JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
4, **36**, 1998

KINEMATICS AND DYNAMICS OF MULTIBODY SYSTEM BASED ON NATURAL AND JOINT COORDINATES USING VELOCITY TRANSFORMATIONS

SAAD M. ISSA

KRZYSZTOF P. ARCZEWSKI

Institute of Aeronautics and Applied Mechanics, Warsaw University of Technology
e-mail: krisarcz@grape.meil.pw.edu.pl

The equations of motion are initially formulated using natural coordinates. These coordinates are used to define the position of the system, kinematic joints between bodies, and forcing functions on and between the bodies. When using this system of coordinates the definition of initial system is straightforward. The equations of motion are then expressed in terms of relative joint coordinates with the use of velocity transformation method. The velocity transformation matrix relates the relative coordinates to the natural ones. The kinematic relationships for each joint type and the graph theory for identifying the system topology are used in constructing the velocity transformation matrix. Use of both natural and relative coordinates produces an efficient set of equations without loss of generality, then the equations of motion can be efficiently integrated.

Key words: analytical mechanics, multibody systems, mathematical modelling

1. Introduction

Recently, the derivation of equations of motion for the Multibody System (MBS) were presented in a variety of forms by Nikravesh (1988), Shabana (1989), Huston (1990), Garcia de Jalon and Bayo (1994). For the rigid MBS, some techniques allow us to generate the equations of motion in terms of a large set of dependent coordinates in the form of a large set of mixed differential-algebraic equations. Other techniques yield the equations of motion as a minimal set of ordinary differential equations. Many other in between approaches provide us with various alternatives.

The equations of motion can be easily expressed in terms of a large number of absolute accelerations (natural coordinates). These formulation is attractive because of its simplicity and easiness of manipulation, however the drawback is that the coordinates form a large set of mixed differential-algebraic equations, the numerical solution of these equations is computationally inefficient and, furthermore, special procedures must be followed to avoid, or to correct, a phenomenon known as constraint violation.

A method which provides computational efficiency is the so-called joint coordinates (relative coordinates). With these coordinates, the equations of motion are expressed in terms of a small set of accelerations. In fact, when an open loop system is considered, the number of coordinates equals the number of degrees of freedom of the system. The major disadvantage of this method is complicated derivation of the equations of motion. It is also difficult to incorporate general constraints and forcing function, moreover, additional work is needed in order to determine the absolute values of positions, velocities, and accelerations of the MBS.

This paper presents a method of formulation of MBS equations of motion based on natural and joint coordinates. An approach to constructing the velocity transformation matrix based on system topology and its kinematics is developed. Frst, the equations of motion are formulated in terms of natural coordinates, this procedure is straightforward and general.

Then, they are transformed into the joint coordinates using a velocity transformation method. We believe, this enables easy computer implementation, yields a minimal set of differential equations and allows for an efficient solution of the equations of motion.

2. Graph theory for topological definition

The order and how the bodies are connected together in the system is called "system topology". The graph theory is an effective method to identify the topological structure of large scale multibody dynamic systems. The topology and kinematical properties of the large scale multibody system can be efficiently represented by a graph. In the graphical representation of the MBS, each body in the system is represented by a node (or vertex) while each kinematic joint is represented by an edge. Each open loop is represented by a branch (tree structure), but each closed loop may be represented by one or two branches depending on the location of a cut joint in the closed loop (spanning tree). Each tree starts from a root toward a leaf corresponding to the topological path which starts from the base body toward the leaf body (Arczewski and Pietrucha, 1993). The graphical representation employs a preliminary path matrix which contains the characteristics of the system topology.

The preliminary path matrix called π, is constructed in this way, row i and column j of the preliminary path matrix correspond to the node i and the edge j, respectively. If we denote the elements of the preliminary path matrix by π_{ij}, then π_{ij} is defined as

$$\pi_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is directed away from node } i \\ -1 & \text{if edge } j \text{ is directed toward node } i \\ 0 & \text{otherwise} \end{cases}$$

A base is defined as the main body of the system. For a system fixed to the ground, ground is the base body. For a floating system, selection of the base is arbitrary, although in most systems a central body in the system is a natural choice. For convenience, the ground is considered as node 0 which is not included in the preliminary path matrix π. The floating base body is assumed to be connected to node 0 through an edge.

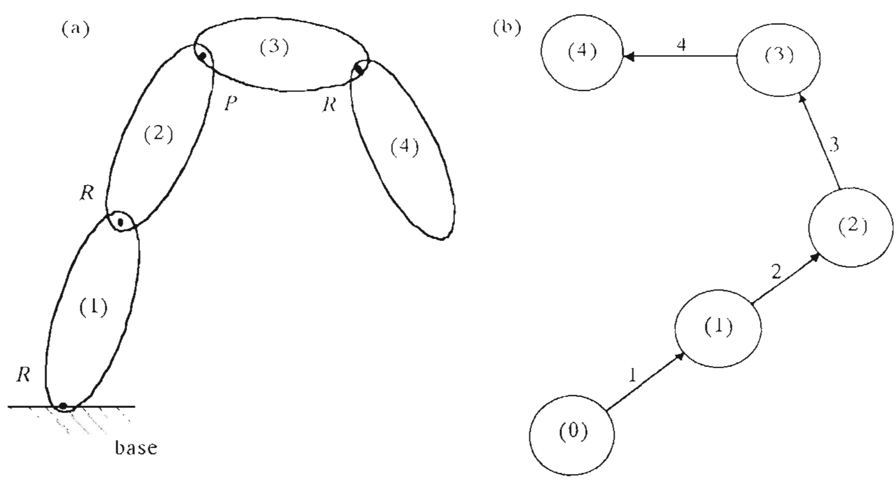

Fig. 1. (a) System connected to ground; (b) its graphical representation

Consider the two examples shown in Fig.1 and Fig.2, the first is connected to the ground, and the other is floating. Both systems are an example of a tree

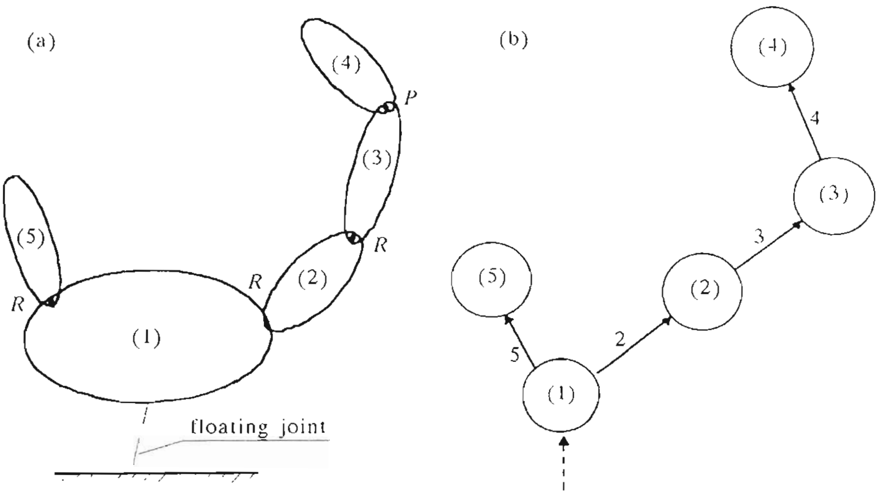

Fig. 2. (a) Floating system; (b) its graphical representation

structure. The first system contains four moving bodies and three revolute (R) and one prismatic (P) joints. The second system contains five moving bodies and three revolute (R) and one prismatic (P) joints.

Based on the graph, the preliminary path matrix $\boldsymbol{\pi}$ is obtained for both the above systems as

$$\boldsymbol{\pi}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4\times4} \qquad \boldsymbol{\pi}_2 = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}_{5\times5}$$

where matrix $\boldsymbol{\pi}_1$ is for the system connected to ground and matrix $\boldsymbol{\pi}_2$ is for the floating system.

In order to get all the information on the connectivity and path flow of the system, the preliminary path matrix $\boldsymbol{\pi}$ will be modified to a secondary path matrix $\boldsymbol{\pi}^*$. An efficient computer implementation of the algorithm given by Gim and Nikravesh (1992) has been modified and used in this paper. Therefore, the secondary path matrices for both systems are

$$\pi_1^* = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ -3 & -2 & -1 & 0 \\ -4 & -3 & -2 & -1 \end{bmatrix}_{4\times4} \qquad \pi_2^* = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 \\ -4 & -3 & -2 & -1 & 0 \\ -2 & 0 & 0 & 0 & -1 \end{bmatrix}_{5\times5}$$

Recall that i is refers to the body (or node) and j refers to the joint (or edge), the numeric order of negative integers indicates the path flow of the system. The columns of the secondary path matrix indicate the nodes connected to the root by the edges. For example, we have element $\pi_{41} = -4$, this means that there are four paths between body four and edge one. Matrix π^* contains all the necessary information on the connectivity of the MBS, it can be used to construct the velocity transformation.

3. Dependent and independent coordinates

When modelling the motion of the MBS, the first problem to consider is that of finding an appropriate system of independent coordinates whose number coincides with the number of degrees of freedom; in this case their number is minimum. Another choice is the dependent coordinates the number of which is larger than that of the degrees of freedom. With these dependent coordinates, the MBS can be described more easily, the positions of all the bodies in the system are determined, while only the positions of the input bodies or the values of the externally driven coordinates are determined when the independent coordinates are used. The dependent coordinates are interrelated through scleronomic constraint equations of the form

$$\boldsymbol{\Psi}(\boldsymbol{q}) = \mathbf{0} \tag{3.1}$$

The first and second time derivatives of the constraint equations yield the kinematic velocity and acceleration equations

$$\begin{aligned} \boldsymbol{\Psi}_q(\boldsymbol{q})\dot{\boldsymbol{q}} &= \mathbf{0} \\ \boldsymbol{\Psi}_q(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Psi}}_q(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} &= \mathbf{0} \end{aligned} \tag{3.2}$$

where $\boldsymbol{\Psi}_q$ is the Jacobain matrix. The equations of motion are written as

$$\mathsf{M}\ddot{\boldsymbol{q}} - \boldsymbol{\Psi}_q^\top \boldsymbol{\lambda} = \boldsymbol{Q} \tag{3.3}$$

where

- M – inertia matrix containing the mass and the inertia tensors of all bodies
- $\boldsymbol{\lambda}$ – vector of Lagrange multipliers
- $\boldsymbol{Q}$ – vector of the forces acting on the system bodies.

Eqs (3.1)÷(3.3) represent a set of differential-algebraic equations of motion for the MBS when the natural coordinates are used.

Denoting by $\boldsymbol{\theta}$ the vector of joint coordinates of the system containing all of the joint coordinates and the natural coordinates of base body if the base body is not the ground (floating base). Therefore, the vector $\boldsymbol{\theta}$ has the dimension equal to the number of degrees of freedom of the system. The vector of joint velocities is defined as $\dot{\boldsymbol{\theta}}$, which is the time derivative of $\boldsymbol{\theta}$. A general and simple method for formulation of the dynamic equations of the MBS in terms of its degrees of freedom, is based on the so called *velocity transformation*. It can be shown that, there is a linear transformation between $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{q}}$ as

$$\dot{\boldsymbol{q}} = \mathsf{R}\dot{\boldsymbol{\theta}} \tag{3.4}$$

Matrix R is orthogonal to the Jacobian matrix Ψ_q. Therefore substituting Eq (3.4) into Eq $(3.2)_1$, yields $\Psi_q\mathsf{R}\dot{\boldsymbol{\theta}} = 0$. Since $\dot{\boldsymbol{\theta}}$ is a vector of independent velocities, then

$$\Psi_q\mathsf{R} = 0 \tag{3.5}$$

The time derivative of Eq (3.4) gives the transformation formula for acceleration

$$\ddot{\boldsymbol{q}} = \mathsf{R}\ddot{\boldsymbol{\theta}} + \dot{\mathsf{R}}\dot{\boldsymbol{\theta}} \tag{3.6}$$

Substituting Eq (3.6) into Eq (3.3), premultiplying by $\mathsf{R}^\top$, and using Eq (3.5) yields

$$\mathsf{R}^\top\mathsf{M}\mathsf{R}\ddot{\boldsymbol{\theta}} = \mathsf{R}^\top(\boldsymbol{Q} - \mathsf{M}\dot{\mathsf{R}}\dot{\boldsymbol{\theta}}) \tag{3.7}$$

Eq (3.7) represents the generalized equations of motion for an open loop MBS when the number of the selected coordinates is equal to the number of degrees of freedom.

4. Velocity transformation matrix

The velocity transformation matrix R depends on the kinematics and the topology of the MBS. If block matrix R_{ij} is defined to represent the local properties of a kinematic joint, then the matrix R can be obtained by assembling block matrices R_{ij}'s based on the system topology. The block matrices are determined for a floating base body and various kinematic joints (Kim and Vanderploeg, 1986), where the subscript i stands for body i (or node i) and the subscript j stands for floating base body j or kinematic joint j (or edge j). It should be noted that the block matrix R_{ij} for a composite joint is determined as a combination of those joints. For a floating base body, a vector of natural coordinates is defined as the joint coordinates, while for kinematic joints the relative joint coordinates are used. The vector $\boldsymbol{d}_{ij}$ is defined as a vector toward the attachment point of body i from the center of mass of floating base body j, or from the attachment point of joint j. If body i is the same as floating base body j, then $\boldsymbol{d}_{ij} = \mathbf{0}$.

To illustrate this method, the above two examples are considered. In example one (cf Fig.1), θ_j, for $j = 1, ..., 5$ are defined as joint coordinates shown in Fig.3. The joint coordinates, θ_1, θ_2, and θ_5, are relative rotational coordinates about the revolute joints R_1, R_2, and R_5, while θ_3 is a relative translational coordinate along a prismatic joint P_3. The absolute velocities of the bodies of the system can be determined in terms of the joint velocities.

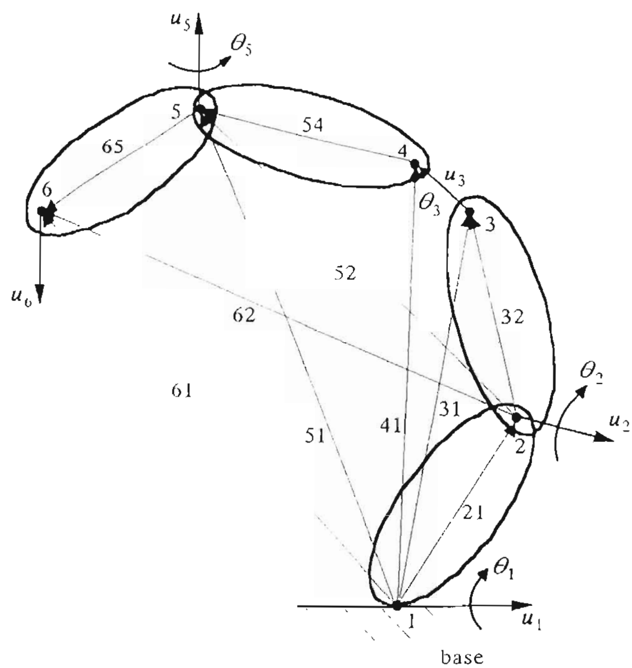

Fig. 3. Schematic representation of the vectors $\boldsymbol{d}_{ij}$ for kinematic joints of the grounded system

We have numbered the basic points from 1 to 6. Basic points 1, 2, 4, 5, are considered to be the reference points for bodies 1, 2, 3, 4, respectively. The formulae for the absolute translational velocities can be written as follows

$$\begin{aligned} \dot{\boldsymbol{r}}_2 &= \boldsymbol{\omega}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1) & \dot{\boldsymbol{r}}_5 &= \dot{\boldsymbol{r}}_4 + \boldsymbol{\omega}_3 \times (\boldsymbol{r}_5 - \boldsymbol{r}_4) \\ \dot{\boldsymbol{r}}_3 &= \dot{\boldsymbol{r}}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{r}_3 - \boldsymbol{r}_2) & \dot{\boldsymbol{r}}_6 &= \dot{\boldsymbol{r}}_5 + \boldsymbol{\omega}_4 \times (\boldsymbol{r}_6 - \boldsymbol{r}_5) \\ \dot{\boldsymbol{r}}_4 &= \dot{\boldsymbol{r}}_3 + \boldsymbol{\omega}_2 \times \boldsymbol{u}_3 + \boldsymbol{u}_3\dot{\theta}_3 \end{aligned} \tag{4.1}$$

where

$\boldsymbol{u}_3$ – unit vector along the prismatic joint axis

$\boldsymbol{\omega}_i$ – angluar velocities of bodies, $i = 1, ..., 4$.

Recall that, since the incoming joint for body 1 is fixed to ground, the basic point 1 is not considered in the vector of velocities because its velocity remains zero.

The above absolute translational velocities can be put in terms of the vector $\boldsymbol{d}_{ij}$ ($\boldsymbol{d}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$) and performing the vector product operation one can get

$$\begin{aligned} \dot{\boldsymbol{r}}_2 &= -\mathsf{D}_{21}\boldsymbol{\omega}_1 & \dot{\boldsymbol{r}}_5 &= \dot{\boldsymbol{r}}_4 - \mathsf{D}_{54}\boldsymbol{\omega}_3 \\ \dot{\boldsymbol{r}}_3 &= \dot{\boldsymbol{r}}_2 - \mathsf{D}_{32}\boldsymbol{\omega}_2 & \dot{\boldsymbol{r}}_6 &= \dot{\boldsymbol{r}}_5 - \mathsf{D}_{65}\boldsymbol{\omega}_4 \\ \dot{\boldsymbol{r}}_4 &= \dot{\boldsymbol{r}}_3 - \mathsf{D}_{43}\boldsymbol{\omega}_2 + \boldsymbol{u}_3\dot{\theta}_3 \end{aligned} \tag{4.2}$$

where D is a skew-symmetric matrix associated with the components of vector $\boldsymbol{d} = [d_1, d_2, d_3]^\top$, which is defined for vector product operation as

$$\mathsf{D} = \begin{bmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{bmatrix}_{3\times 3}$$

The angular velocities of the four bodies are

$$\begin{aligned} \boldsymbol{\omega}_1 &= \boldsymbol{u}_1\dot{\theta}_1 & \boldsymbol{\omega}_3 &= \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_2 &= \boldsymbol{\omega}_1 + \boldsymbol{u}_2\dot{\theta}_2 & \boldsymbol{\omega}_4 &= \boldsymbol{\omega}_3 + \boldsymbol{u}_5\dot{\theta}_5 \end{aligned} \tag{4.3}$$

We now substitute Eqs (4.3) into Eqs (4.2) in a forward process from the base toward the leaf, and simplify the equations by using relationships such as $\boldsymbol{d}_{ik} + \boldsymbol{d}_{kj} = \boldsymbol{d}_{ij}$, the absolute translational velocities can be written in the form

$$\begin{aligned} \dot{\boldsymbol{r}}_2 &= -\mathsf{D}_{21}\boldsymbol{u}_1\dot{\theta}_1 \\ \dot{\boldsymbol{r}}_3 &= -\mathsf{D}_{31}\boldsymbol{u}_1\dot{\theta}_1 - \mathsf{D}_{32}\boldsymbol{u}_2\dot{\theta}_2 \\ \dot{\boldsymbol{r}}_4 &= -\mathsf{D}_{41}\boldsymbol{u}_1\dot{\theta}_1 - \mathsf{D}_{42}\boldsymbol{u}_2\dot{\theta}_2 + \boldsymbol{u}_3\dot{\theta}_3 \\ \dot{\boldsymbol{r}}_5 &= -\mathsf{D}_{51}\boldsymbol{u}_1\dot{\theta}_1 - \mathsf{D}_{51}\boldsymbol{u}_2\dot{\theta}_2 + \boldsymbol{u}_3\dot{\theta}_3 \\ \dot{\boldsymbol{r}}_6 &= -\mathsf{D}_{61}\boldsymbol{u}_1\dot{\theta}_1 - \mathsf{D}_{62}\boldsymbol{u}_2\dot{\theta}_2 + \boldsymbol{u}_3\dot{\theta}_3 - \mathsf{D}_{65}\boldsymbol{u}_5\dot{\theta}_5 \end{aligned} \tag{4.4}$$

Since the absolute translational velocities for the basic points are determined, it remains to determine the absolute velocities for the unit vectors. In general, the absolute velocity of any vector $\boldsymbol{u}_j$ located at joint j relative to joint i is defined as

$$\dot{\boldsymbol{u}}_j = \boldsymbol{\omega}_i \times \boldsymbol{u}_j \tag{4.5}$$

or

$$\dot{\boldsymbol{u}}_j = -\mathsf{U}_j \boldsymbol{\omega}_i \tag{4.6}$$

where U_j is the skew-symmetric matrix associated with the vector $\boldsymbol{u}_j$, and $\boldsymbol{\omega}_i = \boldsymbol{u}_i \dot{\theta}_i$ is the angular velocity at joint i, where $\boldsymbol{u}_i$ and θ_i are the unit vector and rotational coordinate at joint i, respectively.

Therefore, the velocities of the unit vectors (Fig.3) are determined by the following equations

$$\begin{aligned} \dot{\boldsymbol{u}}_2 &= -\mathsf{U}_2 \boldsymbol{\omega}_1 \qquad & \dot{\boldsymbol{u}}_5 &= -\mathsf{U}_5 \boldsymbol{\omega}_3 \\ \dot{\boldsymbol{u}}_3 &= -\mathsf{U}_3 \boldsymbol{\omega}_2 \qquad & \dot{\boldsymbol{u}}_6 &= -\mathsf{U}_6 \boldsymbol{\omega}_4 \end{aligned} \tag{4.7}$$

Substitute Eq (4.3) into Eq (4.7), the velocities of the unit vectors are

$$\begin{aligned} \dot{\boldsymbol{u}}_2 &= -\mathsf{U}_2 \boldsymbol{u}_1 \dot{\theta}_1 \qquad & \dot{\boldsymbol{u}}_5 &= -\mathsf{U}_5 \boldsymbol{u}_1 \dot{\theta}_1 - \mathsf{U}_5 \boldsymbol{u}_2 \dot{\theta}_2 \\ \dot{\boldsymbol{u}}_3 &= -\mathsf{U}_3 \boldsymbol{u}_1 \dot{\theta}_1 - \mathsf{U}_3 \boldsymbol{u}_2 \dot{\theta}_2 \qquad & \dot{\boldsymbol{u}}_6 &= -\mathsf{U}_6 \boldsymbol{u}_1 \dot{\theta}_1 - \mathsf{U}_6 \boldsymbol{u}_2 \dot{\theta}_2 - \mathsf{U}_6 \boldsymbol{u}_5 \dot{\theta}_5 \end{aligned} \tag{4.8}$$

Now we have gathered all the information necessary to construct the velocity transformation matrix for the above system, therefore, the velocity transformation equation is obtained in the matrix form

$$\begin{bmatrix} \dot{\boldsymbol{r}}_2 \\ \dot{\boldsymbol{u}}_2 \\ \dot{\boldsymbol{r}}_3 \\ \dot{\boldsymbol{u}}_3 \\ \dot{\boldsymbol{r}}_4 \\ \dot{\boldsymbol{r}}_5 \\ \dot{\boldsymbol{u}}_5 \\ \dot{\boldsymbol{r}}_6 \\ \dot{\boldsymbol{u}}_6 \end{bmatrix} = \begin{bmatrix} -\mathsf{D}_{21}\boldsymbol{u}_1 & 0 & 0 & 0 \\ -\mathsf{U}_2\boldsymbol{u}_1 & 0 & 0 & 0 \\ -\mathsf{D}_{31}\boldsymbol{u}_1 & -\mathsf{D}_{32}\boldsymbol{u}_2 & 0 & 0 \\ -\mathsf{U}_3\boldsymbol{u}_1 & -\mathsf{U}_3\boldsymbol{u}_2 & 0 & 0 \\ -\mathsf{D}_{41}\boldsymbol{u}_1 & -\mathsf{D}_{42}\boldsymbol{u}_2 & \boldsymbol{u}_3 & 0 \\ -\mathsf{D}_{51}\boldsymbol{u}_1 & -\mathsf{D}_{52}\boldsymbol{u}_2 & \boldsymbol{u}_3 & 0 \\ -\mathsf{U}_5\boldsymbol{u}_1 & -\mathsf{U}_5\boldsymbol{u}_2 & 0 & 0 \\ -\mathsf{D}_{61}\boldsymbol{u}_1 & -\mathsf{D}_{62}\boldsymbol{u}_2 & \boldsymbol{u}_3 & -\mathsf{D}_{65}\boldsymbol{u}_5 \\ -\mathsf{U}_6\boldsymbol{u}_1 & -\mathsf{U}_6\boldsymbol{u}_2 & 0 & -\mathsf{U}_6\boldsymbol{u}_5 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_5 \end{bmatrix} \tag{4.9}$$

By comparing Eqs (4.9) and (3.4), the velocity transformation matrix R can be found.

For the second system (Fig.2), which is shown schematically in Fig.4, the body 1 is floating base body, vectors of absolute translational and angular velocities, $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{u}}_1$, are defined as the joint velocities, while at the revolute

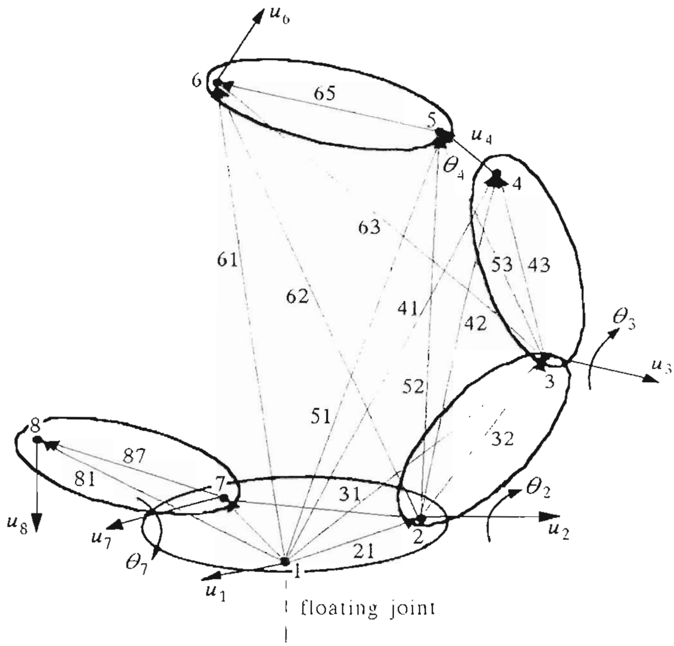

Fig. 4. Schematic representation of vector d_{ij} for kinematic joints of the floating system

joints, R_2, R_3 and R_7, relative rotational velocities, $\dot{\theta}_2$, $\dot{\theta}_3$, and $\dot{\theta}_7$ are defined as the joint velocities. At the prismatic joint P_4, $\dot{\theta}_4$, represents the relative translational velocity between bodies 3 and 4.

The absolute angular velocities of bodies in the system can also be obtained as

$$\begin{aligned} \boldsymbol{\omega}_2 &= \boldsymbol{\omega}_1 + \boldsymbol{u}_2\dot{\theta}_2 \qquad & \boldsymbol{\omega}_4 &= \boldsymbol{\omega}_3 \\ \boldsymbol{\omega}_3 &= \boldsymbol{\omega}_2 + \boldsymbol{u}_3\dot{\theta}_3 \qquad & \boldsymbol{\omega}_5 &= \boldsymbol{\omega}_1 + \boldsymbol{u}_7\dot{\theta}_7 \end{aligned} \tag{4.10}$$

where $\boldsymbol{\omega}_1$ is the angluar velocity of floating base body.

A similar process to that of the previous example yields the translational velocities as

$$\begin{aligned} \dot{\boldsymbol{r}}_2 &= \dot{\boldsymbol{r}}_1 - \mathsf{D}_{21}\boldsymbol{\omega}_1 \qquad \dot{\boldsymbol{r}}_3 = \dot{\boldsymbol{r}}_1 - \mathsf{D}_{31}\boldsymbol{\omega}_1 - \mathsf{D}_{32}\boldsymbol{u}_2\dot{\theta}_2 \\ \dot{\boldsymbol{r}}_4 &= \dot{\boldsymbol{r}}_1 - \mathsf{D}_{41}\boldsymbol{\omega}_1 - \mathsf{D}_{42}\boldsymbol{u}_2\dot{\theta}_2 - \mathsf{D}_{43}\boldsymbol{u}_3\dot{\theta}_3 \\ \dot{\boldsymbol{r}}_5 &= \dot{\boldsymbol{r}}_1 - \mathsf{D}_{51}\boldsymbol{\omega}_1 - \mathsf{D}_{52}\boldsymbol{u}_2\dot{\theta}_2 - \mathsf{D}_{53}\boldsymbol{u}_3\dot{\theta}_3 + \boldsymbol{u}_4\dot{\theta}_4 \\ \dot{\boldsymbol{r}}_6 &= \dot{\boldsymbol{r}}_1 - \mathsf{D}_{61}\boldsymbol{\omega}_1 - \mathsf{D}_{62}\boldsymbol{u}_2\dot{\theta}_2 - \mathsf{D}_{63}\boldsymbol{u}_3\dot{\theta}_3 + \boldsymbol{u}_4\dot{\theta}_4 \\ \dot{\boldsymbol{r}}_7 &= \dot{\boldsymbol{r}}_1 - \mathsf{D}_{71}\boldsymbol{\omega}_1 \qquad \dot{\boldsymbol{r}}_8 = \dot{\boldsymbol{r}}_1 - \mathsf{D}_{81}\boldsymbol{\omega}_1 - \mathsf{D}_{87}\boldsymbol{u}_7\dot{\theta}_7 \end{aligned} \tag{4.11}$$

and the absolute velocities of the unit vectors are

$$\begin{array}{ll} \dot{\boldsymbol{u}}_1 = -\mathsf{U}_1\boldsymbol{\omega}_1 & \dot{\boldsymbol{u}}_6 = -\mathsf{U}_6\boldsymbol{\omega}_1 - \mathsf{U}_6\boldsymbol{u}_2\dot{\theta}_2 - \mathsf{U}_6\boldsymbol{u}_3\dot{\theta}_3 \\ \dot{\boldsymbol{u}}_2 = -\mathsf{U}_2\boldsymbol{\omega}_1 & \dot{\boldsymbol{u}}_7 = -\mathsf{U}_7\boldsymbol{\omega}_1 \\ \dot{\boldsymbol{u}}_3 = -\mathsf{U}_3\boldsymbol{\omega}_1 - \mathsf{U}_3\boldsymbol{u}_2\dot{\theta}_2 & \dot{\boldsymbol{u}}_8 = -\mathsf{U}_8\boldsymbol{\omega}_1 - \mathsf{U}_3\boldsymbol{u}_7\dot{\theta}_7 \\ \dot{\boldsymbol{u}}_4 = -\mathsf{U}_4\boldsymbol{\omega}_3 & \end{array} \tag{4.12}$$

The vectors $\boldsymbol{d}_{ij}$'s are shown in Fig.4. From Eqs (4.11) and (4.12) the vector *of absolute velocities of the system can be written in terms of joint velocities* vector as

$$\begin{bmatrix} \dot{\boldsymbol{r}}_1 \\ \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{r}}_2 \\ \dot{\boldsymbol{u}}_2 \\ \dot{\boldsymbol{r}}_3 \\ \dot{\boldsymbol{u}}_3 \\ \dot{\boldsymbol{r}}_4 \\ \dot{\boldsymbol{u}}_4 \\ \dot{\boldsymbol{r}}_5 \\ \dot{\boldsymbol{r}}_6 \\ \dot{\boldsymbol{u}}_6 \\ \dot{\boldsymbol{r}}_7 \\ \dot{\boldsymbol{u}}_7 \\ \dot{\boldsymbol{r}}_8 \\ \dot{\boldsymbol{u}}_8 \end{bmatrix} = \begin{bmatrix} \mathsf{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathsf{U}_1 & 0 & 0 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{21} & 0 & 0 & 0 & 0 \\ 0 & -\mathsf{U}_2 & 0 & 0 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{31} & -\mathsf{D}_{32}\boldsymbol{u}_2 & 0 & 0 & 0 \\ 0 & -\mathsf{U}_3 & -\mathsf{U}_3\boldsymbol{u}_2 & 0 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{41} & -\mathsf{D}_{42}\boldsymbol{u}_2 & -\mathsf{D}_{43}\boldsymbol{u}_3 & 0 & 0 \\ 0 & -\mathsf{U}_4 & -\mathsf{U}_4\boldsymbol{u}_2 & -\mathsf{U}_4\boldsymbol{u}_3 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{51} & -\mathsf{D}_{52}\boldsymbol{u}_2 & -\mathsf{D}_{53}\boldsymbol{u}_3 & \boldsymbol{u}_4 & 0 \\ \mathsf{I} & -\mathsf{D}_{61} & -\mathsf{D}_{62}\boldsymbol{u}_2 & -\mathsf{D}_{63}\boldsymbol{u}_3 & \boldsymbol{u}_4 & 0 \\ 0 & -\mathsf{U}_6 & -\mathsf{U}_6\boldsymbol{u}_2 & -\mathsf{U}_6\boldsymbol{u}_3 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{71} & 0 & 0 & 0 & 0 \\ 0 & -\mathsf{U}_7 & 0 & 0 & 0 & 0 \\ \mathsf{I} & -\mathsf{D}_{81} & 0 & 0 & 0 & -\mathsf{D}_{87}\boldsymbol{u}_7 \\ 0 & -\mathsf{U}_8 & 0 & 0 & 0 & -\mathsf{U}_8\boldsymbol{u}_7 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{r}}_1 \\ \boldsymbol{\omega}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_7 \end{bmatrix} \tag{4.13}$$

The structure of matrix R in Eq (4.9) or Eq (4.13) shows that R can be constructed from small block matrices R_{ij}. Table 1 shows R_{ij} matrices for a variety of kinematic joints. For automatic construction of R, the negative integer entries $(\pi_{ij} < 0)$ in the secondary path matrix $\boldsymbol{\pi}^*$ are replaced by their corresponding R_{ij} block matrices from Table 1. Then, the zero entries $(\pi_{ij} = 0)$ in secondary path matrix $\boldsymbol{\pi}^*$ are replace by their corresponding zero matrices. For the first and second figures, and give rise to Eqs (4.9) and (4.13), respectively.

The time derivative of the matrix R is needed, matrix $\dot{\mathsf{R}}$ can also be constructed systematically. Table 1 shows $\dot{\mathsf{R}}_{ij}$ block matrices for several kinematic joints.

Table 1. Elementary block matrices

Joint type	Joint velocities	R_{ij}	$\dot{\mathsf{R}}_{ij}$
Floating	$\begin{bmatrix} \dot{\boldsymbol{r}}_j \\ \dot{\boldsymbol{u}}_j \end{bmatrix}$	$\begin{bmatrix} \mathsf{I} & -\mathsf{D}_{ij} \\ 0 & -\mathsf{U}_i \end{bmatrix}_{6\times 6}$	$\begin{bmatrix} 0 & -\dot{\mathsf{D}}_{ij} \\ 0 & -\dot{\mathsf{U}}_i \end{bmatrix}$
Revolute	$\dot{\Theta}_j$	$\begin{bmatrix} -\mathsf{D}_{ij}\boldsymbol{u}_j \\ -\mathsf{U}_i\boldsymbol{u}_j \end{bmatrix}_{6\times 1}$	$\begin{bmatrix} -\dot{\mathsf{D}}_{ij}\boldsymbol{u}_j - \mathsf{D}_{ij}\dot{\boldsymbol{u}}_j \\ -\dot{\mathsf{U}}_i\boldsymbol{u}_j - \mathsf{U}_i\dot{\boldsymbol{u}}_j \end{bmatrix}$
Prismatic	$\dot{\Theta}_j$	$\begin{bmatrix} \boldsymbol{u}_j \\ 0 \end{bmatrix}_{6\times 1}$	$\begin{bmatrix} \dot{\boldsymbol{u}}_j \\ 0 \end{bmatrix}$
Universal	$\begin{bmatrix} \dot{\Theta}_j^1 \\ \dot{\Theta}_j^2 \end{bmatrix}$	$\begin{bmatrix} -\mathsf{D}_{ij}\boldsymbol{u}_j^1 & -\mathsf{D}_{ij}\boldsymbol{u}_j^2 \\ -\mathsf{U}_i^1\boldsymbol{u}_j^1 & -\mathsf{U}_i^1\boldsymbol{u}_j^2 \end{bmatrix}_{6\times 2}$	$\begin{bmatrix} \mathsf{A}^1 & \mathsf{A}^2 \end{bmatrix}$

where

$$\mathsf{A}^s = \begin{bmatrix} -\dot{\mathsf{D}}_{ij}\boldsymbol{u}_j^s - \mathsf{D}_{ij}\dot{\boldsymbol{u}}_j^s \\ -\dot{\mathsf{U}}_i^s\boldsymbol{u}_j^s - \mathsf{U}_i^s\dot{\boldsymbol{u}}_j^s \end{bmatrix} \qquad s = 1, 2$$

5. Conclusion

The joint coordinates method employs a linear velocity transformation matrix between the joint and natural velocities of a multibody system. The graph theory is used for representing the system kinematics and topology. Kinematic relations between natural and relative joint coordinates are derived. Initially, natural coordinates are used straightforward in formulating the constraints, forces acting on the bodies and equations of motion. Then the equations of motion are transformed into the relative joint coordinates using a velocity transformation matrix. This method generates a small or even a minimal set of equations of motion necessary for the dynamics of the multibody systems.

The relative coordinates are not recommended to be used at early steps of formulation, because of their mathematical difficulty. Using the relative coordinates later on, in velocity transformation process, will reduce the number of equations, hence higher computational efficiency will be achieved.

Using a combination of natural and relative joint coordinates yields an efficient modelling method, without loosing generality. The relative joint coordinates method gives rise to easy computer implementation of the algorithms and efficient numerical solution.

References

1. ARCZEWSKI K., PIETRUCHA J., 1993, *Mathematical Modelling of Mechanical Complex System*, Vol 1, *Discrete Models*, Ellis Horwood Limited

2. GARCIA DE JALON J., BAYO E., 1994, *Kinematic and Dynamic Simulation of Multibody Systems, The Real-Time Challenge*, Springer-Verlag

3. GARCIA DE JALON J., UNDA J., AVELLO A., 1986, Natural Coordinates for the Computer Analysis of Multibody Systems, *Computer Methods in Applied Mechanics and Engineering*, **56**

4. GARCIA DE JALON J., UNDA J., AVELLO A., JIMENEZ J.M., 1987, Dynamic Analysis of Three-Dimensional Mechanisms in Natural Coordinates, *ASME Journal of Mechanisms Transmissions and Automation in Design*, **109**

5. GIM G., NIKRAVESH P.E., 1992, Joint Coordinate Method for Analysis and Design of Mulutibody Systems: Part 2. System Topology, *KSME Journal*, **7**, 26-34

6. HUSTON R.L., 1990, *Multibody Dynamics*, Butterworth-Heineman

7. JERKOVSKY W., 1978, The Structure of Multibody Dynamics Equations, *Journal of Guidance and Control*, **1**, 3

8. KIM S.S., VANDERPLOEG M.J., 1986, A General And Efficient Method for Dynamic Analysis of Mechanical Systems Using Velocity Transformations, *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, **108**, 2, 176-182

9. NIKRAVESH P.E., 1988, *Computer-Aided Analysis of Mechanical Systems*, Prentice-Hail International

10. SHABANA A.A., 1989, *Dynamics of Multibody Systems*, John Wiley & Sons

Zastosowanie współrzędnych naturalnych i względnych oraz transformacji prędkości w kinematyce i dynamice układów wieloczłonowych

Streszczenie

W pierwszej fazie modelowania matematycznego równania ruchu formułowane są we współrzędnych naturalnych. Współrzędne te zostały zastosowane do określenia położenia układu, połączeń kinematycznych między ciałami, a także do wyrażenia sił działających na ciała. Następnie równania ruchu wyrażone zostały za pomocą współrzędnych względnych do czego wykorzystano tzw. metodę transformacji prędkości. Macierz transformacji prędkości wiąże oba układy współrzędnych. Macierz ta tworzona jest na podstawie informacji o rodzajach połączeń kinematycznych między ciałami oraz na podstawie informacji o topologii układu. Przy zastosowaniu obu rodzajów współrzędnych otrzymujemy równania ruchu w ogólnej postaci, które następnie mogą być efektywnie całkowane.

Manuscript received October 14, 1997; accepted for print November 25, 1997