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ON VIBRATIONS OF PLATES UNDER PERIODICALLY DISTRIBUTED INERTIAL LOADINGS[1]

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The aim of the paper is twofold. Firstly, it is the application of equations of the refined macro-dynamics of microperiodic plates to calculation of the resonance frequencies of Kirchhoff or Hencky-Reissner periodic plates and comparing the obtained results. Secondly, some from the above results will be compared to those obtained from the known approximation methods and hence we shall prove that the proposed refined models lead to the physically correct results. The presented modelling approach to the linear-elastic plates, having microperiodic structure in planes parallel to the midplane, is based on the assumptions given by Baron and Woźniak (1995) (for the Hencky-Reissner plate theory) and Jędrysiak and Woźniak (1995) (for the Kirchhoff plate theory), and describes the effect of the microstructure length dimensions on the plate macro-behaviour.

Key words: periodic plates vibrations, modelling, length-scale effect

1. Introduction

The first aim of this paper is to use the general equations formulated by Baron and Woźniak (1995) and Jędrysiak and Woźniak (1995) to detect the effect of the microstructure size on the dynamic plate behaviour in the case of resonance frequencies of plate bands subjected to periodically distributed inertial loadings and to compare obtained results.

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The second aim is to show that the presented refined models (based on the Hencky-Reissner or the Kirchhoff assumptions) lead to the physically correct results. For this purpose we shall compare the obtained results to those obtained using the Ritz method.

In these problems we shall distinguish a small repeated element of a plate, called the periodicity unit cell. We assume that the size of the unit cell is sufficiently small compared to the minimum characteristic length dimension of the plate midplane. Moreover, because our considerations related to every periodicity cell will be also based on the Hencky-Reissner or the Kirchhoff theory, the periodicity cell will be assumed to have a form of a medium thickness or a thin plate.

Solving problems in such composite periodic plates meets analytical difficulties. Thus, some simplified models were proposed in which a periodic plate is represented by a certain homogeneous plate structure with a constant effective stiffness and an averaged mass density. These models were presented by Caillerie (1984), Kohn and Vogelius (1984), Lewiński (1992) and others. However, the aforementioned models, called effective stiffness plate theories, were restricted to the static problems only and hence they are not able to describe problems of the plate vibrations. In order to investigate non-stationary problems within the framework of macromechanics certain models (for instance, those based on the concept of the continuum with the extra local degrees of freedom) were proposed. Short wave propagation problems were investigated by Bakhvalov and Panasenko (1984), Maewal (1986) and some refined models describing long wave problems for the microperiodic bodies were presented by Woźniak and Woźniak (1997a,b). These models take into account the effect of the microstructure size on dynamic macrobehaviour of a body. Refined models describing this effect for periodic structures were presented by Baron and Woźniak (1995), Jędrysiak and Woźniak (1995), Michalak et al. (1995) and others.

In this paper, we deal with study the problem of free vibrations of the plate band with periodically distributed loadings. As a tool we apply the refined modelling approach, which takes into account the microstructure scale effect on the dynamic plate response. These approaches to the linear-elastic periodic plates were presented by Baron and Woźniak (1995), for medium thickness plates (the Hencky-Reissner assumptions) and by Jędrysiak and Woźniak (1995), for thin elastic plates (the Kirchhoff assumptions). Using these approaches we obtain the plate theories, which are called "refined" or "structural". Within the framework of these theories we can investigate dynamic processes in periodic plates, where the size of the periodicity unit cell cannot be neglected. Moreover, every periodicity cell has to be treated within the framework of 2D-plate theories. The effect of the microstructure size on the body macrobehaviour is called the length-scale effect.

2. Preliminaries

Throughout the paper subscripts $alpha, \beta, ...$ $(i, j, ...)$ run over $1, 2$ (over 1,2,3), indices $A, B, ...$ run over $1, ..., N$ and indices $a, b, ...$ run over $1, ..., n$. Summation convention holds for all the aforementioned indices.

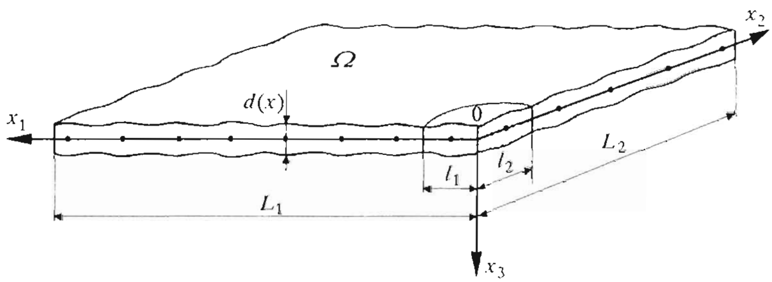

Fig. 1. Sample plate with a microperiodic structure

By $0x_1x_2x_3$ we denote the orthogonal cartesian coordinate system in the physical space. Setting $\boldsymbol{x} \equiv (x_1, x_2)$ and $z \equiv x_3$, the undeformed considered plate (Fig.1) occupies the region $\Omega := \{(\boldsymbol{x}, z) : -d(\boldsymbol{x})/2 < z < d(\boldsymbol{x})/2, \boldsymbol{x} \in \Pi\}$ where Π is the region of midplane and $d(\boldsymbol{x})$ is the plate thickness at a point $\boldsymbol{x} \in \Pi$. By $\Delta := (0, l_1) \times (0, l_2)$ we describe the periodicity unit cell on $0x_1x_2$ plane, where l_1, l_2 are length dimensions. The microstructure parameter l, defined by $l := \sqrt{l_1^2 + l_2^2}$, describing the size of the microstructure, satisfies two conditions. First, it is sufficiently small compared to the minimum characteristic length dimension L_Π of Π $(l \ll L_\Pi)$. Second, it is sufficiently large compared to the maximum plate thickness d. We assume that d is a Δ-periodic function of $\boldsymbol{x}$ and all material and inertial properties of the plate are also Δ-periodic functions of $\boldsymbol{x}$ and even functions of z. For an arbitrary integrable Δ-periodic function f we denote by

$$\langle f \rangle := \frac{1}{l_1 l_2} \int\limits_{\Delta} f(\boldsymbol{x})\, da \qquad da \equiv dx_1 dx_2 \tag{2.1}$$

its averaged (constant) value. We also define t as the time coordinate. Moreover, let u_i, e_{ij}, s_{ij} stand for displacements, strains and stresses, respectively, p^+, p^- be loadings (in the x_3-axis direction) on the top and bottom planes of the plate, respectively, and b be the constant body force (in the x_3-axis direction). By $\rho = \rho(\boldsymbol{x}, z)$ and $a_{ijkl} = a_{ijkl}(\boldsymbol{x}, z)$ we denote a mass density and a tensor of elasticity of the plate material and assume that $z = \text{const}$ are material symmetry planes; at the same time $\rho(\cdot)$ and $a_{ijkl}(\cdot)$ are assumed to be even functions of z and Δ-periodic functions of $\boldsymbol{x}$.

3. Fundamental relations

3.1. Hencky-Reissner plate theory relations

Using the known denotations the above medium thickness plate theory is described by:
— strain-displacement equations

$$e_{ij} = u_{(i,j)} \tag{3.1}$$

— stress-strain relations (for the plane stress condition $s_{33} = 0$)

$$s_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_{\gamma\delta} \qquad\qquad s_{\alpha 3} = 2a_{\alpha 3\beta 3} e_{\beta 3} \tag{3.2}$$

where $c_{\alpha\beta\gamma\delta} := a_{\alpha\beta\gamma\delta} - a_{\alpha\beta 33} a_{\gamma\delta 33} (a_{3333})^{-1}$
— kinematic constrains

$$u_a(\boldsymbol{x}, z, t) = z\theta_\alpha(\boldsymbol{x}, t) \qquad\qquad u_3(\boldsymbol{x}, z, t) = u_3(\boldsymbol{x}, t) \tag{3.3}$$

where u_3, θ_α are deflections of the midplane and rotations, respectively
— equation of motion (in the weak form)

$$\begin{aligned} &\int\limits_{\Pi} \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} \rho \ddot{u}_i \delta u_i \, dzda + \int\limits_{\Pi} \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} (s_{\alpha\beta} \delta e_{\alpha\beta} + 2s_{\alpha 3} \delta e_{\alpha 3}) \, dzda = \\ &= \int\limits_{\Pi} \Big(p^+ \delta u_3\Big|_{\frac{d}{2}} + p^- \delta u_3\Big|_{-\frac{d}{2}}\Big) \, da + b \int\limits_{\Pi} \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} \rho \delta u_3 \, dzda \end{aligned} \tag{3.4}$$

which holds for every admissible virtual displacement $\delta\boldsymbol{u}$, where $\delta u_i\Big|_{\delta\Pi} = 0$.

Using the known modelling procedures from Eqs (3.1) ÷ (3.4) for microperiodic plates, we will obtain differential equations of the medium thickness plate theory with highly oscillating Δ-periodic functions.

3.2. Kirchhoff plate theory relations

The above thin plate theory is described by:
— strain-displacement equations

$$e_{ij} = u_{(i,j)} \tag{3.5}$$

— stress-strain relations (for the plane stress condition $s_{33} = 0$)

$$s_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_{\gamma\delta} \tag{3.6}$$

— kinematic constrains

$$u_\alpha(\boldsymbol{x}, z, t) = -z w_{,\alpha}(\boldsymbol{x}, t) \qquad\qquad u_3(\boldsymbol{x}, z, t) = w(\boldsymbol{x}, t) \tag{3.7}$$

where $w(\boldsymbol{x}, t)$ are *displacements of points of the midplane*
— equation of motion (in the weak form) – in the form (3.4).

For microperiodic plates Eqs (3.5) ÷ (3.7) and (3.4) lead to the governing equations of the thin plate with highly oscillating coefficients.

3.3. Introductory concepts of the modelling approach

In order to formulate the modelling hypotheses, we will introduce two concepts.

The first of them is *the microshape function system.* This is a set of sufficiently smooth functions $h(\boldsymbol{x})$, which are linearly independent, Δ-periodic and such that $\langle|\widehat{h}|\rangle/\langle|\partial_\alpha \widehat{h}|\rangle \in \mathcal{O}(l)$, $\alpha = 1, ..., n$ hold for every $\widehat{h} \in \{h, \partial_\alpha h, ...\}$. The choice of microshape functions depends on a form of micromotions relevant in the problem under consideration; they will be obtained by using an approximate solution to a local vibration problem formulated on the unit cell.

The second from these concepts is *the macrofunction.* Every macrofunction F together with all its derivatives, satisfies, what will be called a long wave approximation, the conditions: $\langle fF\rangle(\boldsymbol{x}) \cong \langle f\rangle(\boldsymbol{x})F(\boldsymbol{x})$, for an arbitrary function f, and $\langle\partial_\alpha(hF)\rangle \cong \langle\partial_\alpha hF\rangle$ for an arbitrary microshape function h. The set of the above macrofunctions will be denoted by LWA(Δ), where LWA stands for a "long wave approximation".

3.4. Modelling hypotheses

The formulation of the refined 2D-theory for the microperiodic thin plates will be based on the following hypotheses.

- *Macrodisplacement Assumption* (A1). This hypothesis is based on the assumption that

$$\exists U \in \mathrm{LWA}(\Delta) \qquad \mathsf{E}\langle U\rangle = \langle \mathsf{E}u\rangle$$

  where u is a displacement field defined on the plate midplane, U is called a macrodisplacement field, and E is the known differential operator such that $\mathsf{E}u$ is the corresponding strain field.

- *Internal Variable Assumption* (A2). This hypothesis is related to the micromotions $v := u - U$ and states that for an arbitrary Δ-periodic function f we have $\langle fv\rangle \cong \langle fh^A V^A\rangle$ and $V^A \in \mathrm{LWA}(\Delta)$, $A = 1,...,N$. Fields V^A are called internal variables.

3.5. Governing equations of the refined Hencky-Reissner model

Let us introduce the following denotations

$$\mu := \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} \rho\, dz \qquad j := \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} \rho z^2\, dz$$
$$d_{\alpha\beta\gamma\delta} := \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} z^2 c_{\alpha\beta\gamma\delta}\, dz \qquad c_{\alpha\beta} := \int\limits_{-\frac{d}{2}}^{\frac{d}{2}} a_{\alpha 3\beta 3}\, dz \tag{3.8}$$

where

μ	–	mean mass density
j	–	plate rotational inertia density
$d_{\alpha\beta\gamma\delta}, c_{\alpha\beta}$	–	were defined in Section 2 and 3.1.

After some transformations we obtain the following equations of refined theory for the Hencky-Reissner plates with U, Θ_α, T^A, Σ^A_α as the basic unknowns:

— equations of motion

$$\widetilde{M}_{\alpha\beta,\beta} - Q_\alpha - \langle j\rangle\ddot{\Theta}_\alpha - \underline{\langle jh^b\rangle}\ddot{\Psi}^b_\alpha = 0$$

$$\begin{aligned} &Q_{\alpha,\alpha} - \langle\mu\rangle\ddot{U} - \underline{\langle\mu r^B\rangle}\ddot{T}^B + p + \langle\mu\rangle b = 0 \\ &\widetilde{M}^a_\alpha + R^a_\alpha + \underline{\langle jh^a h^b\rangle}\ddot{\Psi}^b_\alpha + \underline{\langle jh^a\rangle}\ddot{\Theta}_\alpha = 0 \\ &Q^A + \underline{\langle\mu r^A r^B\rangle}\ddot{T}^B + \underline{\langle\mu r^A\rangle}\ddot{U} = 0 \end{aligned} \tag{3.9}$$

— constitutive equations

$$\begin{aligned} &\widetilde{M}_{\alpha\beta} = \langle d_{\alpha\beta\gamma\delta}\rangle\Theta_{(\gamma,\delta)} + \langle d_{\alpha\beta\gamma\delta}h^b_{,\delta}\rangle\Psi^b_\gamma \\ &Q_\alpha = \langle c_{\alpha\beta}\rangle(\Theta_\beta + U_{,\beta}) + \underline{\langle c_{\alpha\beta}h^b\rangle}\Psi^b_\beta + \langle c_{\alpha\beta}r^B_{,\beta}\rangle T^B \\ &\widetilde{M}^a_\alpha = \langle d_{\alpha\beta\gamma\delta}h^a_{,\beta}\rangle\Theta_{(\gamma,\delta)} + \langle d_{\alpha\beta\gamma\delta}h^a_{,\beta}h^b_{,\delta}\rangle\Psi^b_\gamma \\ &R^a_\alpha = \underline{\langle c_{\alpha\beta}h^a\rangle}(\Theta_\beta + U_{,\beta}) + \underline{\langle c_{\alpha\beta}h^a h^b\rangle}\Psi^b_\beta + \underline{\langle c_{\alpha\beta}r^B_{,\beta}h^a\rangle}T^B \\ &Q^A = \langle c_{\alpha\beta}r^A_{,\alpha}\rangle(\Theta_\beta + U_{,\beta}) + \underline{\langle c_{\alpha\beta}r^A_{,\alpha}h^a\rangle}\Psi^a_\beta + \langle c_{\alpha\beta}r^A_{,\alpha}r^B_{,\beta}\rangle T^B \end{aligned} \tag{3.10}$$

The underlined terms in the above equations depend on the microstructure size.

3.6. Governing equations of the refined Kirchhoff model

The macromodelling procedure based on Eqs (3.5) ÷ (3.7) and (3.4) yields the following system of differential equations with constant coefficients for W and V^A as the basic unknowns:
— equations of motion

$$\begin{aligned} &M_{\alpha\beta,\alpha\beta} + \langle\mu\rangle\ddot{W} - \langle j\rangle\ddot{W}_{,\alpha\alpha} + \underline{\langle\mu g^B\rangle}\ddot{V}^B - \underline{\langle jg^B_{,\alpha}\rangle}\ddot{V}^B_{,\alpha} = p + b\langle\mu\rangle \\ &M^A + \underline{\langle\mu g^A\rangle}\ddot{W} + \underline{\langle\mu g^A g^B\rangle}\ddot{V}^B + \underline{\langle jg^A_{,\alpha}\rangle}\ddot{W}_{,\alpha} + \underline{\langle jg^A_{,\alpha}g^B_{,\alpha}\rangle}\ddot{V}^B = 0 \end{aligned} \tag{3.11}$$

— constitutive equations

$$\begin{aligned} &M_{\alpha\beta} = \langle d_{\alpha\beta\gamma\delta}\rangle W_{,\gamma\delta} + \langle d_{\alpha\beta\gamma\delta}g^B_{,\gamma\delta}\rangle V^B \\ &M^A = \langle d_{\alpha\beta\gamma\delta}g^A_{,\gamma\delta}g^B_{,\alpha\beta}\rangle V^B + \langle d_{\alpha\beta\gamma\delta}g^A_{,\gamma\delta}\rangle W_{,\alpha\beta} \end{aligned} \tag{3.12}$$

The underlined terms represent the length-scale effect on the plate behaviour.

3.7. Important remark

It has to be emphasized that the above equations can be applied to the vibration and wave propagations analysis only accepting the long wave approximation assumption (LWA). The obtained theories make it possible to derive only averaged values of displacements and are not able to describe the plate behaviour inside every periodicity cell.

4. Applications of the refined Hencky-Reissner model

Let us consider a homogeneous, isotropic, periodic plate band with the constant thickness d, mass density ρ, Young modulus E and Poisson ratio ν, simply supported on the opposite edges $x_1 = 0$ and $x_1 = L$, and periodically distributed concentrated masses along the axis x_1 (Fig.2) The cell of periodicity is now one dimensional $(\Delta_1 := (0, l)$, Fig.3). The wave number will be denoted by $k \equiv 2p/L$ and $x \equiv x_1$. Our considerations are related to resonance frequencies and the effect of body forces b and loadings p will be neglected.

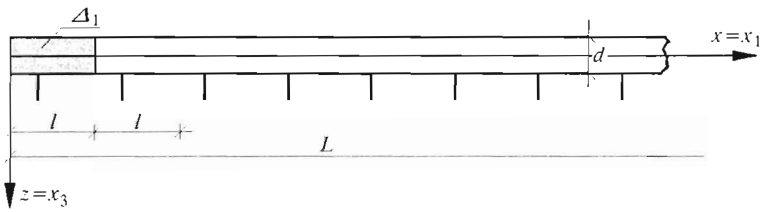

Fig. 2. Plate band with periodically distributed concentrated masses

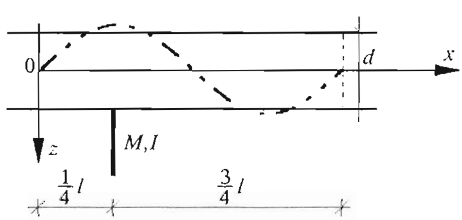

Fig. 3. Cell of periodicity $\Delta_1 = (0, l)$ and the microshape functions g, h, r

Because the unit cell is defined in the form given in Fig.3, we assume only two microshape functions $h = h^1(x)$, $r = r^1(x)$, where each of them has a form $l[\sin(2\pi x/l)]$ obtained by an approximate solution to the eigenvalue periodic problem on the unit cell. We can show that some terms in Eqs (3.10) are neglected: $\langle d_{1111}h_{,1}\rangle = \langle c_{11}r_{,1}\rangle = \langle c_{11}h\rangle = \langle c_{11}hr_{,1}\rangle = 0$. Denoting

$$\begin{array}{lll} B \equiv \langle d_{1111}\rangle & B^{11} \equiv \langle d_{1111}(h_1)^2\rangle & \\ C \equiv \langle c_{11}\rangle & C^{11} \equiv \langle c_{11}(r_{,1})^2\rangle & \widetilde{C}^{11} \equiv \langle c_{11}h^2\rangle \\ m \equiv \langle \mu\rangle & m^1 \equiv \langle \mu r\rangle & \widetilde{m}^{11} \equiv \langle \mu r^2\rangle \\ J \equiv \langle j\rangle & J^1 \equiv \langle jh\rangle & \widetilde{J}^{11} \equiv \langle jh^2\rangle \end{array} \tag{4.1}$$

and defining $T \equiv T^1$, $\Theta \equiv \Theta_1$, $\Psi \equiv \Psi_1^1$ from Eqs (3.9) and (3.10) we obtain the governing system of equations for macrodeflections U, macrorotations Θ and internal variables T, Ψ

$$\begin{aligned} &B\Theta_{,11} - C(\Theta + U_{,1}) - J\ddot{\Theta} - \underline{J^1}\ddot{\Psi} = 0 \\ &C(\Theta_{,1} + U_{,11}) - m\ddot{U} - \underline{m^1}\ddot{T} = 0 \\ &(B^{11} + \underline{\widetilde{C}^{11}})\Psi + \underline{\widetilde{J}^{11}}\ddot{\Psi} + \underline{J^1}\ddot{\Theta} = 0 \\ &C^{11}T + \underline{\widetilde{m}^{11}}\ddot{T} + \underline{m^1}\ddot{U} = 0 \end{aligned} \tag{4.2}$$

where the underlined terms describe the effect of the microstructure length parameter l.

Solution to Eqs (4.2) can be assumed in the form

$$\begin{array}{ll} U(x,t) = A_U \sin(kx)\cos(\omega t) & T(x,t) = A_T \cos(kx)\cos(\omega t) \\ \Theta(x,t) = A_\Theta \cos(kx)\cos(\omega t) & \Psi(x,t) = A_\Psi \sin(kx) cos(\omega t) \end{array} \tag{4.3}$$

where A_U, A_T, A_Θ, A_Ψ are constant vibration amplitudes and ω is a vibration frequency. Substituting the right-hand sides of these above formulae into Eqs (4.2) we obtain the system of four linear algebraic equations formulae for amplitudes A_U, A_T, A_Θ, A_Ψ. This system of equations has nontrivial solutions provided that its determinant is equal to zero. In this way we obtain the characteristic equation of frequencies, from which the values of resonance frequencies λ can be calculated. Denoting

$$\overline{D} \equiv [k^2CJ + m(k^2B + C)]^2 - 4k^4BCJm$$

we arrive at the following formulae for *the macro-resonance frequencies* λ_I, λ_{II}

$$(\lambda_{I,II})^2 = \frac{k^2CJ + m(k^2B + C) \mp \sqrt{\overline{D}}}{2mJ} + \mathcal{O}(l^2) \tag{4.4}$$

and for *the micro-resonance frequencies* λ_{III}, λ_{IV}

$$\begin{aligned}(\lambda_{III})^2 &= \frac{C^{11}m + k^2Cm^{-1}(m^1)^2}{m\widetilde{m}^{11} - (m^1)^2} + \mathcal{O}(l^2)\\ (\lambda_{IV})^2 &= \frac{(B^{11} + \widetilde{C}^{11})J + (k^2B + C)J^{-1}(J^1)^2}{J\widetilde{J}^{11} - (J^1)^2} + \mathcal{O}(l^2)\end{aligned} \tag{4.5}$$

where $\overline{D} > 0$, $m\widetilde{m}^{11} - (m^1)^2 > 0$, $J\widetilde{J}^{11} - (J^1)^2 > 0$.

We can observe that the macro-resonance frequencies described by Eqs (4.4) are independent of a choice of microshape function forms.

5. Applications of the refined Kirchhoff model

Using assumptions and denotations given in Section 4 the refined Kirchhoff model will be investigated. We will confine ourselves only to one form of micromotions for the cell Δ_1 given in Fig.3 and assume one microshape function $g = g^1(x)$ in the form $l^2\sin(2\pi x/l)$. Therefore we have $\langle jg_{,1}\rangle = 0$. Because the plate thickness d is a constant value we obtain $\langle d_{1111}g_{,11}\rangle = 0$. Denoting

$$\begin{aligned}&\widehat{m}^1 \equiv \langle mg\rangle && m^{11} \equiv \langle \mu gg\rangle\\ &J^{11} \equiv \langle j(g_{,1})^2\rangle && D \equiv \langle d_{1111}(g_{,11})^2\rangle\end{aligned} \tag{5.1}$$

defining $V = V^1$, and using notations (4.1) we obtain from Eqs (3.11) and (3.12) the governing system of equations for macrodeflections W and internal variables V in the form

$$\begin{aligned}&BW_{,1111} + m\ddot{W} - J\ddot{W}_{,11} + \underline{\widehat{m}^1}\ddot{V} = 0\\ &\underline{\widehat{m}^1}\ddot{W} + DV + \underline{m^{11}}\ddot{V} + \underline{J^{11}}\ddot{V} = 0\end{aligned} \tag{5.2}$$

where the underlined terms respect the effect of the microstructure length parameter l.

Solution to Eqs (5.2) can be assumed in the form

$$W(x,t) = A_W \sin(kx)\cos(\omega t) \qquad\qquad V(x,t) = A_V \sin(kx)\cos(\omega t) \tag{5.3}$$

where A_W, A_V are constant vibration amplitudes and ω is a vibration frequency. Substituting the right-hand sides of the above formulae into Eqs (5.2) we obtain the system of two linear algebraic equations for amplitudes A_W, A_V. This system of equations has nontrivial solutions provided that its determinant is equal to zero. In this way we obtain the characteristic equation for frequencies, from which the values of resonance frequencies ω can be calculated. Denoting

$$\beta \equiv [B(m^{11} + J^{11})k^4 - D(m + Jk^2)]^2 + 4BD(\widehat{m}^1)^2k^4 \qquad \beta > 0$$

we arrive at the following formulae for lower ω_1 and higher ω_2 resonance frequencies

$$(\omega_{1,2})^2 = \frac{Bk^4(m^{11} + J^{11}) + D(m + Jk^2) \mp \sqrt{\beta}}{2[(m + Jk^2)(m^{11} + J^{11}) - (\widehat{m}^1)^2]} \tag{5.4}$$

In Section 6 the obtained results for the plate band with periodically distributed loadings will be presented.

6. Comparison of results obtained by the refined Hencky-Reissner and Kirchhoff models

In order to compare the applications of the structural model based on the Kirchhoff assumptions with those of the refined model for the Hencky-Reissner assumptions free vibrations of the plate band assumed in Section 4 (Fig.2) will be investigated. This plate is loaded periodically along the x-axis by the system of concentrated masses M, having the inertial moment I about the x-axis per unit length of this band. The cell of periodicity is given in Fig.3.

For the sake of simplicity of calculations we define the reduced loading mass densities $\rho_M \equiv M(ld)^{-1}$, $\rho_I \equiv 12Il^{-1}d^{-3}$. The resonance frequencies λ and ω are calculated under the assumption that the microshape functions have the form:

— for the Hencky-Reissner relations $h = r = l\sin(2\pi x/l)$

— for the Kirchhoff relations $g = l^2\sin(2\pi x/l)$

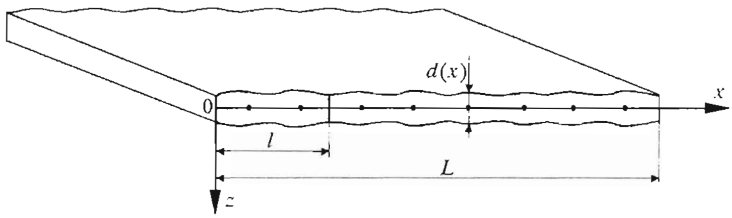

Fig. 4. Diagrams of resonance vibration freauencies

which are presented in Fig.3. The form of assumed microshape functions depends on the form of the micromotions, which we want to investigate, and of the unit cell Δ_1 form.

In Fig.4. we show the diagrams representing the interrelation between the dimensionless resonance vibration frequencies and the dimensionless wave number q ($q \equiv kl$, where $k = 2\pi/L$ is the wave number, l is the microstructure parameter, L is the span of the plate along the x-axis). These dimensionless frequencies are obtained from Eqs (4.4), (4.5), (5.4) and the relations

$$(\Lambda_{I,II,III,IV})^2 \equiv \frac{l^2\rho}{E}(\lambda_{I,II,III,IV})^2 \qquad (\Omega_{1,2})^2 \equiv \frac{l^2\rho}{E}(\omega_{1,2})^2 \tag{6.1}$$

where ρ is the constant value of the plate mass density and E is the Young modulus.

Diagrams are made for the following values of parameters: $\nu = 0.25$, $\rho_M/\rho = \rho_I/\rho = 1$ or $\rho_M/\rho = \rho_I/\rho = 5$.

7. Verification of the presented models by using the Ritz method

7.1. Refined (structural) models

For certain verification of the presented modelling approach free vibrations of special microperiodic plates will be investigated. It will be shown that when assuming only one microshape function it is sufficient to consider the micromotion of the plate described by this function.

Let us assume the plate band simply supported on the opposite edges ($x = 0$ and $x = L$), made of an isotropic homogeneous material and having the l-periodic thickness d, with periodically distributed concentrated masses M.

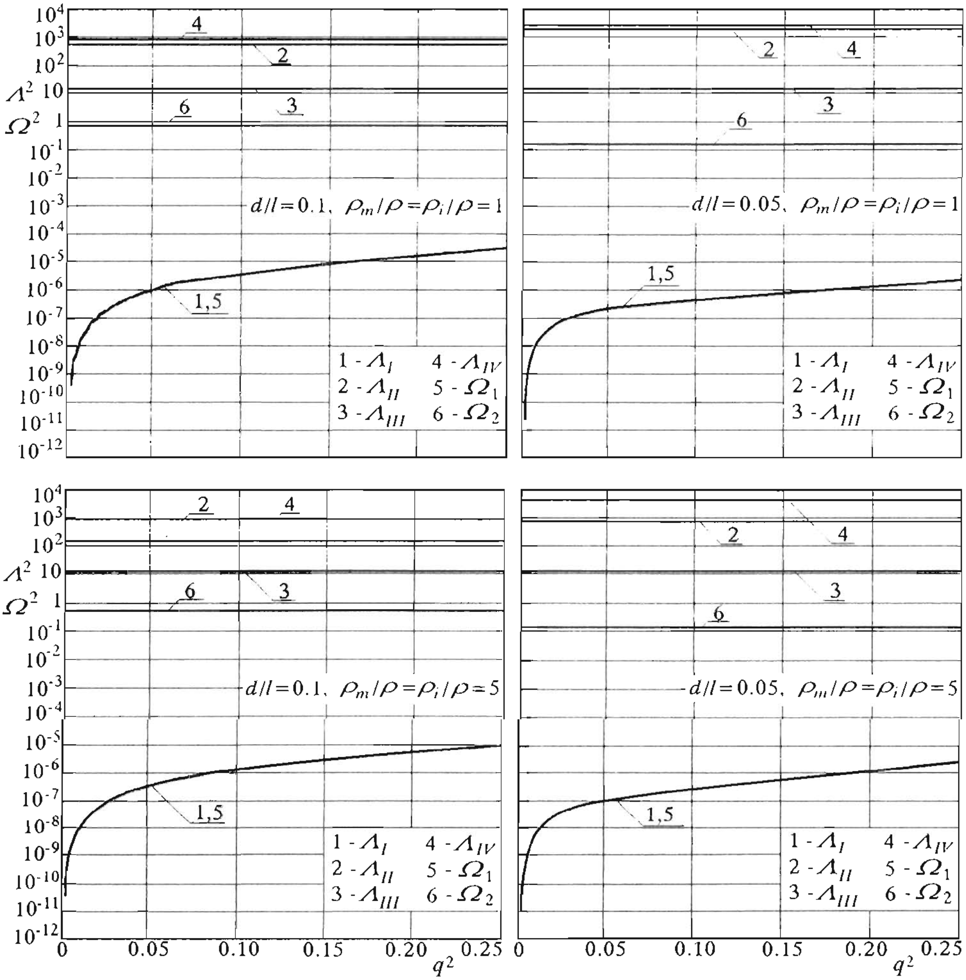

Fig. 5. Plate band with a periodic thickness

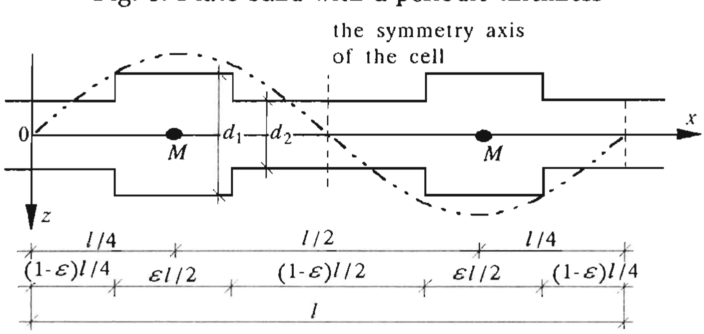

Fig. 6. Unit cell Δ_1 and the microshape functions g, h, r

A sample plate is showed in Fig.5. Moreover, we assume the periodicity unit cell in the form shown in Fig.6, and the thickness d as

$$d(x) = \begin{cases} d_1 & \text{if} \quad x \in \left(\frac{(1-\varepsilon)l}{4}, \frac{(1+\varepsilon)l}{4}\right) \cup \left(\frac{(3-\varepsilon)l}{4}, \frac{(3+\varepsilon)l}{4}\right) \\ d_2 & \text{if} \quad x \in \left[0, \frac{(1-\varepsilon)l}{4}\right] \cup \left(\frac{(1+\varepsilon)l}{4}, \frac{(3-\varepsilon)l}{4}\right) \cup \left[\frac{(3+\varepsilon)l}{4}, l\right] \end{cases} \tag{7.1}$$

where: l is the microstructure parameter, ε is a real number from the range $[0, 1]$. The concentrated masses M will be applied at the two points of the unit cell Δ_1 on the x-axis at $x = l/4$ and $x = 3l/4$. Hence, we will investigate the micromotions described by the microshape functions

$$h(x) = r(x) = l\sin(2\pi x/l) \qquad\qquad g(x) = l^2\sin(2\pi x/l) \tag{7.2}$$

The averaged density per unit area of each mass M will be denoted by $m = \rho H$, where H is an l -periodic function. For the purpose of investigation of free vibrations we will assume that the tractions p on upper and lower boundaries of the plate and the body force b are equal to zero.

7.1.1. Hencky-Reissner model

Below, using the governing equations of the refined Hencky-Reissner model (3.9), (3.10) the plate will be analysed. Describing micromotions by the microshape functions in the form $(7.2)_1$ yields $m^1 = J^1 = 0$. In this way Eqs (4.2) have the form

$$\begin{aligned} &B\Theta_{,11} - C(\Theta + U_{,1}) - J\ddot{\Theta} = 0 \\ &C(\Theta_{,1} + U_{,11}) - m\ddot{U} = 0 \\ &(B^{11} + \underline{\widetilde{C}^{11}})\Psi + \underline{\widetilde{J}^{11}}\ddot{\Psi} = 0 \\ &C^{11}T + \underline{\widetilde{m}^{11}}\ddot{T} = 0 \end{aligned} \tag{7.3}$$

We can observe that the two above equations $(7.3)_{1,2}$ for describe macromotions of the plate band. Eqs $(7.3)_{3,4}$ are independent and each of them describes independent micromotions. The last two of Eqs (7.3) are interesting for us, because those depend on the microstructure parameter l. Hence, we will examine only these equations.

Solutions to Eqs $(7.3)_{3,4}$ satisfying the boundary conditions for the simply supported plate band will be assumed in the form (k is a wave number)

$$\Psi(x,t) = A_\Psi \sin(kx)\cos(\omega t) \qquad\qquad T(x,t) = A_T \sin(kx)\cos(\omega t) \tag{7.4}$$

Substituting the right-hand sides of Eqs (7.4) into Eqs $(7.3)_{3,4}$ we obtain two independent linear algebraic equations for amplitudes A_Ψ, A_T. After some transformations we obtain the following formulae for *the micro-resonance frequencies* λ'_{III}, λ'_{IV}

$$(\lambda'_{III})^2 = \frac{C^{11}}{\widetilde{m}^{11}} + \mathcal{O}(l^2) \qquad\qquad (\lambda'_{IV})^2 = \frac{(B^{11} + \widetilde{C}^{11})}{\widetilde{J}^{11}} + \mathcal{O}(l^2) \tag{7.5}$$

After calculations of coefficients the above formulae can be written in the form

$$\begin{aligned} (\lambda'_{III})^2 &= \frac{4\pi^2 E\{(d_1 - d_2)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2\}}{\rho l^2\{(d_1 - d_2)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2 + 4\pi H\}(1+\nu)} \\ (\lambda'_{IV})^2 &= \frac{4\pi^2 E\frac{1}{1-\nu}\{(d_1^3 - d_2^3)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2^3\}}{\rho l^2\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\}(1+\nu)} + \\ &+ \frac{12\pi^2 E\{(d_1 - d_2)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2\}}{\rho l^2\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\}(1+\nu)} \end{aligned} \tag{7.6}$$

where E, ρ, ν are the Young modulus, the mass density and the Poisson ratio, respectively, d is the plate thickness, $H = M/l\rho$ is the function describing concentrated masses and l is the microstructure length-parameter.

7.1.2. *Refined Kirchhoff model*

Now, using the governing equations of the refined Kirchhoff model (3.11), (3.12) the plate band will be analysed. Describing micromotions by the microshape function in the form $(7.2)_2$ we obtain that $\widehat{m}^1 = 0$, $\langle jg_{,1}\rangle = 0$ and moreover $D_{11} = 0$ (although the plate thickness is not a constant). In this way Eqs (5.2) have the form

$$\begin{aligned} &BW_{,1111} + m\ddot{W} - J\ddot{W}_{,11} = 0 \\ &DV + \underline{m}^{11}\ddot{V} + \underline{J}^{11}\ddot{V} = 0 \end{aligned} \tag{7.7}$$

Above we have the system of two independent differential equations – the first for the macrodeflections W and the second for the internal variable V. The second of these equations is interesting for us, because it depends on the microstructure parameter l. Hence, we will investigate only this equation.

Solutions to Eq $(7.7)_2$ satisfying boundary conditions for the simply supported plate band will be assumed in the form $(5.3)_2$. Substituting the right-hand side of $(5.3)_2$ into Eq $(7.7)_2$ we obtain a linear algebraic equation for the amplitude A_V. After some transformations we obtain the following formula for *the higher resonance frequency*

$$(\omega_2')^2 = \frac{D}{\underline{m}^{11} + \underline{J}^{11}} \tag{7.8}$$

depending on the microstructure of the plate, only.

After some calculations the higher resonance frequency can be written as

$$\begin{aligned}(\omega_2')^2 &= \frac{4\pi^4 E}{\rho l^2(1-\nu^2)}\Big\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\Big\} \cdot \\ &\cdot \Big\{3l^2[(d_1 - d_2)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2 + 4\pi H] + \\ &+ \pi^2[(d_1^3 - d_2^3)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2^3]\Big\}^{-1}\end{aligned} \tag{7.9}$$

where E, ρ, ν are the Young modulus, the mass density and the Poisson ratio, respectively, d is the plate thickness, $H = M/l\rho$ is the function describing concentrated masses and l is the microstructure length-parameter.

As can be seen the higher resonance frequency depends only on the microstructure parameter l and is independent of the wave number k.

7.2. Ritz method

In this section we will investigate the plate band simply supported on the opposite edges ($x = 0$ and $x = l$) and made of an isotropic homogeneous material, with the l-periodic thickness d and two masses M located as in Fig.6. The Ritz method will be used to analyse the resonance frequencies of this plate band. Calculations will be made within the framework of Hencky-Reissner and Kirchhoff theories. The formulae for the potential $\boldsymbol{\mathcal{E}}$ and the kinetic energy $\boldsymbol{\mathcal{K}}$ of this plate band can be written

$$\boldsymbol{\mathcal{E}} = \frac{1}{2}\int_{\Pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} a_{ijkl}e_{ij}e_{kl}\,dzda \qquad \boldsymbol{\mathcal{K}} = \frac{1}{2}\int_{\Pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} \rho(\dot{u}_i)^2\,dzda \tag{7.10}$$

7.2.1. Hencky-Reissner assumptions (Eqs (3.1) ÷ (3.3))

Assuming displacements: $u_1 = zA_1\sin(2\pi x/l)\cos(\omega t)$, $u_2 = 0$, $u_3 = A_2\sin(2\pi x/l)cos(\omega t)$, we can write the formulae for the maximal potential and the kinetic energy

$$
\begin{aligned}
\mathcal{E}_{max} &= \frac{\pi E}{l(1-\nu^2)} A_1^2[(d_1^3 - d_2^3)(\pi\varepsilon - \sin(\pi\varepsilon)) + \pi d_2^3] + \\
&+ \frac{E}{\pi l(1+\nu)}\Big\{\frac{1}{4}A_1^2 l^2[(d_1 - d_2)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2] + \\
&+ \pi^2 A_2^2[(d_1 - d_2)(\pi\varepsilon - \sin(\pi\varepsilon)) + \pi d_2]\Big\} \\
\mathcal{K}_{max} &= \frac{\rho\omega^2 l}{4\pi}\Big\{\frac{1}{12}A_1^2[(d_1^3 - d_2^3)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2^3] + \\
&+ A_2^2[(d_1 - d_2)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2 + 4\pi H]\Big\}
\end{aligned}
$$

The condition of the Ritz method takes the form

$$
\frac{d(\mathcal{E}_{max} - \mathcal{K}_{max})}{dA_1} = 0 \qquad \frac{d(\mathcal{E}_{max} - \mathcal{K}_{max})}{dA_2} = 0
$$

After some transformations we arrive at the formula for resonance frequencies

$$
\begin{aligned}
(\widehat{\lambda}_{III})^2 &= \frac{4\pi^2 E\{(d_1 - d_2)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2\}}{\rho l^2\{(d_1 - d_2)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2 + 4\pi H\}(1+\nu)} \\
(\widehat{\lambda}_{IV})^2 &= \frac{4\pi^2 E\frac{1}{1-\nu}\{(d_1^3 - d_2^3)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2^3\}}{\rho l^2\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\}(1+\nu)} + \\
&+ \frac{12\pi^2 E\{(d_1 - d_2)[\pi\varepsilon - \sin(\pi\varepsilon)] + \pi d_2\}}{\rho l^2\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\}(1+\nu)}
\end{aligned}
\tag{7.11}
$$

We can observe that the above formulae are identical as the higher micro-resonance frequencies (7.6).

7.2.2. Kirchhoff assumptions (Eqs (3.5) and (3.6))

Assuming displacements $w = A\sin(2\pi x/l)\cos(\omega t)$, we can write the formulae for the maximal potential and the kinetic energy

$$
\begin{aligned}
\mathcal{E}_{max} &= A^2\frac{\pi^3 E}{3l^3(1-\nu^2)}[(d_1^3 - d_2^3)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2^3] \\
\mathcal{K}_{max} &= A^2\frac{\rho\omega^2}{4\pi l}\Big\{l^2[(d_1 - d_2)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2 + 4\pi H] + \\
&+ \frac{\pi^2}{3}[(d_1^3 - d_2^3)(\pi\varepsilon - \sin(\pi\varepsilon)) + \pi d_2^3]\Big\}
\end{aligned}
$$

The condition of the Ritz method takes the form

$$\frac{d(\mathcal{E}_{max} - \mathcal{K}_{max})}{dA} = 0$$

After some transformations we arrive at the formula for resonance frequencies

$$\begin{aligned} \widehat{\omega}_2^2 &= \frac{4\pi^4 E}{\rho l^2(1-\nu^2)} \Big\{(d_1^3 - d_2^3)[\pi\varepsilon + \sin(\pi\varepsilon)] + \pi d_2^3\Big\} \cdot \\ &\quad \Big\{3l^2[(d_1 - d_2)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2 + 4\pi H] + \\ &+ \pi^2[(d_1^3 - d_2^3)(\pi\varepsilon + \sin(\pi\varepsilon)) + \pi d_2^3]\Big\}^{-1} \end{aligned} \qquad (7.12)$$

which is identical as the higher frequency (7.9) obtained within the framework of the refined Kirchhoff model.

Summarizing, we can confirm that the above results (7.6) and (7.9) obtained within the framework of the refined models of microperiodic plates are physically correct.

8. Conclusions

We have presented the example, which illustrates applications of refined plate theories to investigations of free vibrations of a microperiodic plate band loaded periodically by the system of concentrated masses. Our considerations are carried out within the framework of *the medium thickness plate theory* (the Hencky-Reissner assumptions) and *the thin plate theory* (the Kirchhoff plate theory assumptions). Using the refined plate theories we can investigate the effect of the microstructure length parameter l on the plate macrobehaviour.

Analysing the results obtained for the refined Hencky-Reissner and Kirchhoff models, respectively, we can formulate the following conclusions:

- Applying the refined plate theories we can get additional higher resonance frequencies: for the Hencky-Reissner assumptions – two microresonance frequencies, for the Kirchhoff assumptions – one resonance frequency

- The effect of the microstructure parameter l on the dynamic plate behaviour should be considered for high frequencies of loadings

- It can be observed from the diagrams shown in Fig.4. that in the case of lower frequencies and the case of higher frequencies we have lower values of them for the Kirchhoff model.

The comparison for the special case of the plate band between the results obtained for the refined models and the Ritz method leads to the conclusion that we can restrict to one microshape function and these results are physically correct.

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O drganiach płyt z periodycznym rozkładem obciążenia inercyjnego

Streszczenie

Cel pracy jest podwójny. Pierwszym celem jest zastosowanie strukturalnej makrodynamiki płyt mikroperiodycznych do badania częstości drgań własnych cienkich płyt Kirchhoffa i płyt średniej grubości Hencky'ego-Reissnera oraz porównanie otrzymanych wyników. Drugi cel to porównanie wyników otrzymanych wg modeli strukturalnych dla pewnych szczególnych przypadków z wynikami uzyskanymi znanymi przybliżonymi metodami oraz pokazanie poprawności stosowanych modeli. Założenia przedstawionych sposobów modelowania liniowo-sprężystych płyt o strukturze mikroperiodycznej w płaszczyznach równoległych do płaszczyzny środkowej płyty zostały przedstawione w pracach Barona i Woźniaka (1995) oraz Jędrysiaka i Woźniaka (1995). Podejście to pozwala uwzględnić wpływ wielkości mikrostruktury na dynamikę płyt (tzw. efekt skali).

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