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NONLINEAR DISCRETE-CONTINUOUS MODELS IN THE ANALYSIS OF LOW STRUCTURES SUBJECT TO KINEMATIC EXCITATIONS CAUSED BY TRANSVERSAL WAVES

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The paper deals with dynamic analysis of low structures subject to kinematic excitations caused by transversal waves when using nonlinear discrete-continuous models. The models consist of rigid bodies and elastic elements which undergo only shear deformations. In these models discrete elements with a damper and a spring of a nonlinear characteristic representing local nonlinearities can be included. In the study a wave approach is used, in terms of the wave solution of the equations of motion. Numerical calculations are performed for model with single, two, three or four rigid bodies. They focus on the determination of amplitude-frequency curves and investigation into the effect of local nonlinearities on displacements of selected cross-sections of elastic elements in the considered models.

Key words: nonlinear dynamics, discrete-continuous models, waves

1. Introduction

In the paper nonlinear discrete-continuous models are proposed for dynamic investigation of low structures subject to transversal kinematic excitations. The discrete-continuous models consist of rigid bodies connected by means of ponderable elastic elements. Continuous elastic elements in these models are assumed to be described by the classical wave equation representing a beam in which shear deformations are the dominating ones. In the discrete-continuous model some additional discrete elements can be included. These elements consist of a spring and a damper. The spring may reveal a nonlinear characteristic.

Linear discrete-continuous models of low structures subject to shear deformations were discussed by Pielorz (1996). In that paper the cases when the use of the classical wave equation is justifiable were also shown. The results obtained confirmed the suggestions put forward by Humar (1990) that many structural systems could be described by the classical wave equation, e.g. low buildings and an isotropic or horizontally layered soil deposit undergoing horizontal deformations.

The aim of the present paper is to generalize the results obtained by Pielorz (1996) by introducing local nonlinearities into the models studied by Pielorz (1996) and to investigate the influence of these nonlinearities on displacements of selected cross-sections of ponderable elastic elements in the models. The inclusion of that type of nonlinearities is justified by many engineering solutions for low structures, see e.g. Humar (1990), Mengi and Dündar (1988), Okamoto (1973), Sackman and Kelly (1979), Su et al. (1989). Generally, from the literature it follows that in each structure one can find the elements which can be taken into account in the dynamic analysis as local nonlinearities.

In the literature there is a lack of papers dealing with nonlinear discrete-continuous models in contrast to the vast literature about nonlinear discrete models, cf Hagedorn (1981), Mickens (1981), Szemplińska-Stupnicka (1990).

In the present paper the approach utilized by Pielorz (1996) for linear models of structures undergoing shear deformations is adopted to nonlinear cases. Investigations are limited to the systems with a nonlinearity represented by a spring of hard characteristic of the Duffing type. In numerical calculations the effect of the local nonlinearity parameters on amplitude-frequency curves for selected multi-mass systems is considered.

2. Assumptions and governing equations

The paper concerns dynamic investigations of low structures subject to kinematic excitation caused by transversal waves. Kinematic excitations can be of the seismic type or can be caused by highway traffic, surface and subsurface railways, or machinery located nearby. In the literature, engineering structures subject to various kinematic excitations are discussed in terms of discrete as well as continuous models, cf Okamoto (1973), Sackman and Kelly (1979) and Mengi and Dündar (1988).

The elastic elements of the structures considered in the present paper have the transverse dimension, alongside of which shear forces act, close to the element length, i.e. they have a low slenderness ratio. Those structures are, e.g., machine supports, bridge piers and low columns in buildings. Many structure elements subject to transversal excitation can be modelled by means of the Timoshenko beam. In the paper by Pielorz (1996) it is shown that in the case of short beams in which the shear forces are predominant, the Timoshenko equations can be replaced by the classical wave equation. Some suggestions about applying the classical wave equations without any discussion on frequencies were given by Humar (1990).

The use of the classical wave equation enables one to discuss the models of engineering structures consisting of many elastic elements and of rigid bodies. The approach applying the classical wave equation and its wave solution is used in dynamic investigations of the nonlinear discrete-continuous model shown in Fig.1. Special cases of this model can be employed in the investigation of the structures mentioned above.

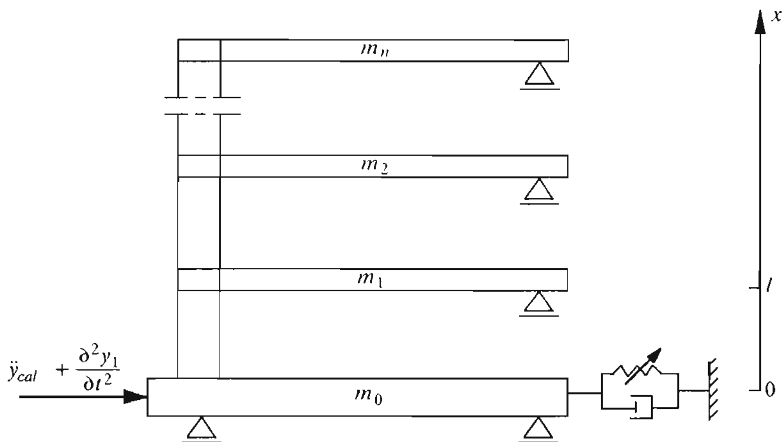

Fig. 1. Nonlinear discrete-continuous model

The studied model consists of n elastic elements connected by rigid bodies. During external excitations all cross-sections of the elastic elements remain flat and parallel to the cross-sections where rigid bodies are located. The elastic elements udergo only shear deformations. They may reveal different mechanical properties, however for the sake of simplicity it is assumed that all the elements are characterized by the shear modulus G, the cross-sectional area A, shear coefficient k, density ρ and length l. To the rigid body m_0 a discrete element with a nonlinear spring can be attached. Such an element may represent any part of the considered structure which needs description by local nonlinearities. For example, it may represent an elastic segment of isolation type, cf Su et al. (1989), Humar (1990). The characteristic of nonlinear spring is assumed to be of a hard type, and the force for the spring is expressed by the third-degree polynomial

$$F(t) = k_1 y_1(x,t) + k_3 y_1^3(x,t) \qquad \text{for} \quad x = 0 \tag{2.1}$$

where

$y_i(x,t)$	–	displacement of the ith elastic element
k_1, k_3	–	linear and nonlinear terms in Eq (2.1), respectively.

The nonlinear function of the type (2.1) is widely exploited in the literature in dynamic investigations into nonlinear discrete systems, cf Hagedorn (1981), Mickens (1981), Szemplińska-Stupnicka (1990). The function (2.1) can be used in the cases of the hard and soft characteristic of the nonlinear spring. Here, the investigations are limited to the nonlinear spring of the hard characteristic, so it is assumed that $k_3 > 0$.

The rigid body m_0 is subject to the absolute acceleration $\partial^2[y_1(0,t) + y_{cal}(t)]/\partial t^2$, where $y_1(0,t)$, is the displacement of the rigid body m_0 relative to the ground and $y_{cal}(t)$ is the ground displacement relative to the fixed spatial system.

Damping in the model is described by means of the equivalent external and internal damping

$$R_{di} = d_i y_{i,t} \qquad R_{Vi} = D_i y_{i,xt} \qquad i = 0,1,...,n \tag{2.2}$$

where the constants d_i and D_i represent the coefficients of external and internal damping, respectively, and the comma denotes partial differentiation. The equivalent damping is taken into account in the boundary conditions. It is assumed that the x-axis direction is normal to the direction of displacements y_i, its origin coincides with the location of the rigid body m_0 in the undisturbed state and velocities and displacements of the cross-sections of all the elastic elements are equal to zero at the instant $t = 0$.

Determination of the displacements, strains and velocities in the cross-sections of the elastic elements for the analysed model reduces to solving the following n classical wave equations

$$y_{i,tt} - c^2 y_{i,xx} = 0 \qquad \text{for} \quad i = 1,2,...,n \tag{2.3}$$

with the initial conditions

$$y_i(x,0) = y_{i,t}(x,0) = 0 \qquad i = 1,2,...,n \tag{2.4}$$

and the nonlinear boundary conditions

$$
\begin{aligned}
& -m_0[\ddot{y}_{cal}(t) + y_{1,tt}] - d_0 y_{1,t} + AkG(D_1 y_{1,xt} + y_{1,x}) + \\
& -k_1 y_1 - k_3 y_1^3 = 0 && \text{for } x = 0 \\
& y_i = y_{i+1} && \text{for } x = il \quad i = 1,2,...,n-1 \\
& -AkG(D_i y_{i,xt} + y_{i,x}) + AkG(D_{i+1} y_{i+1,xt} + y_{i+1,x}) + \\
& -m_i y_{i+1,tt} - d_i y_{i+1,t} = 0 && \text{for } x = il \quad i = 1,2,...,n-1 \\
& -AkG(D_n y_{n,xt} + y_{n,x}) - m_n y_{n,tt} - d_n y_{n,t} = 0 && \text{for } x = nl
\end{aligned}
\tag{2.5}
$$

where $c^2 = kG/\rho$. Eqs (2.5) represent the conditions for displacements and forces acting in the contacting cross-sections of neighbouring elastic elements of the considered model, and $y_{cal}(t)$ is a given time function representing the external excitation which can be irregular (cf Okamoto, 1973; Sackman and Kelly, 1979; Mengi and Dündar, 1988) or regular. Eqs (2.5) differ from the appropriate boundary conditions for linear discrete-continuous models discussed by Pielorz (1996) by the nonlinear term in Eq $(2.5)_1$. When $k_1 = k_3 = 0$ they become the same.

Although the equations of motion which take into account continuously distributed damping would describe the problem more precisely, they cannot be solved as effectively as classical wave equations using the wave method. Moreover, it was shown by Pielorz (1988) that the appropriate equations with continuously distributed damping and with the equivalent damping yield practically the same results beyond a short initial time interval.

Upon the introduction of the nondimensional quantities

$$
\begin{array}{lll}
\bar{x} = \dfrac{x}{l} & \bar{t} = \dfrac{ct}{l} & K_r = \dfrac{A\rho l}{m_r} \\
\bar{D}_i = \dfrac{D_i c}{l} & \bar{d}_i = \dfrac{d_i l}{m_r c} & \bar{y}_i = \dfrac{y_i}{y_r} \\
R_i = \dfrac{m_i}{m_r} & \bar{k}_1 = \dfrac{k_1 l^2}{m_r c^2} & \bar{k}_3 = \dfrac{k_3 y_r^2 l^2}{m_r c^2}
\end{array}
\tag{2.6}
$$

where m_r and y_r are the fixed mass and displacement, respectively, Eqs (2.3) ÷ (2.5) can be rewritten as

$$
y_{i,tt} - y_{i,xx} = 0 \qquad \text{for } i = 1,2,...,n
$$

$$
y_i(x,0) = y_{i,t}(x,0) = 0 \qquad \text{for } i = 1,2,...,n
$$

$$
\begin{aligned}
&R_0\ddot{y}_{cal}(t) + R_0 y_{1,tt} + d_0 y_{1,t} - K_r(D_1 y_{1,xt} + y_{1,x}) + \\
&+k_1 y_1 + k_3 y_1^3 = 0 && \text{for } x = 0 \\
&y_i = y_{i+1} && \text{for } x = i \quad i = 1,2,...,n-1 \\
&K_r(D_i y_{i,xt} + y_{i,x}) - K_r(D_{i+1} y_{i+1,xt} + y_{i+1,x}) + R_i y_{i+1,tt} + \\
&+d_i y_{i+1,t} = 0 && \text{for } x = i \quad i = 1,2,...,n-1 \\
&K_r(D_n y_{n,xt} + y_{n,x}) + R_n y_{n,tt} + d_n y_{n,t} = 0 && \text{for } x = n
\end{aligned}
\tag{2.7}
$$

For convenience, in Eqs (2.7) the bars denoting nondimensional quantities are omitted.

The solutions of Eq $(2.7)_1$ taking into account the initial conditions $(2.7)_2$ are sought in the form

$$
y_i(x,t) = f_i(t-x) + g_i\Big(t + x - 2(i-1)\Big) \qquad i = 1,2,...,n \tag{2.8}
$$

where unknown functions f_i and g_i represent the waves caused by kinematic excitation, propagating in the ith elastic element of the discrete-continuous model in the directions consistent and opposite to the x-axis direction, respectively. In the sought solution (2.8) it is assumed that the first disturbance occurs in the ith element at the instant $t = i-1$ in the cross-section $x = i-1$ for $i = 1,2,...,n$. The functions f_i and g_i are continuous and for negative arguments identical to zero.

Upon substituting the solution (2.8) into the boundary conditions $(2.7)_{3-6}$ and denoting the largest argument of functions appearing in each equality by z, the following nonlinear equations are obtained for the functions f_i and g_i

$$
\begin{aligned}
&g_i(z) = f_{i+1}(z-2) + g_{i+1}(z-2) - f_i(z-2) && i = 1,2,...,n-1 \\
&r_{n+1,1}g_n''(z) + r_{n+1,2}g_n'(z) = r_{n+1,3}f_n''(z-2) + r_{n+1,4}f_n'(z-2) \\
&r_{11}f_1''(z) = -R_0\ddot{y}_{cal}(z) + r_{12}g_1''(z) + r_{13}f_1'(z) + r_{14}g_1'(z) + \\
&-k_1[f_1(z) + g_1(z)] - k_3[f_1(z) + g_1(z)]^3 \\
&r_{i1}f_i''(z) + r_{i2}f_i'(z) = r_{i3}g_i''(z) + r_{i4}g_i'(z) + r_{i5}f_{i-1}''(z) + \\
&+r_{i6}f_{i-1}'(z) && i = 2,3,...,n
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
&r_{11} = K_r D_1 + R_0 && r_{12} = K_r D_1 - R_0 \\
&r_{13} = -K_r - d_0 && r_{14} = K_r - d_0 \\
&r_{i1} = K_r D_i + K_r D_{i-1} + R_{i-1} && r_{i2} = 2K_r + d_{i-1} \\
&r_{i3} = K_r D_i - K_r D_{i-1} - R_{i-1} && r_{i4} = -d_{i-1} \\
&r_{i5} = 2K_r D_{i-1} && r_{i6} = 2K_r \qquad i = 2, 3, ..., n \\
&r_{n+1,1} = K_r D_n + R_n && r_{n+1,2} = K_r + d_n \\
&r_{n+1,3} = K_r D_n - R_n && r_{n+1,4} = K_r - d_n
\end{aligned}
\tag{2.10}
$$

Eqs (2.9) are nonlinear differential equations with a retarded argument. Though appropriate equations for linear models can be solved analytically or numerically by means of the finite difference method, cf Nadolski and Pielorz (1980) and (1992), Pielorz (1996), the nonlinear equations (2.9) can be solved only numerically using e.g. the Runge-Kutta method. Having obtained from Eqs (2.9) the functions $f_i(z)$ and $g_i(z)$ and their derivatives, one can determine displacements, strains and velocities in an arbitrary cross-section of the elastic elements in the considered model at an arbitrary instant. The solution can be obtained in both transient and steady states.

3. Numerical results

The numerical analysis is carried out for the model presented in Fig.1 when $n = 1, 2, 3$ with one, two, three or four rigid bodies, respectively. An arbitrary function of the external excitation $\ddot{y}_{cal}(t)$ can be irregular or regular, periodic or nonperiodic. In the paper it is assumed in the form

$$
\ddot{y}_{cal}(t) = a_0 \sin(pt) \tag{3.1}
$$

and the considerations focus on the determination of displacements in the steady states. The function (3.1) represents various direct and indirect external excitations, where p is the nondimensional frequency of the external excitation.

The considered discrete-continuous systems representing low structures are described by the nondimensional parameters R_i, K_r, see Eqs (2.6). These parameters can take various values. The constants R_i are the ratios between masses m_i and the foundation mass m_0 while the constant K_r is the ratio between the mass of columns and m_0. For real structures those parameters are usually smaller than 1. In the presented calculations they are assumed to be equal 0.5 and 0.3, respectively (cf Pielorz, 1996), where the linear models were studied and the efficiency of the wave method was demonstrated.

For that reason, in numerical calculations in the present paper we concentrate on the representation of the influence of local nonlinearity on displacements in selected cross-sections. The effect of the assumed nonlinearity is represented by the parameter k_3 standing by the nonlinear term in Eq (2.1) and by changes in the damping coefficient d_0 and amplitude a_0 of the external excitation (3.1). This is done for the nondimensional value of k_1 in (2.1) equal to 0.3.

3.1. Single-mass system

The form of the external excitation (3.1) enables us also to perform some comparative numerical analysis. Such an analysis can be done in a special case of the discrete-continuous model presented in Fig.1; namely, for the system consisting of a single elastic element and a single rigid body with the higher end being fixed (cf Pielorz, 1995), using the wave approach and the method of variables separation and neglecting the internal damping.

When applying nondimensional quantities (2.6), the analysis of steady motion of the system with a single mass is reduced to solving the equation of motion

$$K_r(y_{1,tt} - y_{1,xx}) = 0 \tag{3.2}$$

with the nonlinear boundary conditions

$$\begin{aligned} &R_0 y_{1,tt} + d_0 y_{1,t} - K_r y_{1,x} + k_1 y_1 + k_3 y_1^3 = a_0 \sin(pt) && \text{for } x = 0 \\ &y_1(x,t) = 0 && \text{for } x = 1 \end{aligned} \tag{3.3}$$

where the bars are omitted for convenience.

When using the wave approach, the considerations are reduced to finding the solution of Eqs (2.9) in the steady state with $n = 1$ and $D_1 = 0$, $R_1 = \infty$. On the other hand, seeking a single modal solution of Eq (3.2) in the form

$$y_1(x,t) = X_1(x)T(t) \tag{3.4}$$

where

$$X_1(x) = (k_1 - R_0\omega_1^2)(\omega_1 K_r)^{-1}\sin(\omega_1 x) + \cos(\omega_1 x) \tag{3.5}$$

is the first eigenfunction in the linear case with $X_1(0) = 1$, according to the Galerkin method the following nonlinear equation for the unknown function $T(t)$ from (3.2) and (3.3) is derived

$$\ddot{T} + d_0'\dot{T} + \omega_1^2 T + k_3' T^3 = a_0' \sin(pt) \tag{3.6}$$

with

$$d_0' = \frac{d_0}{S_0} \qquad k_3' = \frac{k_3}{S_0} \qquad a_0' = \frac{a_0}{S_0} \qquad S_0 = K_r \int\limits_0^1 X_1^2\, dx + R_0 \tag{3.7}$$

The first approximation of the solution of eq (3.6) according to the Duffing method is sought in the form, cf Hagedorn (1981), Szemplińska-Stupnicka (1990)

$$T(t) = A \sin(pt - \gamma) \tag{3.8}$$

where γ is a phase angle. This leads to the relation between the amplitude a_0' of the external excitation and the amplitude A of the solution (3.8) in the form

$$A^2 \Big[(\omega_1^2 - p^2) + \frac{3k_3' A^2}{4}\Big] + p^2 (d_0')^2 A^2 = (a_0')^2 \tag{3.9}$$

Numerical calculations for the single-mass system are made for the following nondimensional quantities

$$\begin{array}{llll} K_r = 0.3 & \omega_1 = 0.737 & d_0 = 0.05 & n = 1 \\ a_0 = 1 & R_1 = \infty & k_1 = 0.3 & k_3 = 0,\ 0.01,\ 0.05,\ 0.1 \end{array} \tag{3.10}$$

The value of the frequency of free vibration $\omega_1 = 0.737$ is obtained for given K_r and k_1 from the frequency equation in the linear case of the examined system.

Amplitude-frequency curves for the single-mass system, according to Eq (3.9) and when using the wave approach are presented in Fig.2. Continuous and broken lines denote stable and unstable branches, respectively, for the solution (3.8), while stars correspond to the steady solution in the cross-section $x = 0$ for the single-mass system with the parameters (3.10). Diagrams in Fig.2 show a good agreement between the results obtained for the nonlinear equation of Duffing type and the nonlinear single-mass discrete-continuous model. The above comparative numerical analysis was made not only in order to show the agreements with available analytical solutions but also in order to estimate the values of the coefficient k_3 representing local nonlinearities of the type (2.1). From Fig.2 it follows that for the assumed value of the parameter $k_1 = 0.3$ the value of k_3 should not exceeded 0.1. Similar comparable calculations can be done for other values of k_1.

3.2. Multi-mass systems

The equations with a retarded argument (2.9) describe more complex systems than the nonlinear equation (3.6). Eq (3.6) concerns vibrations in the first

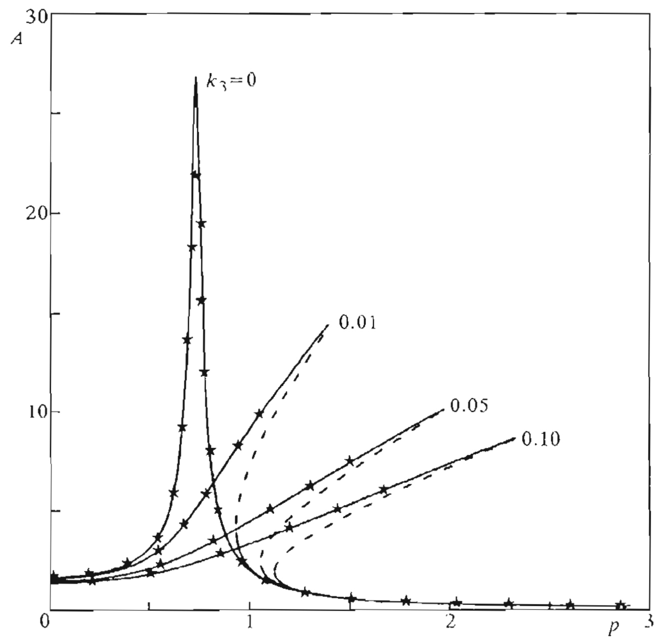

Fig. 2. Amplitude-frequency curves of the single-mass system according to the Duffing equation (continuous and broken lines) and according to the wave approach (stars)

resonant region for the single-mass system and the solution (3.8) concerns the steady motion only, while the nonlinear equations (2.9) enable us to obtain solutions in the first and further resonant regions and in transient states. For that reason, in further numerical analysis of nonlinear discrete-continuous systems with more than one rigid body only the wave approach is adopted, i.e., Eqs (2.9) are employed.

Eqs (2.9) are solved from $z = 0$ until the steady state for displacements expressed by (2.8) with zero initial conditions for the functions f_i, g_i and their derivatives. Then, for each frequency p of the external excitation (3.1) there exists a value p_0 for which the displacement amplitudes jump from the upper to lower curves. However, Eqs (2.9) can be also solved with nonzero initial conditions. Eqs (2.9) have a retarded argument with the argument shift equal to 2, so the nonzero values of functions $f_i(z)$, $g_i(z)$ and their derivatives should be known in the interval $< -2, 0 >$. If Eqs (2.9) with $p > p_0$ are solved up to $z_k = 2\pi m/(p_0 + k\Delta p)$, $k = 1, 2, \ldots$ and m is a fixed integer, and next the values of the functions $f_i(z)$, $g_i(z)$ and their derivatives from the interval $< z_k - 2, z_k >$ are assumed to be the initial values of these functions in the interval $< -2, 0 >$ for $p = p_0 + (k+1)\Delta p$, then the displacement amplitudes y_A with these initial conditions lay on the prolongation of the upper amplitude-frequency curves up to the next jump.

Sample numerical calculations are made for two-, three- and four-mass nonlinear discrete-continuous systems shown in Fig.1. They are done for the following nondimensional parameters (cf Pielorz, 1996)

$$
\begin{array}{lll}
R_0 = 1.0 & a_0 = 1.0 & \\
K_r = 0.3 & m_r = m_0 & \\
R_i = 0.5 & d_i = D_i = d_0 = 0.05 & i = 1, 2, ..., n \\
k_1 = 0.3 & k_3 = 0,\ 0.01,\ 0.025,\ 0.05,\ 0.075,\ 0.1 &
\end{array}
\tag{3.11}
$$

The efficiency of the method applied in the paper was shown by Pielorz(1996) in the case of linear discrete-continuous systems for low structures subject to shear deformations. For that reason, in the present paper we concentrate on the representation of the influence of local nonlinearity on amplitude-frequency curves for selected cross-sections of the elastic elements in the considered models.

3.2.1. Two-mass system

Eqs (2.9) together with Eqs (2.8) enable one to determine displacements in arbitrary cross-sections of elastic elements of the multi-mass system shown in Fig.1. Amplitude-frequency curves for the two-mass system $(n = 1)$ for the cross-sections $x = 0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$ and for $k_3 = 0.05$ are plotted in Fig.3 for a frequency $p < 3.8$ of the external excitation (3.1) including 3 resonant regions $(\omega_1 = 0.379,\ \omega_2 = 0.975,\ \omega_3 = 3.407)$. From Fig.3 it follows that the maximal displacement amplitudes in the first resonant region increase with the increase of x and that such regularity does not occur in the remaining resonant regions. Amplitude jumps, typical for the nonlinear discrete systems (cf Hagedorn, 1981) occur in the second resonant region. The jumps appear place in all considered cross-sections of the model for the same frequencies of external excitations. In the third resonant region no effects of the local nonlinearity are observed. The solution in this region coincide with the solution in the linear case.

From Fig.3 it follows that the effect of the local nonlinearity can be investigated for an arbitrary cross-section of the elastic element in the considered model. In order to avoid plotting too many diagrams, we concentrate on the study of this effect only for the cross-section $x = 0$ for $p \leqslant 2$. The effect of the assumed nonlinearity is represented by the parameter k_3 standing by the

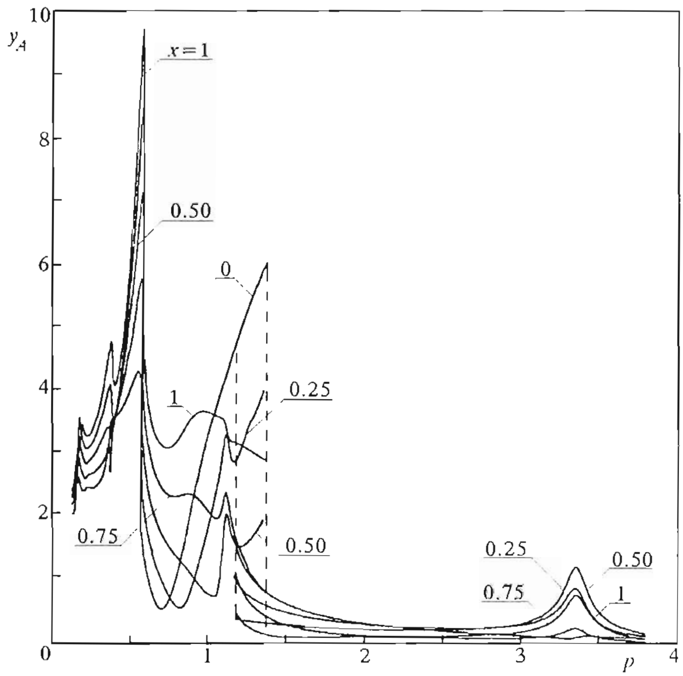

Fig. 3. Amplitude-frequency curves for the two-mass system for $x = 0, 0.25, 0.5, 0.75, 1$ with $k_3 = 0.05$

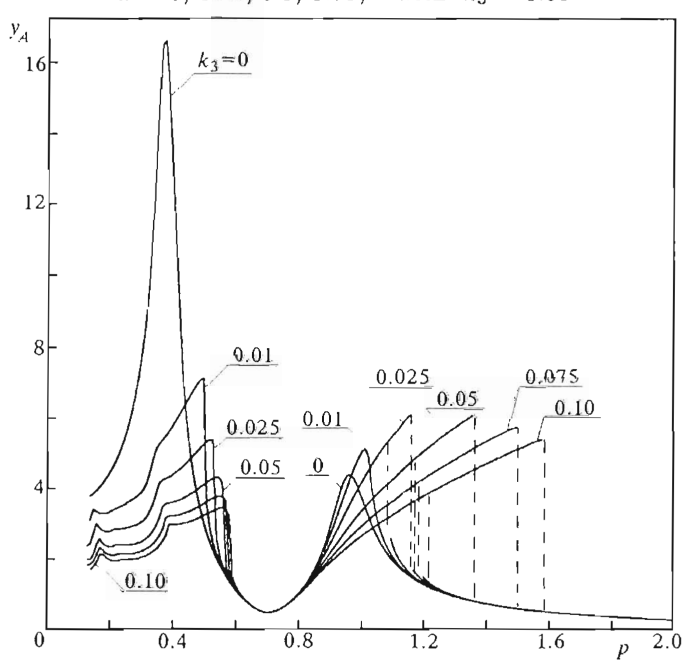

Fig. 4. The effect of k_3 for the two-mass system with $d_0 = 0.05$, $a_0 = 1$

nonlinear term in Eq (2.1) and by the changes of the damping coefficient d_0 and amplitude a_0 of the external excitation (3.1).

The amplitude-frequency curves for the displacement in the cross-section $x = 0$ are plotted in Fig.4 in the linear case as well as for the nonlinear model with $k_3 = 0.01$, 0.025, 0.05, 0.075, 0.1. From this figure it follows that in the first resonant region the displacement amplitude decreases with the increase in the parameter k_3 while in the second resonant region the amplitudes in nonlinear cases are higher than those in the linear case. The highest amplitude in the second resonant region is observed for $k_3 = 0.05$. The jump phenomenon typical for nonlinear discrete systems occurs for $k_3 > 0.01$. The distance between jumps for a fixed k_3 grows and the angle of diagrams inclination to p-axis decreases with the increase in k_3.

Diagrams in Fig.5 and Fig.6 show the influence of damping ($d_0 = 0.05$, 0.075, 0.1) and the amplitude of the external excitation ($a_0 = 0.5$, 1.0, 1.5), respectively, on amplitude-frequency curves for $x = 0$ for the two-mass system. From these figures it follows that in the both considered resonant regions the displacement amplitudes increase with the decrease of damping and with the increase in the amplitude a_0 of the external excitation. Amplitude jumps can occur in the second resonant region and the distances between jumps increase with the decrease of damping and with the increase in a_0, respectively.

3.2.2. Three-mass system

Diagrams in Fig.7 $\div$ Fig.9 concern the three-mass system with $n = 2$ and parameters (3.11). The amplitude-frequency curves for cross-sections $x = 0$, 0.5, 1.0, 1.5, 2.0, $k_3 = 0.05$, $d_0 = 0.05$, $a_0 = 1$ and for $p < 4$ are shown in Fig.7 including 5 resonant regions ($\omega_1 = 0.28$, $\omega_2 = 0.727$, $\omega_3 = 1.187$, $\omega_4 = 3.275$, $\omega_5 = 3.595$). One can notice that in the first resonant region the maximal amplitudes increase with the increase in x while in the remaining resonant regions no regularities of that type occur. The maximum amplitudes in the second, third and fourth resonant regions are obtained for $x = 2.0$, $x = 1.0$ and $x = 0.5$, respectively. In the second resonant region the jump phenomenon typical of the nonlinear discrete systems appears. It appears for the same frequencies of the external excitation for all considered cross-sections.

The effect of the parameter k_3 in the case of the three-mass system is shown in Fig.8 and Fig.9 for amplitudes of displacements in $x = 0$ and for amplitudes of nonlinear forces (2.1) applied in the cross-section $x = 0$. The diagrams are plotted out with $d_0 = 0.05$ and $a_0 = 1.0$ and for $p < 2$. From diagrams in Fig.8 it follows that displacement amplitudes in the first resonant region decrease with the increase in k_3, in the second resonant region

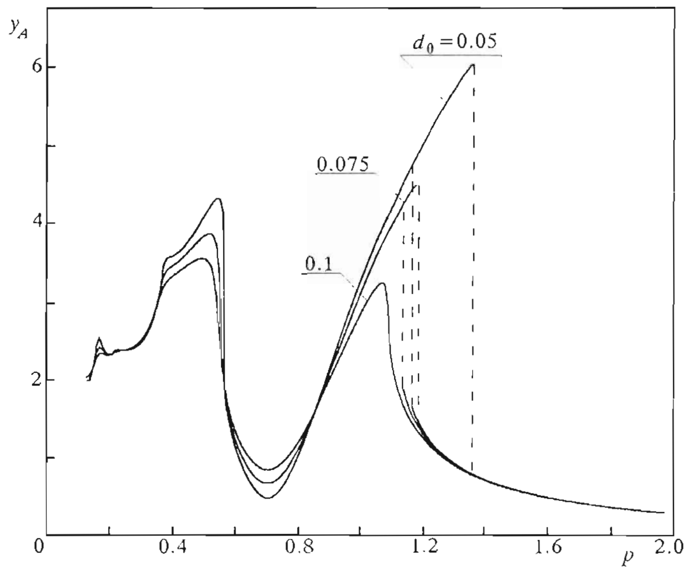

Fig. 5. The effect of damping for the two-mass system with $k_3 = 0.05$, $a_0 = 1$

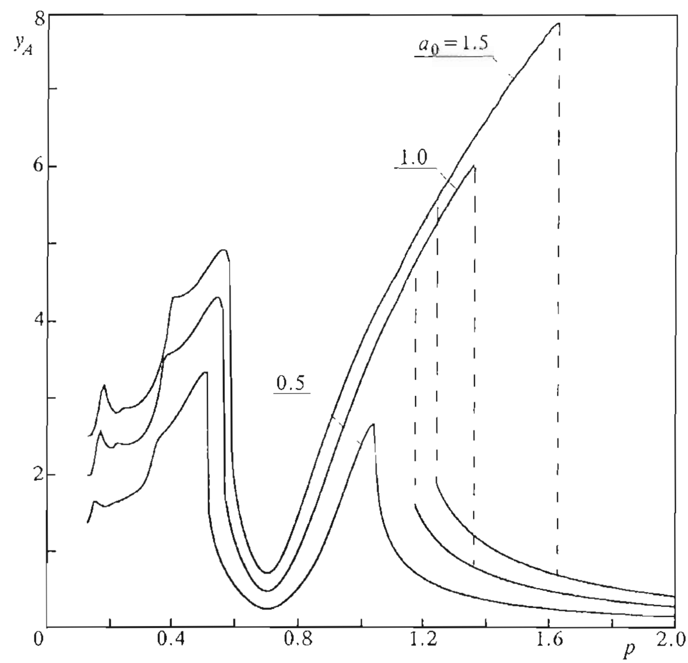

Fig. 6. The amplitude effect of the external excitation for the two-mass system with $k_3 = 0.05$, $d_0 = 0.05$

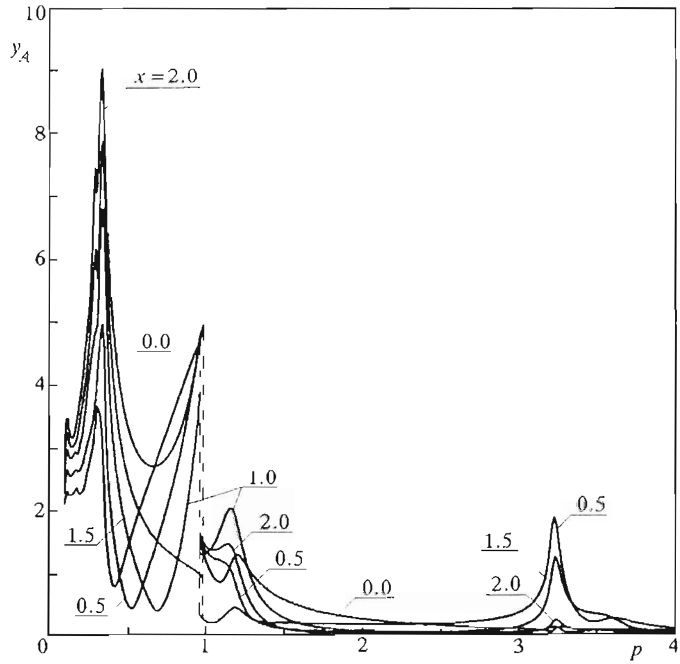

Fig. 7. Amplitude-frequency curves for the three-mass system for $x = 0,\ 0.5,\ 1,\ 1.5,\ 2$ with $k_3 = 0.05$

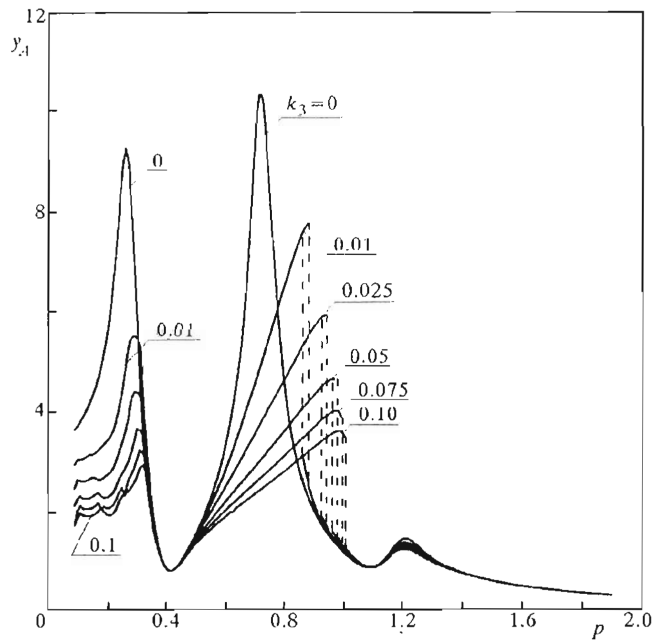

Fig. 8. The effect of k_3 on the displacement amplitudes for the three-mass system with $d_0 = 0.05$, $a_0 = 1$

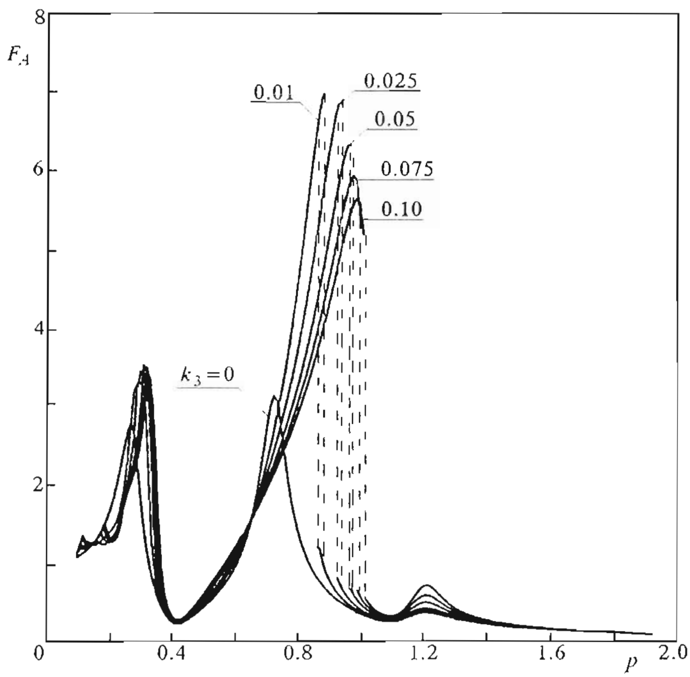

Fig. 9. The effect of k_3 on the force amplitudes for the three-mass system with $d_0 = 0.05$, $a_0 = 1$

they also decrease with the increase in k_3 and in the third resonant region its influence is rather weak. The jump phenomenon typical of the nonlinear discrete systems occurs in the second resonant region. From the amplitude-frequency diagrams for the spring force (2.1) shown in Fig.9 it follows that the strongest effect of the parameter k_3 is noticed in the second resonant region where the amplitude jumps occur. The amplitudes in nonlinear cases are much higher than for $k_3 = 0$.

The effects of damping and the amplitude of the external excitation are presented in Fig.10 and Fig.11. Respective diagrams of amplitude-frequency curves for $x = 0$ in Fig.10 obtained with $d_0 = 0.05$, 0.1, 0.15, 0.2 and diagrams in Fig.11 obtained with $a_0 = 0.1$, 0.25, 0.5, 1.0, 1.5 inform that in all the considered resonant regions displacement amplitudes of the cross-section $x = 0$ in the three-mass system decrease with the increase of damping and with the decrease in external excitation amplitude. In the second resonant region these amplitudes may suffer the jump.

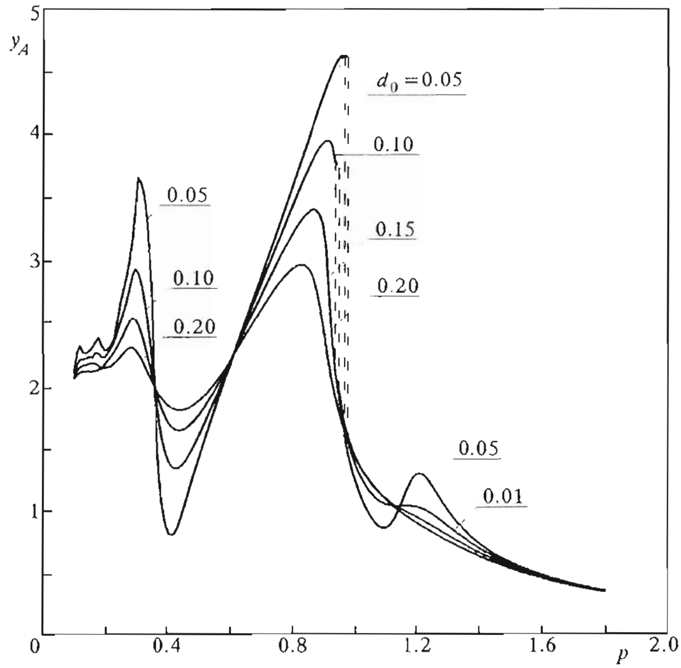

Fig. 10. The effect of damping for the three-mass system with $k_3 = 0.05$, $a_0 = 1$

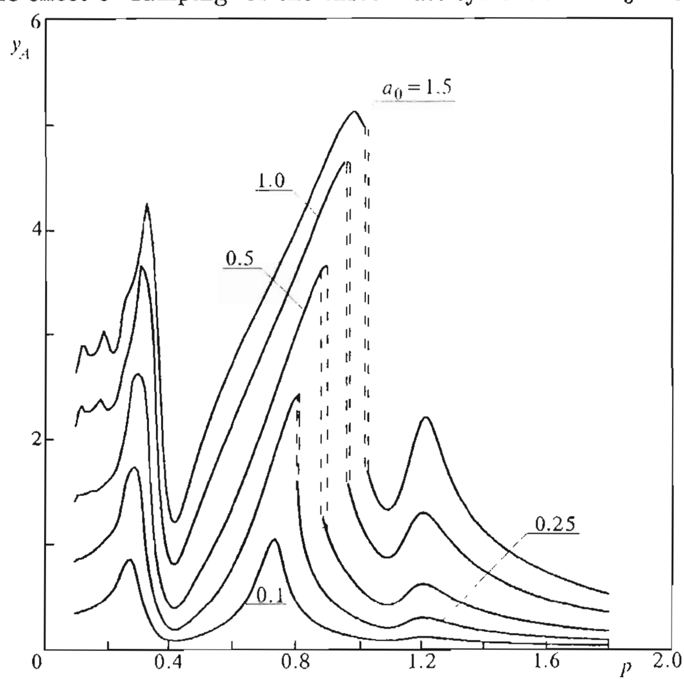

Fig. 11. The amplitude effect of the external excitation for the three-mass system with $k_3 = 0.05$, $d_0 = 0.05$

3.2.3. Four-mass system

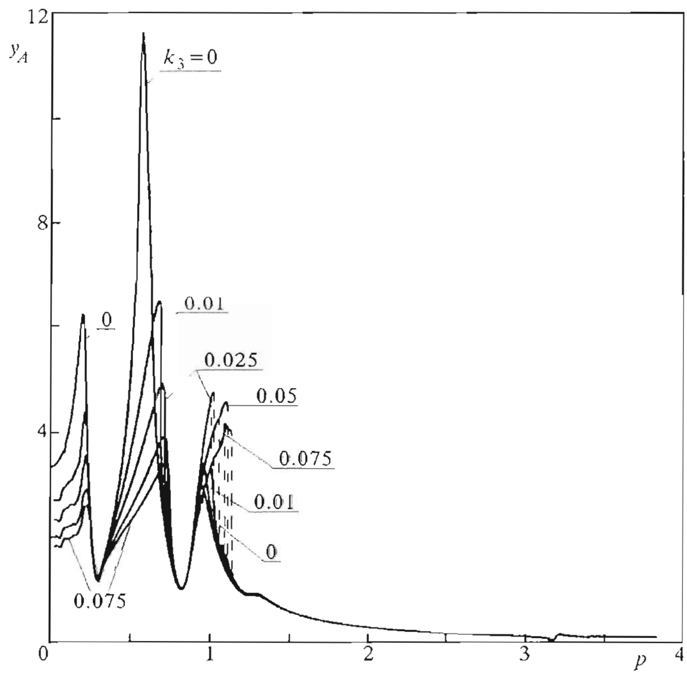

Fig. 12. The effect of k_3 on the displacement amplitudes for the four-mass system with $d_0 = 0.05$, $a_0 = 1$

Similar numerical results can be obtained for more complex discrete-continuous models having more than 3 rigid bodies. Such calculations were performed for four-mass and five-mass systems. Some of these results for the four-mass system are presented in Fig.12. Namely, those diagrams show the effect of the parameter k_3 on amplitude-frequency curves for $x = 0$ including 7 resonant regions ($\omega_1 = 0.22$, $\omega_2 = 0.594$, $\omega_3 = 0.949$, $\omega_4 = 1.278$, $\omega_5 = 3.222$, $\omega_6 = 3.442$, $\omega_7 = 3.667$). One can notice that the amplitude jumps may appear in the third resonant region. Similarly as in Fig.4 and Fig.8 showing the effect k_3 for the two-mass and three-mass systems, respectively, displacement amplitudes in resonant regions before the jumps decrease with the increase of k_3 while after the amplitude jumps no effect of the local nonlinearity is observed. In the case of the five-mass system, amplitude jumps are noticed also in the third resonant region. Suitable amplitude-frequency diagrams for the five-mass system are not given here.

It seems to be interesting to compare the above results with the numerical results presented by Pielorz (1995) for nonlinear discrete-continuous models of mechanical systems undergoing torsional deformations. From those results it follows that for the assumed parameters corresponding to drive systems jump phenomenon are observed in the ith resonant region for the system having i rigid bodies. In the systems under considerations such regularity does not occur. Here, for the two-mass and three-mass systems the amplitude jumps appear in the second resonant region and for the four-mass and the five-mass systems in the third resonant region.

4. Final remarks

The considerations presented in this paper have a theoretical character. From them it follows that the method utilizing the wave solution of the equations of motion can be an efficient tool in dynamic investigations not only of linear but also of nonlinear vibrations of dicrete-continuous models of systems subject to transverse kinematic excitations. The accuracy of the method has been examined in the case of the single-mass system the analysis of which can be reduced to studying a nonlinear equation of the Duffing type.

In the discussed discrete-continuous systems with the local nonlinearity, some properties of the solution corresponding to the Duffing equation are observed. Namely, in the amplitude-frequency curves determined for multi-mass systems the jump phenomenon may appear in the second or third resonant region. The presented diagrams show not only the effect of the parameters representing the local nonlinearity, but also give some information about the regions where the local nonlinearity can be neglected.

Numerical calculations concerning the amplitude-frequency curves are performed for selected values of the constant parameters describing the analysed discrete-continuous models of low structures. Analogous calculations can be executed for other values of these parameters. However, these results will differ from those given in the present paper only quantitatively and not qualitatively. The main conclusions about the effect of the local nonlinearity on the behaviour of discrete-continuous systems remain valid.

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Nieliniowe modele dyskretno-ciągłe w analizie niskich obiektów poddanych wymuszeniom kinematycznym wywołanym falami poprzecznymi

Streszczenie

Praca dotyczy analizy dynamicznej niskich obiektów poddanych poprzecznym wymuszeniom kinematycznym wykorzystując nieliniowe modele dyskretno-ciągłe. Modele te składają się z brył sztywnych i elementów sprężystych poddanych tylko odkształceniom ścinającym. W modelach tych można uwzględnić dyskretne elementy ze sprężyną o nieliniowej charakterystyce reprezentujące lokalne nieliniowości. W rozważaniach zastosowano metodę falową, w której wykorzystuje się rozwiązanie falowe równań ruchu. Obliczenia numeryczne wykonano dla modeli z jedną, dwiema, trzema i czterema bryłami sztywnymi. Koncentrują się one na wyznaczaniu krzywych amplitudowo-częstościowych i badaniu wpływu parametrów repezentujących lokalną nieliniowość na przemieszczenia w wybranych przekrojach poprzecznych elementów sprężystych rozważanych układów dyskretno-ciągłych.

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