, we have to determine the variations $\delta\overset{0}{E}_{ij}$, $\delta\overset{1}{E}_{ij}$ from the integral I_1.

In order to simplify the expressions which will occur next we involve the substitutions

$$
\overset{1}{\psi} = 1 + \frac{1}{2}\,\overset{1}{w} - \frac{1}{2}(\overset{0}{u})' \qquad\qquad \overset{2}{\psi} = 1 - \frac{1}{2}\,\overset{1}{w} + \frac{1}{2}(\overset{0}{u})'
\tag{4.7}
$$

Calculating the variations of (4.5) we get

$$
\begin{aligned}
\delta\overset{0}{E}_{11} &= \delta\overset{0}{u}{}' + \frac{1}{4}(3\,\overset{0}{w}{}' - \overset{1}{u})\delta\overset{0}{w}{}' - \frac{1}{4}\,\overset{2}{\varphi}\,\delta\overset{1}{u}\\
\delta\overset{1}{E}_{11} &= \frac{1}{2}\,\overset{1}{w}{}'\delta\overset{0}{w}{}' - \frac{1}{2}\,\overset{1}{w}{}'\delta\overset{1}{u} + \delta\overset{1}{u}{}' + \frac{1}{2}\,\overset{1}{\varphi}\,\delta\overset{1}{w}{}'\\
\delta\overset{0}{E}_{13} &= -\frac{1}{4}\,\overset{1}{\varphi}\,\delta\overset{0}{u}{}' + \frac{1}{2}\,\overset{1}{\psi}\,\delta\overset{0}{w}{}' + \frac{1}{2}\,\overset{2}{\psi}\,\delta\overset{1}{u} + \frac{1}{4}\,\overset{1}{\varphi}\,\delta\overset{1}{w}\\
\delta\overset{1}{E}_{13} &= -\frac{1}{4}\,\overset{1}{u}{}'\delta\overset{0}{w}{}' - \frac{1}{4}\,\overset{1}{u}{}'\delta\overset{1}{u} - \frac{1}{4}\,\overset{1}{\varphi}\,\delta\overset{1}{u}{}' + \frac{1}{2}\delta\overset{1}{w}{}'\\
\delta\overset{0}{E}_{33} &= -\frac{1}{4}\,\overset{2}{\varphi}\,\delta\overset{0}{w}{}' + \frac{1}{4}(3\,\overset{1}{u} - \overset{0}{w}{}')\delta\overset{1}{u} + \delta\overset{1}{w}
\end{aligned}
\tag{4.8}
$$

Let us change now the notation of coordinate axes

$$
\Theta^1 = x \qquad\qquad \Theta^3 = z \tag{4.9}
$$

and introduce the stress resultants as follows

$$
{}^n\mathcal{L}^{ij}(x) = b\int\limits_{-h/2}^{h/2} z^n S^{ij}(x,z)\,dz \qquad\qquad n = 0,1 \tag{4.10}
$$

Then

$$
\begin{aligned}
I_1 &= \int\limits_0^l \overset{0}{\mathcal{L}}{}^{ij}\delta\overset{0}{E}_{ij}\;dx + \int\limits_0^l \overset{1}{\mathcal{L}}{}^{ij}\delta\overset{1}{E}_{ij}\;dx =\\
&= \int\limits_0^l \overset{0}{\mathcal{L}}{}^{11}\delta\overset{0}{E}_{11}\;dx + \int\limits_0^l \overset{0}{\mathcal{L}}{}^{33}\delta\overset{0}{E}_{33}\;dx + 2\int\limits_0^l \overset{0}{\mathcal{L}}{}^{13}\delta\overset{0}{E}_{13}\;dx +\\
&+ \int\limits_0^l \overset{1}{\mathcal{L}}{}^{11}\delta\overset{1}{E}_{11}\;dx + 2\int\limits_0^l \overset{1}{\mathcal{L}}{}^{13}\delta\overset{1}{E}_{13}\;dx
\end{aligned}
\tag{4.11}
$$

Taking into account Eqs (4.8) and (4.10) we obtain

$$
\begin{aligned}
I_1 &= \int\limits_0^l \Big(\overset{0}{S}_u\,\delta\overset{0}{u}{}' + \overset{1}{S}_u\,\delta\overset{1}{u}{}' + \overset{0}{S}_w\,\delta\overset{0}{w}{}' + \overset{1}{S}_w\,\delta\overset{1}{w}{}'\Big)\,dx +\\
&+ \int\limits_0^l \Big(\overset{1}{R}_u\,\delta\overset{1}{u} + \overset{1}{R}_w\,\delta\overset{1}{w}\Big)\,dx
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
\overset{0}{S}_u &= \overset{0}{\mathcal{L}}{}^{11} - \frac{1}{2}\overset{0}{\mathcal{L}}{}^{13}(\overset{0}{w}{}' - \overset{1}{u}) \\
\overset{1}{S}_u &= \overset{1}{\mathcal{L}}{}^{11} - \frac{1}{2}\overset{1}{\mathcal{L}}{}^{13}(\overset{0}{w}{}' - \overset{1}{u}) \\
\overset{0}{S}_w &= \frac{1}{4}\Big[\overset{0}{\mathcal{L}}{}^{11}(3\,\overset{0}{w}{}' - \overset{1}{u}) - \overset{0}{\mathcal{L}}{}^{33}(\overset{0}{w}{}' + \overset{1}{u})\Big] + \overset{0}{\mathcal{L}}{}^{13}\Big(1 + \frac{1}{2}\,\overset{1}{w} - \frac{1}{2}\,\overset{0}{u}{}'\Big) + \\
&+ \frac{1}{2}(\overset{1}{\mathcal{L}}{}^{11}\,\overset{1}{w}{}' - \overset{1}{\mathcal{L}}{}^{13}\,\overset{1}{u}{}') \\
\overset{1}{S}_w &= \overset{1}{\mathcal{L}}{}^{13} + \frac{1}{2}\overset{1}{\mathcal{L}}{}^{11}(\overset{0}{w}{}' - \overset{1}{u}) \qquad (4.13)\\
\overset{1}{R}_u &= \frac{1}{4}\Big[\overset{0}{\mathcal{L}}{}^{33}(3\,\overset{1}{u} - \overset{0}{w}{}') - \overset{0}{\mathcal{L}}{}^{11}(\overset{0}{w}{}' + \overset{1}{u})\Big] + \overset{0}{\mathcal{L}}{}^{13}\Big(1 - \frac{1}{2}\,\overset{1}{w} + \frac{1}{2}\,\overset{0}{u}{}'\Big) + \\
&- \frac{1}{2}(\overset{1}{\mathcal{L}}{}^{11}\,\overset{1}{w}{}' - \overset{1}{\mathcal{L}}{}^{13}\,\overset{1}{u}{}') \\
\overset{1}{R}_w &= \overset{0}{\mathcal{L}}{}^{33} + \frac{1}{2}\overset{0}{\mathcal{L}}{}^{13}(\overset{0}{w}{}' - \overset{1}{u})
\end{aligned}
$$

Next, integrating I_1 by parts we get

$$
\begin{aligned}
I_1 = -\int_0^l \Big[(\overset{0}{S}_u)'\delta\overset{0}{u} + (\overset{1}{S}_u)'\delta\overset{1}{u} + (\overset{0}{S}_w)'\delta\overset{0}{w} + (\overset{1}{S}_w)'\delta\overset{1}{w}\Big]\,dx \qquad (4.14)\\
+ \int_0^l (\overset{1}{R}_u\,\delta\overset{1}{u} + \overset{1}{R}_w\,\delta\overset{1}{w})\,dx + (\overset{0}{S}_u\,\delta\overset{0}{u} + \overset{1}{S}_u\,\delta\overset{1}{u} + \overset{0}{S}_w\,\delta\overset{0}{w} + \overset{1}{S}_w\,\delta\overset{1}{w})\Big|_0^l
\end{aligned}
$$

From Eqs (3.12) and (3.13) yields that

$$
I_2 := B = \rho g h b \qquad (4.15)
$$

Let us consider the third integral of Eq (4.6)

$$
I_3 = I_{31} + I_{33} \qquad (4.16)
$$

First, we assign a virtual work of the resultant forces applied to the left and right ends of the beam, i.e.

$$
\begin{aligned}
I_{31} =& \overset{0}{F}_u(0)\delta\overset{0}{u}(0) + \overset{0}{F}_w(0)\delta\overset{0}{w}(0) + \overset{1}{F}_u(0)\delta\overset{1}{u}(0) + \overset{1}{F}_w(0)\delta\overset{0}{w}(0) + \qquad (4.17)\\
&+ \overset{0}{F}_u(l)\delta\overset{0}{u}(l) + \overset{0}{F}_w(l)\delta\overset{0}{w}(l) + \overset{1}{F}_u(l)\delta\overset{1}{u}(l) + \overset{1}{F}_w(l)\delta\overset{0}{w}(l)
\end{aligned}
$$

where, for simplicity, new notation has been used

$$\overset{n}{F}_\alpha(0) = {}^*\overset{n}{\underset{(0)}{\mathcal{L}}}{}^k \qquad \overset{n}{F}_\alpha(l) = {}^*\overset{n}{\underset{(l)}{\mathcal{L}}}{}^k \qquad \begin{aligned} \alpha &= u, w \\ n &= 0, 1 \\ k &= 1, 3 \end{aligned} \tag{4.18}$$

$${}^*\overset{n}{\underset{(0)}{\mathcal{L}}}{}^k = b \int\limits_{-h/2}^{h/2} {}^*t^{1k}\Big|_{x=0} z^n \, dz \qquad {}^*\overset{n}{\underset{(l)}{\mathcal{L}}}{}^k = b \int\limits_{-h/2}^{h/2} {}^*t^{1k}\Big|_{x=l} z^n \, dz \tag{4.19}$$

Next, we determine I_{33} (i.e. the virtual work of external loads applied to the upper and bottom planes, of the beam, respectively) as

$$I_{33} = -\int\limits_0^l ({}^*\overset{0}{P}_u \, \delta\overset{0}{u} + {}^*\overset{0}{P}_w \, \delta\overset{0}{w} + {}^*\overset{1}{P}_u \, \delta\overset{1}{u} + {}^*\overset{1}{P}_w \, \delta\overset{1}{w}) \, dx \tag{4.20}$$

where

$${}^*\overset{n}{P}_\alpha := {}^*\overset{n}{P}{}^k = b\Big[z^{n*}t^{3k}\Big]_{-h/2}^{h/2} \qquad k = 1, 2, 3 \tag{4.21}$$

Since the variations $\delta\overset{0}{u}$, $\delta\overset{1}{u}$, $\delta\overset{0}{w}$, $\delta\overset{1}{w}$ are unrestricted, we obtain

$$\begin{aligned} \delta\overset{0}{u} &: \quad (\overset{0}{S}_u)' + {}^*\overset{0}{P}_u = 0 \\ \delta\overset{1}{u} &: \quad (\overset{1}{S}_u)' + {}^*\overset{1}{P}_u - \overset{1}{R}_u = 0 \\ \delta\overset{0}{w} &: \quad (\overset{0}{S}_w)' + {}^*\overset{0}{P}_w - B = 0 \\ \delta\overset{1}{w} &: \quad (\overset{1}{S}_w)' + {}^*\overset{1}{P}_w - \overset{1}{R}_w = 0 \end{aligned} \qquad \text{(E.Es)} \tag{4.22}$$

The above system of the equations will be called later on as (E.Es). The quantities ${}^*\overset{n}{P}_\alpha$, $\alpha = u, w$, $n = 0, 1$, are components of the external loads applied to $\mathcal{A}_{3+}$ and $\mathcal{A}_{3-}$.

So, we have got the equilibrium equations for elastic beams in the geometrically nonlinear range of deformation. The associated static boundary conditions have been obtained simultaneously. E.g. for the clamped left end of the beam from Eqs (4.14) and (4.17) yields

$$\begin{aligned} \overset{0}{S}_u(l) - \overset{0}{F}_u(l) &= 0 \qquad & \overset{1}{S}_u(l) - \overset{1}{F}_u(l) &= 0 \\ \overset{0}{S}_w(l) - \overset{0}{F}_w(l) &= 0 \qquad & \overset{1}{S}_w(l) - \overset{1}{F}_w(l) &= 0 \end{aligned} \tag{4.23}$$

These relations together with the kinematic boundary conditions for $x = 0$

$$\overset{n}{u}(0,z) = 0 \qquad \overset{n}{w}(0,z) = 0 \qquad \overset{n}{w}'(0,z) = 0 \tag{4.24}$$

so for $x = 0$ we have

$$\delta\overset{n}{u}(0,z) = 0 \qquad \delta\overset{n}{w}(0,z) = 0 \qquad \delta\overset{n}{w}'(0,z) = 0 \qquad n = 0,1$$

determine completely the boundary quantities for the equilibrium problem represented by (4.22). Naturally, suitable constitutive equations have to be assumed.

Similarly, the other associated static boundary conditions can be derived and the appropriate question can be formulated.

5. Variational formulation of a theory of geometrically nonlinear beams. Elastic-plastic problem

In the case of elastic-plastic material of the beam our considerations are based on the incremental variational principle due to Neal (1972), cf also Telega (1976), $\delta I = 0$, for

$$\begin{aligned} I(\dot{V}) &= \int_{\mathcal{V}} \Big[\dot{S}^{ij}(\dot{V})\dot{E}_{ij}(\dot{V}) + \frac{1}{2}S^{ij}(\dot{V})\ddot{E}_{ij}(\dot{V}) - \rho\dot{F}^{i}\dot{V}_i - W\Big]\, d\mathcal{V} + \\ &- \int_{\mathcal{A}_s} {}^{*}\dot{t}^{k}\dot{V}_k\, d\mathcal{A} \end{aligned} \tag{5.1}$$

where $W = W(\dot{\mathsf{S}})$ is a potential function depending on the state of stress rates (cf Neal (1972)).

The rates $\dot{V}_i$ are independent kinematical variables subjected to the variational procedure only. The second rates $\ddot{E}_{ij}(\dot{V})$ are calculated under an additional assumption of the quasi-static process: $\ddot{V}_i = 0$ thus $\ddot{E}_{ij}(\dot{V}) = \dot{V}^{k}{}_{,i}\dot{V}_{k,j}$.

Let us pass to the dimensional reduction of the functional (5.1).

The first term: $I_{1a} = \int\limits_{\mathcal{V}} \dot{S}^{ij}(\dot{V})\dot{E}_{ij}(\dot{V})\, d\mathcal{V}$ becomes

$$I_{1a} = \int_0^l \sum_n \overset{n}{\dot{E}}_{ij}\overset{n}{\dot{\mathcal{L}}}{}^{ij}\, d\Theta^1 \tag{5.2}$$

where

$$\overset{n}{\dot{\mathcal{L}}}{}^{ij} = b \int_{-h/2}^{h/2} (\Theta^3)^n \dot{S}^{ij} \, d\Theta^3 \tag{5.3}$$

The next term: $I_{1b} = \frac{1}{2} \int_{\mathcal{V}} S^{ij} \ddot{E}_{ij} \, d\mathcal{V}$ may be expressed as

$$I_{1b} = \int_0^l \frac{1}{2} \sum_n \overset{n}{\ddot{E}}_{ij} \overset{n}{\mathcal{L}}{}^{ij} \, d\Theta^1 \tag{5.4}$$

where

$$\overset{n}{\mathcal{L}}{}^{ij} = b \int_{-h/2}^{h/2} (\Theta^3)^n S^{ij} \, d\Theta^3 \qquad n = 0, 1 \qquad i, j = 1, 3 \tag{5.5}$$

The body-forces term is given by

$$I_2 = -\int_{\mathcal{V}} \rho \dot{F}^i \dot{V}_i \, d\mathcal{V} \tag{5.6}$$

It is natural to assume here that $\dot{F}^i = 0$ for $i = 1, 2, 3$ (whereas in the elastic case it was: $F^1 = 0$, $F^2 = 0$, $F^3 = -F$). So, we have $I_2 = 0$.

Let us consider the surface integral

$$I_3 = -\int_{\mathcal{A}_s} {}^{*}\dot{t}^k \dot{V}_k \, d\mathcal{A} \tag{5.7}$$

which, in fact, consists of two terms (for one pair of rectangular faces and for the ends of the beam). Using the same notation as in the elastic case, we have

$$I_{33} = -\int_0^l \sum_n \overset{n}{\dot{\mathcal{P}}}{}^k \overset{n}{\dot{V}}_k(\Theta^1) \, d\Theta^1 \tag{5.8}$$

where $\overset{n}{\mathcal{P}}{}^k$ is given by Eq (3.17). Similarly,

$$I_{31} = b \sum_n {}^{*}\overset{n}{\underset{(0)}{\dot{\mathcal{L}}}}{}^k \overset{n}{\dot{V}}_k(0) - b \sum_n {}^{*}\overset{n}{\underset{(l)}{\dot{\mathcal{L}}}}{}^k \overset{n}{\dot{V}}_k(l) \tag{5.9}$$

where ${}^{*}\overset{n}{\underset{(0)}{\mathcal{L}}}{}^k$ and ${}^{*}\overset{n}{\underset{(l)}{\mathcal{L}}}{}^k$ are defined in Eq (3.20).

So, in general, Neal's variational principle (cf Neal (1972)), for quasi-static processes of elastic-plastic beams, may be written down as

$$\delta(I_{1a} + I_{1b} + I_{33} + I_{31}) = 0 \tag{5.10}$$

It becomes effective, when $\dot{\boldsymbol{V}}$ is expressed by the rates $\overset{0}{\dot u}$, $\overset{1}{\dot u}$, $\overset{0}{\dot w}$, $\overset{1}{\dot w}$ with the help of Eqs (2.2) and (2.3). Obviously, the appropriate rate constitutive relations for the elastic-plastic material should be used.

6. Rate-equilibrium equations of geometrically nonlinear elastic-plastic beams; plane deformation

In this section we will present a treatment of geometrically nonlinear elastic-plastic beams that is based on the 1D version, see (5.10), of Neal's formulation. We restrict ourselves to the quasi-static processes (i.e. we consider the case: $\ddot V_i = 0$, $i = 1, 3$). The rate principle: $\delta I = 0$, for the functional $I(\dot{\boldsymbol{V}})$, defined in Eq (5.1), will yield the *rate equilibrium equations*, (R.E.Es), and static boundary conditions of the considered beams at moderate rotations. The notations introduced in the previous sections are obligatory here. The strain-displacement relations (4.5) yield the following first-order rates of the Green strain tensor components

$$\begin{aligned}
\overset{0}{\dot E}_{11} &= \overset{0}{\dot u}{}' + \frac{3}{4}\,\overset{0}{w}{}'\,\overset{0}{\dot w}{}' - \frac{1}{4}\,\overset{1}{u}\overset{1}{\dot u} - \frac{1}{4}\big(\overset{1}{u}\overset{0}{\dot w}{}' + \overset{0}{w}{}'\,\overset{1}{\dot u}\big)\\
\overset{1}{\dot E}_{11} &= \overset{1}{\dot u}{}' + \frac{1}{2}\,\overset{1}{\dot w}{}'\,\overset{1}{\varphi} + \frac{1}{2}\,\overset{1}{w}{}'\,\overset{1}{\dot\varphi}\\
\overset{0}{\dot E}_{13} &= \frac{1}{2}\,\overset{2}{\dot\varphi} + \frac{1}{4}\,\overset{1}{w}\overset{1}{\dot\varphi} + \frac{1}{4}\,\overset{1}{\dot w}\overset{1}{\varphi} - \frac{1}{4}\,\overset{0}{u}{}'\,\overset{1}{\dot\varphi} - \frac{1}{4}\,\overset{0}{\dot u}{}'\,\overset{1}{\varphi}\\
\overset{1}{\dot E}_{13} &= \overset{1}{\dot w}{}' - \frac{1}{4}\,\overset{1}{u}{}'\,\overset{1}{\dot\varphi} - \frac{1}{4}\,\overset{1}{\dot u}{}'\,\overset{1}{\varphi}\\
\overset{0}{\dot E}_{33} &= \overset{1}{\dot w} - \frac{1}{4}\,\overset{0}{\dot w}{}'\,\overset{2}{\varphi} + \frac{1}{4}\,\overset{1}{\dot u}\,\big(3\,\overset{1}{u} - \overset{0}{w}{}'\big)
\end{aligned} \tag{6.1}$$

So, we get the variations of $\overset{n}{\dot E}_{ij}$, $n = 0, 1$, $i, j = 1, 3$, as follows

$$
\begin{aligned}
\delta\overset{0}{\dot{E}}_{11} &= \delta\overset{0}{\dot{u}}{}' - \frac{1}{4}\overset{2}{\dot{\varphi}}\,\delta\overset{1}{\dot{u}} + \frac{1}{4}(3\,\overset{0}{\dot{w}}{}' - \overset{1}{\dot{u}})\delta\overset{0}{\dot{w}}{}' \\
\delta\overset{1}{\dot{E}}_{11} &= -\frac{1}{2}\overset{1}{\dot{w}}{}'\delta\overset{1}{\dot{u}} + \delta\overset{1}{\dot{u}}{}' + \frac{1}{2}\overset{1}{\dot{w}}{}'\delta\overset{0}{\dot{w}}{}' + \frac{1}{2}\overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{w}}{}' \\
\delta\overset{0}{\dot{E}}_{13} &= -\frac{1}{4}\overset{1}{\dot{\varphi}}\,\delta\overset{0}{\dot{u}}{}' + \frac{1}{2}\overset{2}{\dot{\psi}}\,\delta\overset{1}{\dot{u}} + \frac{1}{2}\overset{1}{\dot{\psi}}\,\delta\overset{0}{\dot{w}}{}' + \frac{1}{4}\overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{w}} \\
\delta\overset{1}{\dot{E}}_{13} &= \frac{1}{4}\overset{1}{\dot{u}}{}'\delta\overset{1}{\dot{u}} - \frac{1}{4}\overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{u}}{}' - \frac{1}{4}\overset{1}{\dot{u}}{}'\delta\overset{0}{\dot{w}}{}' + \frac{1}{2}\delta\overset{1}{\dot{w}}{}' \\
\delta\overset{0}{\dot{E}}_{33} &= \frac{1}{4}(3\,\overset{1}{\dot{u}} - \overset{0}{\dot{w}}{}')\delta\overset{1}{\dot{u}} - \frac{1}{4}\overset{2}{\dot{\varphi}}\,\delta\overset{0}{\dot{w}}{}' + \delta\overset{1}{\dot{w}}
\end{aligned}
\tag{6.2}
$$

Next, making use of Eqs (6.1) we obtain the second-order rates of the Green strain tensor components

$$
\begin{aligned}
\overset{0}{\ddot{E}}_{11} &= \frac{3}{4}(\overset{0}{\dot{w}}{}')^2 - \frac{1}{4}(\overset{1}{\dot{u}})^2 - \frac{1}{2}\overset{0}{\dot{w}}{}'\,\overset{1}{\dot{u}} \\
\overset{1}{\ddot{E}}_{11} &= \overset{1}{\dot{w}}{}'\,\overset{1}{\dot{\varphi}} \\
\overset{0}{\ddot{E}}_{13} &= \frac{1}{2}\overset{1}{\dot{\varphi}}\,(\overset{1}{\dot{w}} - \overset{0}{\dot{u}}{}') \\
\overset{1}{\ddot{E}}_{13} &= -\frac{1}{2}\overset{1}{\dot{\varphi}}\,\overset{1}{\dot{u}}{}' \\
\overset{0}{\ddot{E}}_{33} &= \frac{3}{4}(\overset{1}{\dot{u}})^2 - \frac{1}{4}(\overset{0}{\dot{w}}{}')^2 - \frac{1}{2}\overset{0}{\dot{w}}{}'\,\overset{1}{\dot{u}}
\end{aligned}
\tag{6.3}
$$

and their variations

$$
\begin{aligned}
\delta\overset{0}{\ddot{E}}_{11} &= -\frac{1}{2}\overset{2}{\dot{\varphi}}\,\delta\overset{1}{\dot{u}} + \frac{1}{2}(3\,\overset{0}{\dot{w}}{}' - \overset{1}{\dot{u}})\delta\overset{0}{\dot{w}}{}' \\
\delta\overset{1}{\ddot{E}}_{11} &= -\,\overset{1}{\dot{w}}{}'\delta\overset{1}{\dot{u}} + \overset{1}{\dot{w}}{}'\delta\overset{0}{\dot{w}}{}' + \overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{w}}{}' \\
\delta\overset{0}{\ddot{E}}_{13} &= -\frac{1}{2}\overset{1}{\dot{\varphi}}\,\delta\overset{0}{\dot{u}}{}' + \frac{1}{2}(\overset{0}{\dot{u}}{}' - \overset{1}{\dot{w}})\delta\overset{1}{\dot{u}} - \frac{1}{2}(\overset{0}{\dot{u}}{}' - \overset{1}{\dot{w}})\delta\overset{0}{\dot{w}}{}' + \frac{1}{2}\overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{w}} \\
\delta\overset{1}{\ddot{E}}_{13} &= \frac{1}{2}\overset{1}{\dot{u}}{}'\delta\overset{1}{\dot{u}} - \frac{1}{2}\overset{1}{\dot{\varphi}}\,\delta\overset{1}{\dot{u}}{}' - \frac{1}{2}\overset{1}{\dot{u}}{}'\delta\overset{0}{\dot{w}}{}'
\end{aligned}
\tag{6.4}
$$

$$\delta\overset{0}{\ddot{E}}_{33}=\frac{1}{2}(3\overset{1}{\dot{u}}-\overset{0}{\dot{w}}{}')\delta\overset{1}{\dot{u}}-\frac{1}{2}\overset{2}{\dot{\varphi}}\,\delta\overset{0}{\dot{w}}{}'$$

The way of getting the equilibrium equations (R.E.Es) for elastic-plastic beams is similar to that applied in the elastic case (see Section 4) (we mean here the partial integration etc.). Substituting Eqs (6.2) and (6.4) into Eq (5.10) we obtain (R.E.Es) as the Euler equations for functional (5.1). Besides, the natural boundary conditions result in this treatment simultaneously. We have to mention here that in the present rate problem the quantities: rate stress resultants $\overset{n}{\dot{\mathcal{L}}}{}^{ij}$, $(n=0,1;\ i,j=1,3)$, rate external surface loads $\overset{n}{\dot{\mathcal{P}}}_u$, $\overset{n}{\dot{\mathcal{P}}}_w$; rate external loads ${}^{*}\overset{n}{\dot{\mathcal{L}}}{}^{ij}$ (at the ends of the beam) should be understood as the rates of the corresponding quantities which were defined in Section 4.

The system of equilibrium equations (R.E.Es) for geometrically nonlinear elastic-plastic beams which have been undergone a quasi-static state of loading is given below (note that its form is similar to that of Eqs (4.22))

$$\begin{aligned}
&\delta(\overset{0}{\dot{u}}):\quad (\overset{0}{\mathcal{S}}_u)'+\overset{0}{\dot{\mathcal{P}}}_u=0\\
&\delta(\overset{1}{\dot{u}}):\quad (\overset{1}{\mathcal{S}}_u)'+\overset{1}{\dot{\mathcal{P}}}_u-\overset{1}{\dot{\mathcal{R}}}_u=0\\
&\delta(\overset{0}{\dot{w}}):\quad (\overset{0}{\mathcal{S}}_w)'+\overset{0}{\dot{\mathcal{P}}}_w=0\\
&\delta(\overset{1}{\dot{w}}):\quad (\overset{1}{\mathcal{S}}_w)'+\overset{1}{\dot{\mathcal{P}}}_w-\overset{1}{\dot{\mathcal{R}}}_w=0
\end{aligned}\qquad \text{(R.E.Es)} \tag{6.5}$$

where

$$\begin{aligned}
\overset{0}{\mathcal{S}}_u&=\overset{0}{\dot{\mathcal{L}}}{}^{11}-\frac{1}{2}(\overset{1}{\varphi}\overset{0}{\dot{\mathcal{L}}}{}^{13}+\overset{1}{\dot{\varphi}}\overset{0}{\mathcal{L}}{}^{13})\\
\overset{1}{\mathcal{S}}_u&=\overset{1}{\dot{\mathcal{L}}}{}^{11}-\frac{1}{2}(\overset{1}{\varphi}\overset{1}{\dot{\mathcal{L}}}{}^{13}+\overset{1}{\dot{\varphi}}\overset{1}{\mathcal{L}}{}^{13})\\
\overset{0}{\mathcal{S}}_w&=\frac{1}{4}\Big[(3\overset{0}{w}{}'-\overset{1}{u})\overset{0}{\dot{\mathcal{L}}}{}^{11}+\overset{0}{\mathcal{L}}{}^{11}(3\overset{0}{\dot{w}}{}'-\overset{1}{\dot{u}})-\overset{2}{\varphi}\overset{0}{\dot{\mathcal{L}}}{}^{33}-\overset{2}{\dot{\varphi}}\overset{0}{\mathcal{L}}{}^{33}\Big]+\\
&\quad+\overset{1}{\psi}\overset{0}{\dot{\mathcal{L}}}{}^{13}+\dot{\psi}\overset{0}{\mathcal{L}}{}^{13}+\frac{1}{2}(\overset{1}{w}{}'\overset{1}{\dot{\mathcal{L}}}{}^{11}+\overset{1}{\dot{w}}{}'\overset{1}{\mathcal{L}}{}^{11})-\frac{1}{2}(\overset{1}{u}{}'\overset{1}{\dot{\mathcal{L}}}{}^{13}+\overset{1}{\dot{u}}{}'\overset{1}{\mathcal{L}}{}^{13})\\
\overset{1}{\mathcal{S}}_w&=\overset{1}{\dot{\mathcal{L}}}{}^{13}+\frac{1}{2}(\overset{1}{\varphi}\overset{1}{\dot{\mathcal{L}}}{}^{11}+\overset{1}{\dot{\varphi}}\overset{1}{\mathcal{L}}{}^{11})
\end{aligned}\tag{6.6}$$

$$\overset{1}{\mathcal{R}}_u = \frac{1}{4}\Big[(3\overset{1}{\dot{u}} - \overset{0}{\dot{w}}{}') \overset{0}{\dot{\mathcal{L}}}{}^{33} + \overset{0}{\mathcal{L}}{}^{33}(3\overset{1}{\dot{u}} - \overset{0}{\dot{w}}{}') - \overset{2}{\varphi}\overset{0}{\dot{\mathcal{L}}}{}^{11} - \overset{2}{\dot{\varphi}}\overset{0}{\mathcal{L}}{}^{11}\Big] +$$

$$+ \overset{2}{\psi}\overset{0}{\dot{\mathcal{L}}}{}^{13} + \overset{2}{\dot{\psi}}\overset{0}{\mathcal{L}}{}^{13} - \frac{1}{2}(\overset{1}{w}{}'\overset{1}{\dot{\mathcal{L}}}{}^{11} + \overset{1}{\dot{w}}{}'\overset{1}{\mathcal{L}}{}^{11}) + \frac{1}{2}(\overset{1}{u}{}'\overset{1}{\dot{\mathcal{L}}}{}^{13} + \overset{1}{\dot{u}}{}'\overset{1}{\mathcal{L}}{}^{13})$$

$$\overset{1}{\mathcal{R}}_w = \overset{0}{\dot{\mathcal{L}}}{}^{33} + \frac{1}{2}(\overset{1}{\varphi}\overset{0}{\dot{\mathcal{L}}}{}^{13} + \overset{1}{\dot{\varphi}}\overset{0}{\mathcal{L}}{}^{13})$$

7. Concluding remarks

The aim of this work was to consider the problem of the geometrical nonlinearity in beam theory in the case when the moderate rotations have been employed in the nonlinear strain-displacement relations. Important results of the variational analysis applied in the paper are the nonlinear equilibrium equations derived for two cases: elastic and elastic-plastic beams, separately.

The approach developed in the present paper can also be applied to a beam with variable cross-section.

The next stage of this work will be a numerical solution to the mathematical problems which have been discussed in this paper. The variational functionals have to be introduced into numerical procedures and used directly to calculate the increments of the field quantities. Obviously, a particular form of the constitutive equations describing the beam material will be assumed then and included to the calculation process. The systems of nonlinear differential equations (E.Es) or (R.E.Es) with appropriate boundary conditions will be used for an equilibrium check. The numerical calculations should be programmed as an iterative process.

Acknowledgement: This paper was supported by the State Committee for Scientific Research (KBN), Poland, under grant No 3 P404 013 06.

References

1. ERINGEN A.C., 1962, *Nonlinear Theory of Continuous Media,* Mc Graw-Hill Book Company, Inc. New York, Toronto, London
2. GAŁKA A., TELEGA J.J., BIELSKI W., 1994, Contribution to Dual Variational Principles for Nonlinear Elastic Beams, *Control and Cybernetics,* **23**

3. MIKKOLA M.J., 1989, Complementary Energy Theorem in Geometrically Non-Linear Structural Problems, *Int. J. Non-Lin. Mech.*,**24**, 499-508

4. NEAL K.W., 1972, A General Variational Theorem for the Rate Problem in Elasto-Plasticity, *Int. J. Solids Structures,* **8**, 865-876

5. PIETRASZKIEWICZ W., 1980, *Finite Rotations in the Non-Linear Theory of Thin Shells,* in OLSZAK (edit.): *Thin Shell Theory,* Springer-Verlag, Wien

6. REISSNER E., 1972, On One-Dimensional Finite Strain Beam Theory: the Plane Problem, *J. Appl. Math. and Phys.*,**23**

7. RYCHTER Z., 1993, A Family of Shear-Deformation Beam Theories and a Refined Bernoulli-Euler Theory, *Int. J. Eng. Sci.,* 4, **31**

8. SCHMIDT R., REDDY J.N., 1988, A Refined Small Strain and Moderate Rotation Theory of Elastic Anisotropic Shells, *J. Appl. Mech.,* **55**

9. SCHMIDT R., WEICHERT D., 1989, A Refined Theory of Elastic-Plastic Shells at Moderate Rotations, *Z.A.M.M.,* **69**, 1, 11-21

10. SHIELD R.T., 1992, Bending of a Beam or Wide Strip, *Quart. J. Mech. Appl. Math.,* **45**

11. TELEGA J.J., 1976, O operatorowym podejściu do formułowania zasad wariacyjnych dla ośrodków plastycznych, *Mech. Teor. i Stos.,* 4, **14**

12. WASHIZU K., 1975, *Variational Methods in Elasticity and Plasticity,* Pergamon Press

Geometrycznie nieliniowe modele sprężystych i sprężysto-plastycznych belek

Streszczenie

Rozpatrzono zagadnienie płaskiej deformacji sprężystych i sprężysto-plastycznych belek przy założeniu małych odkształceń lecz umiarkowanie dużych obrotów.

Przyjęto, że belka odkształca się w płaszczyźnie pionowej pod wpływem obciążeń zewnętrznych działających na górną i dolną płaszczyznę oraz jej przekroje końcowe.

Sformułowane zasady wariacyjne pozwoliły uzyskać geometrycznie nieliniowe równania równowagi dla statycznego problemu belek sprężystych oraz dla quasi-statycznego zagadnienia belek sprężysto-plastycznych.

Manuscript received June 14, 1996; accepted for print July 30, 1996