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NUMERICAL MODELLING OF BIO-HEAT TRANSFER USING THE BOUNDARY ELEMENT METHOD

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Numerical models of heat transfer processes proceeding in a biological tissue subjected to a strong external thermal interaction are discussed. In this case one can consider diametrically different phenomena; such as burns resulting from thermal contact of the skin with an external heat source, or the freezing process of biological tissue used in cryosurgery. From the mathematical point of view these processes belong to the group of boundary-initial problems described by the diffusion equation and adequate boundary-initial conditions. At the stage of numerical realization the boundary element method can be applied and such an approach is discussed in this paper.

Key words: boundary element method, bio-heat transfer, numerical simulation

1. Introduction

Non-steady temperature field in a domain of biological tissue is decsribed by the following equation (cf Comini and Del Guidice, 1976)

$$x \in \Omega : \quad c(T)\frac{\partial T(x,t)}{\partial t} = \mathrm{div}[\lambda(T)\mathrm{grad}T(x,t)] + Q_p + Q_m \tag{1.1}$$

where

c	–	specific heat per unit of volume
λ	–	thermal conductivity
Q_p	–	perfusion heat source
Q_m	–	metabolic heat source
T, x, t	–	temperature, spatial co-ordinates and time, respectively.

The perfusion heat source is given by the formula

$$Q_p = G_b c_b [T_b - T(x,t)] \tag{1.2}$$

where

G_b	–	tissue perfusion [m^3 blood/s/m^3 tissue]
c_b	–	specific heat of blood per unit of volume
T_b	–	arterial blood temperature.

The value of metabolic heat source lies within the range of $245 \div 24500\ \mathrm{W/m^3}$ (cf Brinck and Werner, 1994).

The boundary conditions given on the outer surface of the system can be written in a general form

$$x \in \Gamma : \quad \Phi[T(x,t), \boldsymbol{n} \cdot \mathrm{grad}T(x,t)] = 0 \tag{1.3}$$

where $\boldsymbol{n} \cdot \mathrm{grad}T$ denotes the normal derivative at a boundary point x. For $t = 0$ the initial temperature field is known, namely

$$t = 0 : \quad T(x,0) = T_0(x) \tag{1.4}$$

2. Analysis of freezing and thawing processes – governing equations

The cryosurgery has a variety of applications to medical treatment; e.g., to causing a local necrosis of a tissue, the detaching a pathological tissue, destructing the cancer cells, etc. These methods are widely used in dermatology, gynecology, proctology, oncology and also laryngology. The cryoprobe being in thermal contact with a biological tissue causes that the freezing process proceeds in the domain considered. The problem belongs to the group of moving boundary ones because the shape and dimensions of frozen region are time-dependent. The energy equation (1.1) must be supplemented by the term connected with the latent heat evolution, while the metabolic and perfusion sources can be neglected (cf Budman et al., 1995). So, the following equation is taken into account

$$x \in \Omega : \quad c(T)\frac{\partial T(x,t)}{\partial t} = \mathrm{div}[\lambda(T)\mathrm{grad}T(x,t)] + L\frac{\partial S(x,t)}{\partial t} \tag{2.1}$$

where

L – latent heat [J/m^3]

S – solid state fraction at the point considered x.

Let us assume that the solid state fraction is a known function of temperature $S = S(T)$ from the interval $[T_2, T_1]$ (the beginning and end of the freezing process, respectively) and then

$$L\frac{\partial S(x,t)}{\partial t} = L\frac{dS(T)}{dT}\frac{\partial T(x,t)}{\partial t} \tag{2.2}$$

Eq (2.1) can be written in the form

$$x \in \Omega : \quad C(T)\frac{\partial T(x,t)}{\partial t} = \operatorname{div}[\lambda(T)\operatorname{grad}T(x,t)] \tag{2.3}$$

where

$$C(T) = c(T) - L\frac{\partial S(x,t)}{\partial t} \tag{2.4}$$

is called the substitute thermal capacity of intermediate region. The energy equation in the form (2.3) can be extended on the whole domain considered, because for $T > T_1 : \ S(T) = 0$, while for $T < T_2 : \ S(T) = 1$ and $C(T) \to c(T)$.

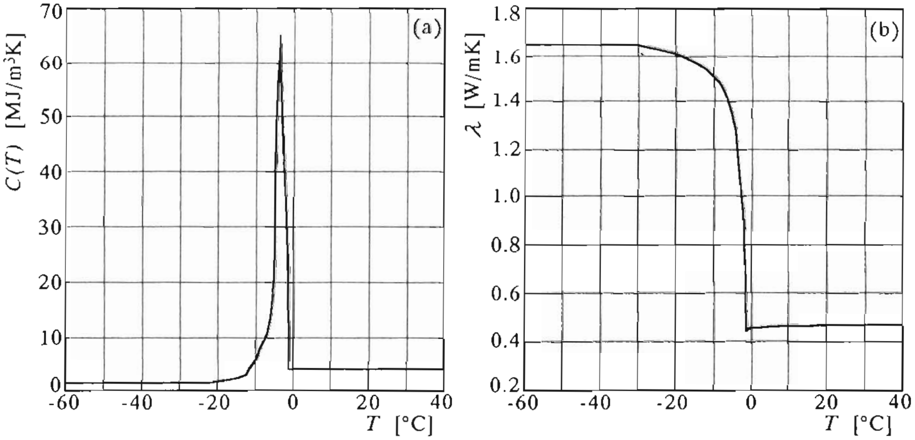

Fig. 1. (a) – Function $C(T)$; (b) – thermal conductivity $\lambda(T)$

This property of Eq (2.3) constitutes a base of the so-called fixed domain approach (cf Idelsohn et al., 1994; Voller, 1991). Summing up, the equation discussed describes the heat transfer processes in the whole conventionally homogenous domain. The problem is strongly non-linear – both the parameters

$C(T)$ and $\lambda(T)$ are temperature dependent, the courses of these functions (cf Budman et al., 1995) are shown in Fig.1.

In order to use the boundary element method, linearization of the task discussed must be taken into account. One of the possibilities is application of the alternating phase truncation method – APTM (cf Rogers et al., 1979; Kapusta and Mochnacki, 1988) (at the stage of numerical computations). The method requires the adapting of the governing equations to the enthalpy convention. The physical enthalpy related to a unit of volume is defined as follows

$$H(T) = \int_{T_r}^{T} C(\mu)\, d\mu \tag{2.5}$$

where T_r is an arbitrary assumed reference level (e.g. $T_r = -60°\mathrm{C}$). Additionally, both $C(T)$ and $\lambda(T)$ should be approximated by the stair-case functions and constant values of above functions are assumed for successive sub-domains, i.e. the frozen region, intermediate zone and unfrozen phase. The course of enthalpy function for the material considered is shown in Fig.2.

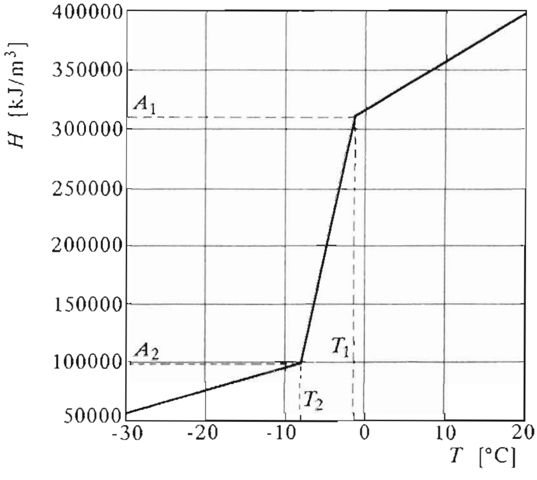

Fig. 2. Course of the enthalpy function

The equation (2.3) written using the enthalpy convention is of the form

$$x \in \Omega\ : \quad \frac{\partial H(x,t)}{\partial t} = \mathrm{div}[a(H)\mathrm{grad}H(x,t)] \tag{2.6}$$

where $a(H) = \lambda(H)/C(H)$ is the thermal diffusion coefficient.

The boundary conditions given on the outer surface are the following

$$x \in \Gamma : \quad \Phi[H(x,t), n \cdot \mathrm{grad} H(x,t)] = 0 \tag{2.7}$$

The initial condition $H(x,0) = H_0(x)$ is also given.

3. Analysis of freezing and thawing processes – numerical model

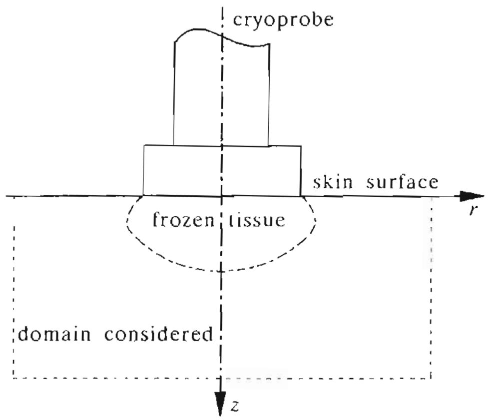

Fig. 3. Domain considered

Taking into account the cryoprobe geometry (see Fig.3) the energy equation (2.6) should be rewritten in the form corresponding to the cylindrical co-ordinate system

$$\frac{\partial H(r,z,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\Big[ra(H)\frac{\partial H(r,z,t)}{\partial r}\Big] + \frac{\partial}{\partial z}\Big[a(H)\frac{\partial H(r,z,t)}{\partial z}\Big] \tag{3.1}$$

Assuming a constant value of $a(H)$ (it results from the *philosophy* of the APTM) we have

$$\frac{\partial H(r,z,t)}{\partial t} = a\Big[\frac{\partial^2 H(r,z,t)}{\partial r^2} + \frac{\partial^2 H(r,z,t)}{\partial z^2}\Big] + Q \tag{3.2}$$

where

$$Q = \frac{a}{r}\frac{\partial H(r,z,t)}{\partial r} \tag{3.3}$$

The form of Eq (3.2) results from the concept of application of typical BEM algorithm for objects oriented in the Cartesian system in order to find a numerical solution of the problem discussed. The term Q can be treated as an artificial source function and its value can be determined using a simple iterative procedure.

On the contact surface between cryoprobe and skin the 1st type of the boundary condition is assumed, namely

$$(r,z) \in \Gamma_c : \quad H(r,z,t) = H(T_s - wt) \tag{3.4}$$

where
w – cooling rate [°C/s]
T_s – initial temperature of the skin surface.

In numerical computations it is assumed that $w = 1/6$ and $T_s = 37°C$.

For the remaining parts of the boundary (skin surface and conventionally assumed limits of the domain considered) the adiabatic condition can be accepted

$$(r,z) \in \Gamma_0 : \quad q(r,z,t) = -a\boldsymbol{n} \cdot \mathrm{grad}H(r,z,t) = 0 \tag{3.5}$$

The condition

$$t = 0 : \quad H(r,z,0) = H(37.0) \tag{3.6}$$

determines the initial enthalpy of the domain considered.

One of the possibilities of freezing model *linearization* is the algorithm called the alternating phase truncation method. The method was presented by Rogers et al. (1979), while its generalization by Mochnacki and Kapusta (1988) and next by Mochnacki et al. (1991). In the paper a situation corresponding to the course of enthalpy presented in Fig.2 is discussed.

The APTM consists in an approximate solution of the freezing problem by conventional reduction of the domain considered to the homogenous one thermophysical parameters of which are constant values. For every step Δt resulting from the time mesh assumed, three boundary-initial problems are solved. The first concerns the unfrozen region, the second the intermediate phase whereas in the last one the frozen tissue is taken into account. The successive solutions are in a certain way modified. The APTM is especially effective as a supplement to the BEM algorithm because computations are realized for homogenous domains and generally the problem is reduced to the linear one.

Let us consider the multiphase domain Ω, being the union of sub-domains $\Omega_1 \cup \Omega_2 \cup \Omega_3$. The limits of enthalpy corresponding to isotherms T_1 and T_2 are denoted as A_1 and A_2 (see Fig.2). Additionally, it is assumed that the enthalpy field for time t^f is known, while the enthalpy field for time $t^{f+1} = t^f + \Delta t$ is searched.

The first stage of computations deals with the hottest phase (unfrozen tissue). The given enthalpy distribution in the domain Ω at time t^f is transformed in this way

$$V_1(r, z, t^f) = \max[A_1, H(r, z, t^f)] \tag{3.7}$$

This new pseudo-initial condition corresponds to a structural reduction of the whole area Ω to the unfrozen domain. Next, on the basis of BEM the transition $t^f \to t^{f+1}$ is calculated assuming that the diffusion coefficient $a(H) = a_1 =$ const. The solution for time t^{f+1} : $V_1^*(r, z, t^{f+1})$ is corrected according to the formula

$$V_1(r, z, t^{f+1}) = V_1^*(r, z, t^{f+1}) + H(r, z, t^f) - V_1(r, z, t^f) \tag{3.8}$$

If we consider the second stage corresponding to the intermediate phase then the results of previous computations $V_1(r, z, t^{f+1})$ are known and they are transformed to the new pseudo-initial condition, namely

$$V_2(r, z, t^f) = \min\{A_1, \max[A_2, V_1(r, z, t^{f+1})]\} \tag{3.9}$$

The transition $t^f \to t^{f+1}$ for the domain thermal diffussivity of which is equal to a_2 is calculated, and the result obtained $V_2^*(r, z, t^{f+1})$ is corrected in following way

$$V_2(r, z, t^{f+1}) = V_2^*(r, z, t^{f+1}) + V_1(r, z, t^{f+1}) - V_2(r, z, t^f) \tag{3.10}$$

At the last stage (frozen tissue) the pseudo-initial condition in the form

$$V_3(r, z, t^f) = \min[A_2, V_2(r, z, t^{f+1})] \tag{3.11}$$

is assumed and for $a(H) = a_3$ the solution for the time t^{f+1} : $V_3^*(r, z, t^{f+1})$ is found.

The enthalpy field for the time t^{f+1} results from the formula

$$H(r, z, t^{f+1}) = V_3^*(r, z, t^{f+1}) + V_2(r, z, t^{f+1}) - V_3(r, z, t^f) \tag{3.12}$$

The transition $t^f \to t^{f+1}$ requires the solution of three linear diffusion problems in structurally homogenous domains, but in this way the well known difficulties associated with a strongly non-linear mathematical model can be eliminated. It should be pointed out that each step of time in the APTM is done three times, i.e. it is necessary to *join* the boundary conditions adequately because they should act only during one interval Δt. In this connection for two of the stages the boundary Γ should be insulated.

Now, the BEM algorithm for linear Fourier's equation is shortly discussed. The integral equation corresponding to the boundary-initial problem considered is the following (cf Brebbia et al., 1984; Majchrzak, 1991).

$$B(\xi,\eta)H(\xi,\eta,t^{f+1}) + \int_{t^f}^{t^{f+1}}\int_{\Gamma} q(r,z,t)H^*\,d\Gamma dt = \int_{t^f}^{t^{f+1}}\int_{\Gamma} H(r,z,t)q^*\,d\Gamma dt + \int_{\Omega} H(r,z,t^f)H^*\,d\Omega + \int_{t^f}^{t^{f+1}}\int_{\Omega}\frac{a}{r}\frac{\partial H(r,z,t)}{\partial r}H^*\,d\Omega dt \tag{3.13}$$

where

$B(\xi,\eta)$	–	coefficient from the interval $[0,1]$
$H(r,z,t), q(r,z,t)$	–	given boundary conditions for Γ_c and Γ_0
$H(r,z,t^f)$	–	pseudo-initial condition
H^*	–	fundamental solution and

$$H^* = \frac{1}{4\pi a(t^{f+1}-t)}\exp\Big[-\frac{\rho^2}{4a(t^{f+1}-t)}\Big] \tag{3.14}$$

In this formula ρ is the distance from the point considered (r,z) to the point (ξ,η) where the concentrated heat source is applied, this means

$$\rho = \sqrt{(r-\xi)^2+(z-\eta)^2} \tag{3.15}$$

Heat flux resulting from the fundamental solution is equal to

$$q^* = -a\frac{\partial H^*}{\partial \boldsymbol{n}} = \frac{d}{8\pi a(t^{f+1}-t)^2}\exp\Big[-\frac{\rho^2}{4a(t^{f+1}-t)}\Big] \tag{3.16}$$

At the same time

$$\boldsymbol{n} = [\cos\alpha, \cos\beta] \qquad d = (r-\xi)\cos\alpha + (z-\eta)\cos\beta \tag{3.17}$$

Assuming constant values of H and q during the time interval $[t^f, t^{f+1}]$, substituting the boundary Γ by a sum of boundary elements Γ_j, $j = 1,2,...,N$ and the interior Ω by a sum of internal cells Ω_l, $l = 1,2,...,L$ we obtain the following form of Eq (3.13)

$$
\begin{aligned}
&B(\xi_i,\eta_i)H(\xi_i,\eta_i,t^{f+1}) + \sum_{j=1}^{N}\int_{\Gamma_j} q(r,z,t^{f+1}) \int_{t^f}^{t^{f+1}} H^*\, dt d\Gamma_j = \\
&= \sum_{j=1}^{N}\int_{\Gamma_j} H(r,z,t^{f+1}) \int_{t^f}^{t^{f+1}} q^*\, dt d\Gamma_j + \sum_{l=1}^{L}\int_{\Omega_l} H(r,z,t)H^*\, d\Omega_l + \\
&+ \sum_{l=1}^{L}\int_{\Omega_l} \frac{a}{r}\frac{\partial H(r,z,t^{f+1})}{\partial r} \int_{t^f}^{t^{f+1}} H^*\, dt d\Omega_l
\end{aligned}
\tag{3.18}
$$

Time integration is possible in an analytic way. So

$$
\int_{t^f}^{t^{f+1}} H^*\, dt = \frac{1}{4\pi a}\mathrm{Ei}\Big(\frac{\rho^2}{4a\Delta t}\Big) \qquad \int_{t^f}^{t^{f+1}} q^*\, dt = \frac{d}{2\pi\rho^2}\exp\Big(-\frac{\rho^2}{4a\Delta t}\Big) \tag{3.19}
$$

where $\mathrm{Ei}(\zeta)$ is the exponential integral function, $\mathrm{Ei}(\zeta) = \int_{\zeta}^{\infty} \frac{1}{\zeta}\exp(-\zeta)\, d\zeta$. In numerical realization the function $\mathrm{Ei}(\zeta)$ is approximated by interpolating polynomials (cf Cody and Thacher, 1968).

Putting Eqs (3.19) into (3.18) one obtains the following system of algebraic equations ($i = 1, 2, ..., N$)

$$
\sum_{j=1}^{N} G_{ij} q(r_j,z_j,t^{f+1}) = \sum_{j=1}^{N} W_{ij} H(r_j,z_j,t^{f+1}) + \sum_{l=1}^{L} P_{il} H(r_l,z_l,t^f) + \sum_{l=1}^{L} Z_{il} \tag{3.20}
$$

where

$$
\begin{aligned}
&G_{ij} = \frac{1}{4\pi a}\int_{\Gamma_j} \mathrm{Ei}\Big(\frac{\rho^2}{4a\Delta t}\Big)\, d\Gamma_j \qquad P_{il} = \frac{1}{4\pi a\Delta t}\int_{\Omega_l} \exp\Big(-\frac{\rho^2}{4a\Delta t}\Big)\, d\Omega_l \\
&W_{ij} = \begin{cases} \frac{1}{2\pi}\int_{\Gamma_j} \exp\Big(-\frac{\rho^2}{4a\Delta t}\Big)\, d\Gamma_j & i \neq j \\ -B(\xi_j,\eta_j) & i = j \end{cases} \\
&Z_{il} = \frac{1}{4\pi a}\int_{\Omega_l} \frac{a}{r}\frac{\partial H(r,z,t^{f+1})}{\partial r}\mathrm{Ei}\Big(\frac{\rho^2}{4a\Delta t}\Big)\, d\Omega_l
\end{aligned}
\tag{3.21}
$$

The integrals (3.21) are calculated using standard Gaussian quadratures.

From Eqs (3.20) unknown values of the enthalpy $H(r_j, z_j, t^{f+1})$ at boundary nodes $(r_j, z_j) \in \Gamma_0$ and unknown values of the heat flux $q(r_j, z_j, t^{f+1})$ at nodes $(r_j, z_j) \in \Gamma_c$ are determined using the Gauss elimination method.

After determinig the boundary values one can find the enthalpy field at the internal nodes using the following formula $(i = N + 1, N + 2, ..., N + L)$

$$H(\xi_i, \eta_i, t^{f+1}) = \sum_{j=1}^{N} W_{ij} H(r_j, z_j, t^{f+1}) - \sum_{j=1}^{N} G_{ij} q(r_j, z_j, t^{f+1}) + \\ + \sum_{l=1}^{L} P_{il} H(r_l, z_l, t^f) + \sum_{l=1}^{L} Z_{il} \tag{3.22}$$

The last component in Eqs (3.20) and (3.22) results from integration of the artificial source term. So, on the basis of solution obtained for time t^{f+1} one can estimate the derivative $\partial H/\partial r$ and using a simple iterative process correct the local values of this expression. The test computations show that the number of iterations assuring a sufficient accuracy is small. It should be pointed out that because of singularity appearing in this term for $r = 0$ a cylinder is replaced by the *pipe* with very small hole around the singular point $r = 0$.

4. Analysis of freezing and thawing processes – example of computations

Cryoprobe of diameter 15 mm being in ideal thermal contact with the skin generates on the adequate part of the skin surface the Dirichlet condition (3.4). On the remaining parts of the boundary the Neumann condition in the form (3.5) is assumed. Additionally, for time $t = 0$ the initial condition (3.6) is given. The numerical solution of the boundary-initial problem considered is shown in Fig.4.

5. Prediction of skin temperatures under the flash fire conditions – governing equations

From the mathematical point of view thermal processes proceeding in the multi-layer domain, i.e. epidermis-dermis-subcutaneous-region are described

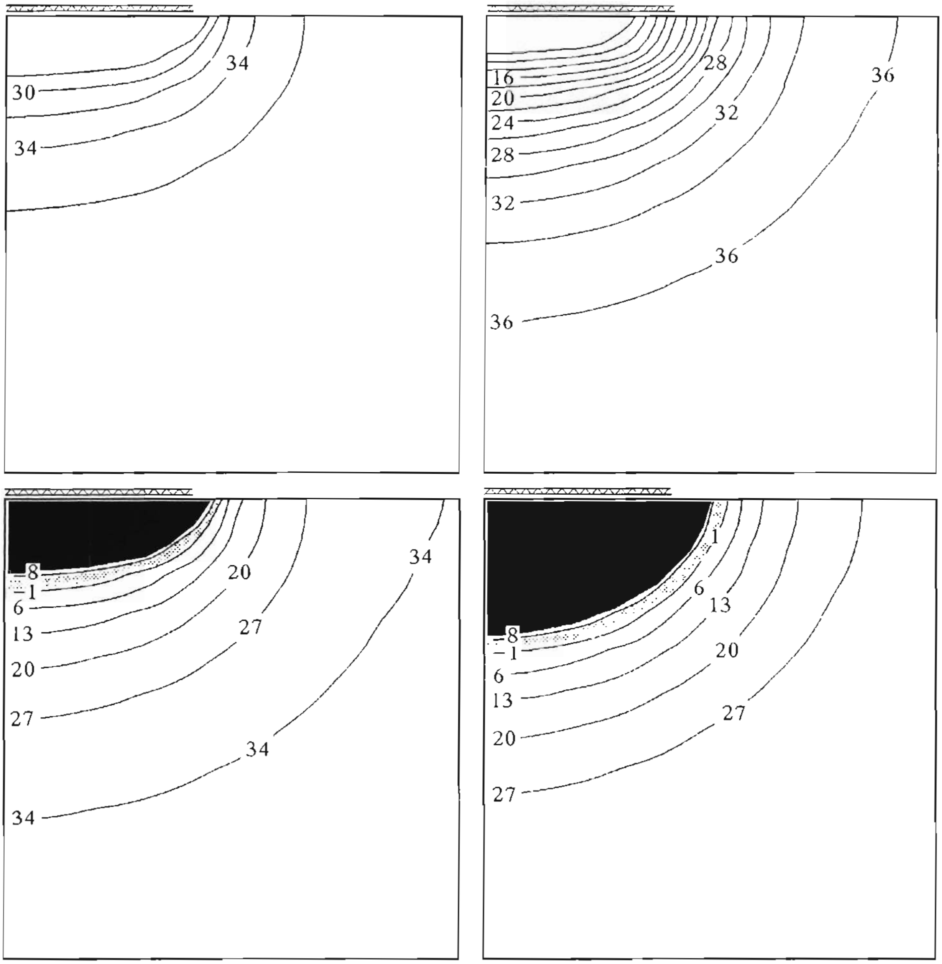

Fig. 4. Temperature field and position of the frozen region after 2, 4, 6 and 8 minutes, respectively

by a system of diffusion equations, at the same time the internal heat sources must be taken into account (dermis and sub-cutaneous region). The equations are supplemented by adequate boundary and initial conditions. The thermal action is simulated by the Neumann boundary condition and the so-called exposure time t_{ex} (cf Torvi and Dale, 1994). Taking into account small thicknesses of successive skin layers, the 1D model in a sufficiently accurate way describes the thermal processes proceeding in the domain considered.

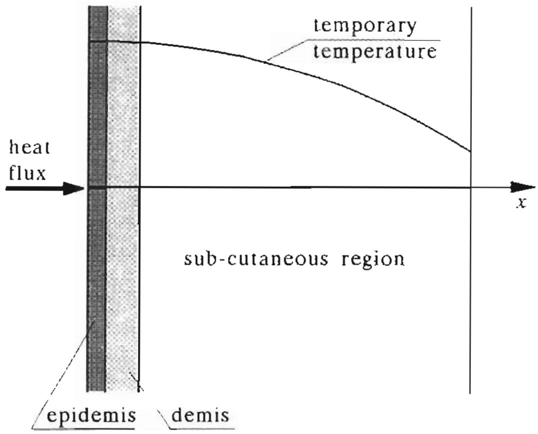

Fig. 5. Skin domain

In the skin domain the following sub-domains are taken into account (Fig.5):

- Epidermis of the thickness L_1 [m] and thermophysical parameters λ_1 [W/(m °C)] (thermal conductivity) and c_1 [J/(m^{3}°C)] (specific heat)
- Dermis of the thickness L_2 and thermophysical parameters λ_2, c_2
- Sub-cutaneous region of the thickness L_3 and thermophysical parameters λ_3, c_3.

The non-steady temperature field in the heterogeneous area $0 < x < L_1 + L_2 + L_3 = L$ is represented by the system of equations (cf Eq (1.1))
— for the epidermis sub-domain, $0 = L_0 < x < L_1$

$$c_1 \frac{\partial T_1(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 T_1(x,t)}{\partial x^2} \tag{5.1}$$

— for the dermis sub-domain, $L_1 < x < L_1 + L_2$

$$c_2 \frac{\partial T_2(x,t)}{\partial t} = \lambda_2 \frac{\partial^2 T_2(x,t)}{\partial x^2} + G_2 c_b [T_b - T_2(x,t)] \tag{5.2}$$

— for the sub-cutaneous region, $L_1 + L_2 < x < L$

$$c_3 \frac{\partial T_3(x,t)}{\partial t} = \lambda_3 \frac{\partial^2 T_3(x,t)}{\partial x^2} + G_3 c_b [T_b - T_3(x,t)] \tag{5.3}$$

The system of equations is supplemented by the following boundary-initial conditions:

- The Neumann and next the Robin condition on the skin surface

$$x = 0 : \quad \begin{cases} q_1(x,t) = q_b & t \leq t_{ex} \\ q_1(x,t) = \alpha[T_1(x,t) - T^{\infty}] & t > t_{ex} \end{cases} \tag{5.4}$$

  where q_b is the given heat flux (flash fire), while α is the heat transfer coefficient and T^{∞} is the ambient temperature

- Continuity condition on the contact surface between the epidermis and dermis

$$x = L_1 : \quad \begin{cases} q_1(x,t) = q_2(x,t) = q_{12}(x,t) \\ T_1(x,t) = T_2(x,t) = T_{12}(x,t) \end{cases} \tag{5.5}$$

- Continuity condition on the contact surface between the dermis and subcutaneous region

$$x = L_1 + L_2 : \quad \begin{cases} q_2(x,t) = q_3(x,t) = q_{23}(x,t) \\ T_2(x,t) = T_3(x,t) = T_{23}(x,t) \end{cases} \tag{5.6}$$

- The Dirichlet condition on the conventionally assumed right-hand limit of the system

$$x = L : \quad T_3(x,t) = T_b \tag{5.7}$$

- The initial condition, $t = 0$

$$T_1(x,0) = T_{10}(x) \qquad T_2(x,0) = T_{20}(x) \qquad T_3(x,0) = T_{30}(x) \tag{5.8}$$

6. Prediction of skin temperatures under the flash fire conditions – numerical model

The BEM approach in the case of 1D problem (cf Mochnacki and Suchy, 1995) leads to the following equations (for successive layers of the skin $e = 1,2,3$)

$$T_e(\xi, t^{f+1}) + \left[\frac{1}{c_e} \int_{t^f}^{t^{f+1}} T_e^*(\xi, x, t^{f+1}, t) q_e(x,t)\, dt \right]_{x=L_{e-1}}^{x=L_e} =$$
$$= \left[\frac{1}{c_e} \int_{t^f}^{t^{f+1}} q_e^*(\xi, x, t^{f+1}, t) T_e(x,t)\, dt \right]_{x=L_{e-1}}^{x=L_e} +$$
$$+ \int_{L_{e-1}}^{L_e} T_e^*(\xi, x, t^{f+1}, t) T_e(x, t^f)\, dx + \tag{6.1}$$
$$+ \frac{c_b}{c_e} G_e \int_{L_{e-1}}^{L_e} [T_b - T_e(x, t^f)] \int_{t^f}^{t^{f+1}} T^*(\xi, x, t^{f+1}, t)\, dt dx$$

where T_e^* are the fundamental solutions for $L_{e-1} < x,\ \xi < L_e$ given by the formulas

$$T_e^*(\xi, x, t^{f+1}, t) = \frac{1}{2\sqrt{\pi a_e (t^{f+1} - t)}} \exp\left[-\frac{(x-\xi)^2}{4a_e(t^{f+1} - t)} \right] \tag{6.2}$$

while ξ is the point at which the concentrated heat source is applied. One can notice that for $e = 1$: $G_1 = 0$.

The heat fluxes resulting from the fundamental solutions are the following

$$q_e^*(\xi, x, t^{f+1}, t) = -\lambda_e \frac{\partial T_e^*(\xi, x, t^{f+1}, t)}{\partial x} =$$
$$= \frac{\lambda_e (x-\xi)}{4\sqrt{\pi [a_e (t^{f+1} - t)]^3}} \exp\left[-\frac{(x-\xi)^2}{4a_e(t^{f+1} - t)} \right] \tag{6.3}$$

Assuming that for $t \in [t^f, t^{f+1}]$: $T_e(x,t) = T_e(x, t^{f+1})$ and $q_e(x,t) = q_e(x, t^{f+1})$ one obtains the following form of Eqs (6.1)

$$T_e(\xi, t^{f+1}) + \left[\frac{q_e(x, t^{f+1})}{c_e} \int_{t^f}^{t^{f+1}} T_e^*(\xi, x, t^{f+1}, t)\, dt \right]_{x=L_{e-1}}^{x=L_e} =$$

$$
\begin{aligned}
&= \left[\frac{T_e(x,t^{f+1})}{c_e} \int\limits_{t^f}^{t^{f+1}} q_e^*(\xi,x,t^{f+1},t)\,dt \right]_{x=L_{e-1}}^{x=L_e} + \\
&+ \int\limits_{L_{e-1}}^{L_e} T_e^*(\xi,x,t^{f+1},t)T_e(x,t^f)\,dx + \\
&+ \frac{c_b}{c_e}G_e \int\limits_{L_{e-1}}^{L_e} [T_b - T_e(x,t^f)] \int\limits_{t^f}^{t^{f+1}} T^*(\xi,x,t^{f+1},t)\,dtdx
\end{aligned}
\tag{6.4}
$$

The integration with respect to time can be done in an analytical way

$$
h_e(\xi,x) = \frac{1}{c_e}\int\limits_{t^f}^{t^{f+1}} T_e^*(\xi,x,t^{f+1},t)\,dt = \frac{\mathrm{sgn}(x-\xi)}{2}\mathrm{erfc}\Big(\frac{|x-\xi|}{2\sqrt{a_e\Delta t}}\Big)
\tag{6.5}
$$

and

$$
\begin{aligned}
g_e(\xi,x) &= \frac{1}{c_e}\int\limits_{t^f}^{t^{f+1}} T_e^*(\xi,x,t^{f+1},t)\,dt = \\
&= \frac{\sqrt{\Delta t}}{b_e\sqrt{\pi}}\exp\Big[-\frac{(x-\xi)^2}{4a_e\Delta t}\Big] - \frac{|x-\xi|}{2\lambda_e}\mathrm{erfc}\Big(\frac{|x-\xi|}{2\sqrt{a_e\Delta t}}\Big)
\end{aligned}
\tag{6.6}
$$

where $b_e = \sqrt{\lambda_e c_e}$ and $\mathrm{erfc}(\cdot) = 1 - \mathrm{erf}(\cdot)$, while $\mathrm{erf}(\cdot)$ is the error function.

The Eqs (6.4) can be written in the form

$$
\begin{aligned}
&T_e(\xi,t^{f+1}) + g_e(\xi,L_e)q_e(L_e,t^{f+1}) - g_e(\xi,L_{e-1})q_e(L_{e-1},t^{f+1}) = \\
&= h_e(\xi,L_e)T_e(L_e,t^{f+1}) - h_e(\xi,L_{e-1})T_e(L_{e-1},t^{f+1}) + p_e(\xi) + z_e(\xi)
\end{aligned}
\tag{6.7}
$$

at the same time

$$
\begin{aligned}
p_e(\xi) &= \int\limits_{L_{e-1}}^{L_e} T_e^*(\xi,x,t^{f+1},t)T(x,t^f)\,dx = \\
&= \frac{1}{2\sqrt{\pi a_e\Delta t}}\int\limits_{L_{e-1}}^{L_e} \exp\Big[-\frac{(x-\xi)^2}{4a_e\Delta t}\Big]T_e(x,t^f)\,dx
\end{aligned}
\tag{6.8}
$$

and

$$z_e(\xi) = \frac{c_b}{c_e} G_e \int\limits_{L_{e-1}}^{L_e} [T_b - T_e(x, t^f)] \int\limits_{t^f}^{t^{f+1}} T^*(\xi, x, t^{f+1}, t)\, dt dx =$$
$$= c_b G_e \int\limits_{L_{e-1}}^{L_e} [T_b - T_e(x, t^f)] g_e(\xi, x)\, dx \tag{6.9}$$

The integrals $p_e(\xi)$ and $z_e(\xi)$ can be determined using Gaussian quadratures.

For $\xi \to L_{e-1}^+$ and $\xi \to L_e^-$ for each domain considered one obtains the system of equations

$$\begin{bmatrix} g_{11}^e & g_{12}^e \\ g_{21}^e & g_{22}^e \end{bmatrix} \begin{bmatrix} q_e(L_{e-1}, t^{f+1}) \\ q_e(L_e, t^{f+1}) \end{bmatrix} =$$
$$= \begin{bmatrix} h_{11}^e & h_{12}^e \\ h_{21}^e & h_{22}^e \end{bmatrix} \begin{bmatrix} T_e(L_{e-1}, t^{f+1}) \\ T_e(L_e, t^{f+1}) \end{bmatrix} + \begin{bmatrix} p_e(L_{e-1}) \\ p_e(L_e) \end{bmatrix} + \begin{bmatrix} z_e(L_{e-1}) \\ z_e(L_e) \end{bmatrix} \tag{6.10}$$

The final form of resolving system results from the continuity conditions (5.5), (5.6) and conditions given for $x = 0$ and $x = L$, namely

$$\begin{bmatrix} -h_{11}^1 & -h_{12}^1 & g_{12}^1 & 0 & 0 & 0 \\ -h_{21}^1 & -h_{22}^1 & g_{22}^1 & 0 & 0 & 0 \\ 0 & -h_{11}^2 & g_{11}^2 & -h_{12}^2 & g_{12}^2 & 0 \\ 0 & -h_{21}^2 & g_{21}^2 & -h_{22}^2 & g_{22}^2 & 0 \\ 0 & 0 & 0 & -h_{11}^3 & g_{11}^3 & g_{12}^3 \\ 0 & 0 & 0 & -h_{21}^3 & g_{21}^3 & g_{22}^3 \end{bmatrix} \begin{bmatrix} T_1(0, t^{f+1}) \\ T_{12}(L_1, t^{f+1}) \\ q_{12}(L_1, t^{f+1}) \\ T_{23}(L_1 + L_2, t^{f+1}) \\ q_{23}(L_1 + L_2, t^{f+1}) \\ q_3(L, t^{f+1}) \end{bmatrix} =$$
$$= \begin{bmatrix} -g_{11}^1 q_b + p_1(0) \\ -g_{21}^1 q_b + p_1(L_1) \\ z_2(L_1) + p_2(L_1) \\ z_2(L_1 + L_2) + p_2(L_1 + L_2) \\ h_{12}^3 T_b + p_3(L_1 + L_2) + z_3(L_1 + L_2) \\ h_{22}^3 T_b + p_3(L) + z_3(L) \end{bmatrix} \tag{6.11}$$

This system of equations corresponds to $t < t_{ex}$ (see Eq (5.4)). The resolving system for $t > t_{ex}$ is somewhat different. The solution of Eq (6.11) determines the boundary temperatures and heat fluxes for time t^{f+1} for

$x = 0, L_1, L_1 + L_2, L$ and next the internal temperature can be found using the formula

$$T_e(\xi, t^{f+1}) = g_e(\xi, L_{e-1})q_e(L_{e-1}, t^{f+1}) - g_e(\xi, L_e)q_e(L_e, t^{f+1}) + \\ + h_e(\xi, L_e)T_e(L_e, t^{f+1}) - h_e(\xi, L_{e-1})T_e(L_{e-1}, t^{f+1}) + p_e(\xi) + z_e(\xi) \tag{6.12}$$

7. Prediction of skin temperatures under the flash fire conditions – sample computations

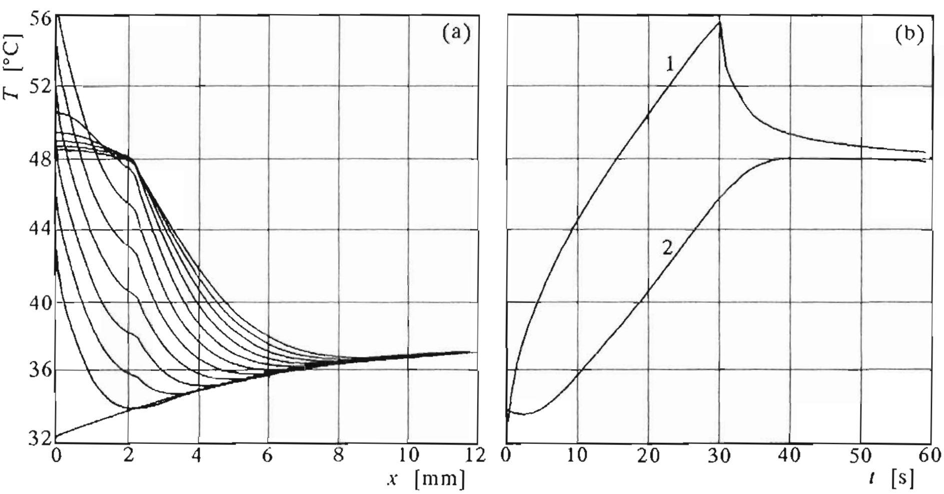

Fig. 6. (a) – Temperature field for times 0,5,10,...,55 s ($q_b = 4186\,\mathrm{W/m^2}$, $t_{ex} = 30\,\mathrm{s}$); (b) – heating (cooling) curves ($x = L_1$ and $x = L_1 + L_2$)

On the basis of the algorithm discussed the computations for the following input data have been done (cf Torvi and Dale, 1994): $L_1 = 0.0001\,\mathrm{m}$, $L_2 = 0.001, L_3 = 0.02, \lambda_1 = 0.23, c_1 = 4.3 \cdot 10^6, \lambda_2 = 0.45, c_2 = 3.96 \cdot 10^6$, $\lambda_3 = 0.18, c_3 = 2.6 \cdot 10^6, c_b = 4.0 \cdot 10^6, G_2 = G_3 = 0.00125, T_b = 37°\mathrm{C}$. For $x = 0$: $q_b = 4186\,\mathrm{W/m^2}, t_{ex} = 30\,\mathrm{s}$. At time $t = 0$ it is assumed that the temperature changes in a parabolic way (on the skin surface $T = 32.5°\mathrm{C}$, while for $x = L$: $T = T_b = 37°\mathrm{C}$).

In Fig.6a the temperature field for times 0,5,...,55 s is shown, while in Fig.6b the heating (cooling) curves for $x = L_1$ and $x = L_1 + L_2$ are presented.

The results obtained allow one to predict the degree of burns (cf Torvi and Dale, 1994).

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Modelowanie numeryczne przepływu bio-ciepła za pomocą metody elementów brzegowych

Streszczenie

W pracy przedstawiono opisy mwtematyczne i modele numeryczne procesów cieplnych zachodzących w tkance biologicznej poddanej silnym termicznym oddziaływaniom zewnętrznym. Można tu rozpatrywać skrajnie różne zjawiska, takie jak oparzenia wynikające z kontaktu skóry z zewnętrznym źródłem ciepła, lub też proces zamrażania tkanki w czasie zabiegu kriochirurgicznego. Z matematycznego punktu widzenia procesy te należą do grupy zadań brzegowo-początkowych opisanych równaniami dyfuzji i odpowiednimi warunkami jednoznaczności. Na etapie realizacji numerycznej można wykorzystać metodę elementów brzegowych i takie właśnie podejście jest prezentowane w niniejszej pracy.

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