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ON APPLICATION OF THE BOUNDARY ELEMENT METHODS FOR ANALYSIS OF SHAKEDOWN PROBLEMS

WITOLD CECOT

Faculty of Civil Engineering, Cracow University of Technology

e-mail: plcecot@cyf-kr.edu.pl

The paper aims at presentation of the numerical model of shakedown analysis based on the direct and indirect versions of the boundary element method. Comparison of numerical results with those obtained by the finite element and finite difference methods proved the efficiency and reliability of the boundary elements. A posteriori error estimate and adaptive meshing are discussed and proposed as further improvement of the BEM shakedown analysis.

Key words: boundary element methods, shakedown analysis, railroad rails

1. Introduction

The Boundary Element Method (BEM) like other general numerical methods; e.g., the finite element or the finite difference methods may be applied to solution of nonelastic problems, Banerjee and Butterfield (1979), (1981), Maier and Polizotto (1983), Maier et al. (1987). In spite of the boundary, also some volume integrals are present in this case. Nevertheless, this complication usually only slightly impairs the efficiency of the boundary methods.

The BEM approach is applied to analysis of the residual stresses and strains in railroad rails. Railroad rails are subject to repeated loading which frequently exceeds their elastic limit. As a result of plastic deformation, residual stresses and strains are present in the body and affect mechanical behavior of the rail. The objective is an analysis allowing one to estimate the stress, strain and displacement fields in the residual steady-state of the body, whenever such a state can be reached for the repeated loading, approximated here by cyclic one.

2. Formulation of the problem

The final shakedown state is considered as infinitesimal, nonelastic deformation developed quasistatically. The problem may be mathematically formulated in many equivalent ways. Let us recall here a sequence of such formulations resulting in the boundary integral equations. The purpose of such a presentation is to show the way of derivation of the boundary formulation for nonelastic bodies and to compare it with some other, better known mathematical models.

2.1. Local formulation

A system of partial differential equations can be derived after making use of the principle of conservation of momentum and some experimentally justified assumptions like short range of the intermolecular forces, validity of Hook's law for the elastic part of the strain, linear formula for the total strain. Excluding body forces, for the sake of simplicity, the displacement field of an elastic-plastic body Ω with nonelastic strain ε^*, is the solution to the following boundary value problem

Find $\boldsymbol{u} \in C^2(\Omega)$ such that

$$\begin{aligned} &\mu u_{i,jj} + (\mu+\lambda)u_{j,ji} = 2\mu\varepsilon^*_{ij,j} && \text{in} \quad \Omega \\ &u_i = 0 && \text{on} \quad \partial\Omega_D \\ &\sigma_{ij}n_j = g_i && \text{on} \quad \partial\Omega_N \end{aligned} \tag{2.1}$$

where we have used

— momentum equations in the form

$$\sigma_{ij,j} = 0 \tag{2.2}$$

— linear geometric equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{in} \quad \Omega \tag{2.3}$$

— Hook's law for the elastic part of the strain

$$\sigma_{ij} = E_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) \qquad \text{in} \quad \Omega \tag{2.4}$$

and where $\Omega \subset I\!R^N$ is the material domain, $N = 1, 2, 3$, $i, j = 1, \ldots, N$, λ, μ are the Lame elastic constants, $\mathsf{E}(\lambda, \mu)$ is the elastic module tensor, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, ε is the total strain tensor, $\partial\Omega_D, \partial\Omega_N$ are the parts of the boundary with the Dirichlet and the Neuman type boundary conditions, respectively, n_j are the components of unit vector, normal to the boundary.

2.2. Variational formulation

Multiplying the equilibrium equation $(2.1)_1$ by a test function, integrating it over Ω and making use of the divergence theorem one obtains the weak statement, equivalent in a sense to Eq (2.1) and known as the principle of virtual work.

Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\int\limits_{\Omega} v_{i,j} E_{ijkl} u_{k,l}\, d\Omega = \int\limits_{\Omega} v_{i,j} E_{ijkl} \varepsilon_{kl}^* + \int\limits_{\partial\Omega_N} v_i g_i\, ds \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1 \tag{2.5}$$

where $\boldsymbol{H}_0^1$ is the Sobolev space of functions satisfying the Dirichlet type boundary conditions. Note that due to the symmetry of E the identities $v_{i,j} E_{ijkl} u_{k,l} = \varepsilon_{ij}(\boldsymbol{v})\sigma_{ij}(\boldsymbol{u})$ and $v_{i,j} E_{ijkl} \varepsilon_{kl}^* = \sigma_{ij}(\boldsymbol{v})\varepsilon_{ij}^*$ hold.

Since, the bilinear functional in the variational formulation (2.5) is symmetric, it is possible to formulate the problem as the minimization of a functional (total potential energy). However, this formulation will not be used in further transformations, so it is not discussed here.

2.3. Reciprocal formulation

The left-hand side of Eq (2.5) can be transformed again by the divergence theorem. If the test functions are not forced to satisfy the Dirichlet boundary conditions one obtains the Betti reciprocal theorem:

Find $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ such that

$$\begin{aligned} &\int\limits_{\Omega} \sigma_{kl,l}(\boldsymbol{v}) u_k\, d\Omega + \int\limits_{\partial\Omega} t_k(\boldsymbol{v}) u_k\, ds = \int\limits_{\Omega} \sigma_{kl,l}(\boldsymbol{u}) v_k\, d\Omega + \\ &+ \int\limits_{\partial\Omega} t_k(\boldsymbol{u}) v_k\, ds \int\limits_{\Omega} \varepsilon_{kl}^* \sigma_{kl,l}(\boldsymbol{v})\, d\Omega \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \end{aligned} \tag{2.6}$$

2.4. Generalized Somigliana identity

Similarly, as for the purely elastic problem. This identity can be obtained by substituting the fundamental solution $(U_{\alpha i})$ for the test function. $U_{\alpha i}$

is the displacement corresponding to a unit force i.e. satisfying the equation $\mu U_{\alpha i,jj} + (\mu + \lambda)u_{\alpha j,ji} = -\delta_\alpha$, where δ_α is the Dirac distribution. Since the fundamental solution is a singular function, one has to subtract a ball with the point of singularity from the Ω domain and perform some limit analysis. Then the generalization of the Somigliana identity for nonelastic bodies is obtained in the form

$$\kappa u_\alpha = \int\limits_{\partial\Omega} (U_{\alpha i}t_i - T_{\alpha i}u_i)\, ds + \int\limits_{\Omega_p} 2\mu\varepsilon^*_{ij} B_{\alpha ij}\, d\Omega \tag{2.7}$$

where

$$\kappa = \begin{cases} 1.0 & \text{for interior points} \\ 0.5 & \text{for smooth boundary points} \end{cases} \tag{2.8}$$

U, **T**, **B** denote the fundamental solution and its derivatives, respectively. The last term in Eq (2.7) may be transformed by the divergence theorem giving the following formula

$$\kappa u_\alpha = \int\limits_{\partial\Omega} (U_{\alpha i}t_i - T_{\alpha i}u_i)\, ds + \int\limits_{\Omega_p} b^*_i U_{\alpha i}\, d\Omega + \int\limits_{\partial\Omega_p} t^*_i U_{\alpha i}\, ds \tag{2.9}$$

where $t^*_i = 2\mu\varepsilon^*_{ij}n_j$, $b^*_i = 2\mu\varepsilon^*_{ij,j}$. The $\boldsymbol{b}^*$ function is called the modified body force and the difference $\boldsymbol{t} - \boldsymbol{t}^*$ is called the modified surface traction, Banerjee and Butterfield (1981). Eq (2.9) has been chosen for the numerical application, since singularity of the kernels of integrals appearing in this equation is smaller than in Eq (2.7). The elastic-plastic deformation is now represented by the integral equation.

Find $\boldsymbol{u} \in \boldsymbol{L}_2(\partial\Omega_N)$ and $\boldsymbol{t} \in \boldsymbol{L}_2(\partial\Omega_D)$ such that

$$\kappa u_\alpha = \int\limits_{\partial\Omega} (U_{\alpha i}t_i - T_{\alpha i}u_i)\, ds + \int\limits_{\Omega_p} b^*_i U_{\alpha i}\, d\Omega + \int\limits_{\partial\Omega_p} t^*_i U_{\alpha i}\, ds \qquad \text{on } \partial\Omega \tag{2.10}$$

2.5. Indirect integral formula

Eq (2.9) is a general formula for the solution to the problem considered. Any function of this form satisfies identically Eq $(2.1)_1$. One can obtain another formula with the same property by replacing the first term of Eq (2.10) with a suitable integral over a curve γ which does not necessary coincide with the boundary. It rather surrounds the Ω domain, in order to avoid some of the singularities. In such a case one obtains the following *indirect* formula for the displacement

$$u_\alpha = \int_\gamma U_{\alpha i}\varphi_i \, ds + \int_{\Omega_p} b_i^* U_{\alpha i} \, d\Omega + \int_{\partial\Omega_p} t_i^* U_{\alpha i} \, ds \tag{2.11}$$

and the corresponding integral equation for the unknown function φ in the following form.

Find $\varphi \in \boldsymbol{L}_2(\gamma)$ such that

$$u_\alpha = \int_\gamma U_{\alpha i}\varphi_i + \int_{\Omega_p} b_i^* U_{\alpha i} \, d\Omega + \int_{\partial\Omega_p} t_i^* U_{\alpha i} \, ds \qquad \text{on } \partial\Omega \tag{2.12}$$

The local formulation (2.1) is the basis for the local finite difference method. The weak statement (2.5) is used mainly in the finite element method as well as in the so called meshless methods (e.g. the global finite difference method). The integral formulations (2.10), (2.12) are used in the boundary element methods. If the function φ in the indirect version is assumed as a set of concentrated loads one obtains a finite sum instead of the integral over γ. Similiar formula is used in the T-complete function method, Zieliński (1988). The difference is that instead of the fundamental solutions, other, nonsingular functions satisfying the differential equation are used as the basis functions.

The nonelastic strain field in the above, equivalent formulations (2.1) $\div$ (2.12) of the elastic-plastic deformation, was treated as a parameter. For evaluation of this field an additional criterion is necessary. Generally, there are two kinds of such criteria. The first ones are constitutive equations, either with an explicit yield hypothesis (cf Banerjee and Butterfield, 1979, 1981; Maier and Novati, 1987; Maier and Polizotto, 1983) or with the internal state variables (see Bodner, 1985). All these models involve incremental analysis of the whole (in our case cyclic) loading history. This would be very time-consuming, especially for 3D bodies. Fortunately, it is also possible to estimate the final shakedown state on the basis of some simplified models. One of the best known is the Melan theorem (Martin, 1975), which allows one to answer the question whether a body shakes down under a given cyclic loading, and which also gives, as a byproduct, the residual stress field corresponding to the extreme loading for which the shakedown takes place. However, this is not enough for our purposes and we use other approaches which allow for estimation of the residual state for any magnitude of the cyclic loading. Let us briefly discuss two of them.

The Orkisz model (see Orkisz, 1992; Orkisz and Cecot, 1997) can be formulated in the following way. Find $\boldsymbol{\varepsilon}^*$ such that realize

$$J_e(\boldsymbol{\varepsilon}^*) = \inf_{\boldsymbol{\epsilon}^* \in \boldsymbol{V}_e} J_e(\boldsymbol{\epsilon}^*) \tag{2.13}$$

where

$$J_e(\boldsymbol{\epsilon}^*) = \frac{1}{2} \int\limits_{\Omega} \boldsymbol{\mathcal{F}}(\boldsymbol{\epsilon}^*)\mathsf{E}^{-1}\boldsymbol{\mathcal{F}}(\boldsymbol{\epsilon}^*)\, d\Omega \tag{2.14}$$

$$\boldsymbol{V}_e = \Big\{\boldsymbol{\epsilon}^* : \quad f[\sigma^e + \boldsymbol{\mathcal{F}}(\boldsymbol{\epsilon}^*) - c\epsilon^p] \leq \sigma_y \ \text{ in } \ \Omega\Big\} \tag{2.15}$$

where f is a yield function (e.q. the Mises one), σ_y is the yield limit, c is the hardening parameter. The above approach is formulated including modification which has been proposed recently by Orkisz and Cecot (1997) and which extends validity of the method on materials with hardening.

The BEM approach has been used (more details are discussed in the next section) for evolution of the relation

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{F}}(\boldsymbol{\varepsilon}^*) \tag{2.16}$$

It expresses the residual stresses $\boldsymbol{\sigma}$ in terms of nonelastic strains $\boldsymbol{\varepsilon}^*$. If this relation is known it can be introduced to the functional (2.14). Then the constrained minimization results in estimation of the plastic strain. The displacements can now be computed from Eq (2.9), and the strains and stresses from similar formulas. However, only the residual stresses obtained in such a way are unique. Eq (2.16) is not bijective, therefore the kinematic quantities are estimated with accuracy to some nonzero fields, since two fields of nonelastic strains which differ by a field of strains resulting from the displacements satisfying homogenous boundary conditions give the same residual stresses. To prove that let us consider a compatible strain field ε_{ij}^0 with the following property

$$\varepsilon_{ij}^0 = \frac{1}{2}(u_{i,j}^0 + u_{j,i}^0) \qquad \text{in } \ \Omega \tag{2.17}$$

where

$$\begin{array}{lll} u_i^0 = 0 & \text{on } \ \partial\Omega_D & \\ \sigma_{ij}^0(\boldsymbol{u}^0)n_j = 0 & \text{on } \ \partial\Omega_N & i,j = 1,\dots,N \end{array} \tag{2.18}$$

The $\boldsymbol{u}^0$ function may be such that its support is a ball centered at an interior point A, and with such a radius length ρ that the ball is contained in the Ω domain. One of the possible formulas for such a function is the following C^3 class spline

$$u_i^0 = \begin{cases} 1 - 6d^2 + 8d^3 - 3d^4 & \text{for } \ d \leq 1 \\ 0 & \text{for } \ d \geq 1 \end{cases} \tag{2.19}$$

where $d = r/\rho$, r is the distance from the center point A, $i = 1, 2, 3$.

One can easily verify that the field u^0 is the solution to the problem discussed here, if the nonelastic strain ε^* is assumed to be equal to this particular strain field ε^0. Note that the stresses resulting from such a solution are identically equal to zero (Eq (2.4) for $\varepsilon^* = \varepsilon^0$ and $\varepsilon = \varepsilon^0$). This proves nonuniqueness of the plastic strains obtained from the approach (2.13). We have not encountered any problems during the numerical analysis with this lack of uniqueness of the nonelastic strain because the ε^0 strain has to be a nonlinear function, while in our numerical approach the nonelastic strain is approximated by linear functions.

Another approach which can be used for the direct estimation of nonelastic strain and, consequently, the residual stress field resulting from the cyclic loading is the Zarka model (Zarka, 1980). One of the main steps of this method is evolution of the same, as previously, relation (2.16). According to the literature there is no problem with nonuniqueness of plastic strains in this approach. Since, BEM was not used together with the Zarka approch, the details of this algorithm will not be discussed here.

3. Algorithm of the BEM application to estimation of the residual stresses in railroad rails

The direct BEM (cf Cecot, 1989; Cecot and Orkisz, 1992, 1993) has been applied to approximation of residual deformation of railroad rails. In order to reduce this 3D problem to a 2D one we have assumed that the nonelastic strain field, which develops during the material yielding, proceding the steady shakedown state, is a function of only two transverse coordinates and does not change along the rail axis (the crossties which support rails have been replaced with a continuous foundation). Moreover, we simulated the contact with the wheels by an arbitrary distributed pressure and we approximated the service loading by a cyclic and quasistatic one. First, the elastic–perfectly plastic model of material was assumed ($c = 0$ in Eq (2.15)). Nevertheless, reasonable numerical results have been obtained.

Our 2D problem is not a typical one, since it is neither the plane strain nor the plane stress state. All components of the displacement field are assumed to be non-zero and depend, like the nonelastic strain, only on the two transverse coordinates (x_1, x_2). Therefore, we obtain fully populated strain and stress tensors, also dependent only on the two coordinates. Such a deformation is governed by three local equations of equilibrium, i.e. Eq (2.1) for $i, j = 1, 2$

(plane strain) and additionally the third equation in the form

$$u_{3,11} + u_{3,22} = -2\varepsilon^*_{33} \tag{3.1}$$

Analogically, despite the two integral equations (2.10) or (2.12) like for the plane strain case, the following third equation exists

$$\kappa u_3 = \int\limits_{\partial\Omega} (\mathcal{U}t_i - \mathcal{T}u_i)\, ds + \int\limits_{\Omega_p} b^*_i\mathcal{U}\, d\Omega + \int\limits_{\partial\Omega_p} t^*_i\mathcal{U}\, ds \quad \text{on } \partial\Omega \tag{3.2}$$

or

$$u_3 = \int\limits_{c} \mathcal{U}\varphi_i\, ds + \int\limits_{\Omega_p} b^*_i\mathcal{U}\, d\Omega + \int\limits_{\partial\Omega_p} t^*_i\mathcal{U}\, ds \tag{3.3}$$

where $\mathcal{U} = L/\mu$. and L is the fundamental solution to the Laplace equation.

General algorithm being used for solution to the shakedown problems by the direct BEM approach may be stated as follows.

3.1. Discretization

The boundary of the computational domain is subdivided into isoparametric finite elements (linear. quadratic and cubic shape functions were used). Additionally the plastic zone is subdivided into triangular linear elements to approximate nonelastic strain (six components at each node). However. this interior degrees of freedom have different character than the boundary ones. These are treated as parameters. and BEM is used to evaluate the residual displacements and stresses as functions of this interior nonelastic strain field. The real plastic zone is not known *a priori* and the computational plastic zone is assumed to be such that it contains the real one.

In order to improve the efficiency and accuracy of the method the whole domain is divided into a few subregions. Therefore, one obtains some additional degrees of freedom along the intersubdomain boundaries and it is necessary to satisfy the continuity condition (for displacement and traction). However, the condition number of the resulting system of algebraic equations is smaller.

3.2. Solution to the integral equations

The integral equations (2.10) can be solved numerically either using the collocation method or the Galerkin approach, i.e. by multiplying the integral equation by a test function and integrating it along the boundary. The Galerkin approach requires an additional integration but results in a symmetric system of the linear. algebraic equations. On the other hand. the collocation method can be easily applied to the integral equations, since they do not require any differentiation of the unknown functions. It is also faster than the Galerkin method, but results in the nonsymmetric system of linear equations. The collocation method is used here, however, the Galerkin approach is worth to be investigated. Both methods give systems of linear algebraic equations of the following type

$$\mathbf{A}z = \mathbf{B}\epsilon^{*} \tag{3.4}$$

where $\mathbf{A}$, $\mathbf{B}$ are known matrices, different for the collocation and the Galerkin approximations, ϵ^{*} is the vector of discrete values of the nonelatic strain tensor and z is the vector of nodal values of the boundary unknown functions. Computation of some entries of the matrices $\mathbf{A}$, $\mathbf{B}$ requires evaluation of singular integrals. It is done either analytically or using the rigid body motion approach (see Burczyński, 1995) or the polar transformation proposed by Li et al. (1985). A special attention is also required while the *almost* singular integrals are being computed. Refinement and enrichment of the Gauss integration scheme is used in this case but it was done in an arbitrary rather than in an automatic way. The adaptive integration was studied and applied by Karafiat (1998).

From the linear system of equations (3.4) the following relation is obtained

$$z = \mathbf{D}\epsilon^{*} \tag{3.5}$$

where $\mathbf{D} = \mathbf{A}^{-1}\mathbf{B}$.

3.3. Error estimation and adaptation

After the numerical solution, i.e. Eq (3.5), is obtained, an answer to the question what the approximation error is and how it can be effectively minimized is very welcome. *A posteriori* error estimation techniques for the finite element method were already developed (cf Ainsworth and Oden, 1993; Babuska et al., 1994; Demkowicz et al., 1991). These provide approximation of the error on each finite element which is the basis for an adaptive meshing procedure designed to minimize the error. Gradient recovery and residual analysis are two main types of the error estimates. Each of them has many versions. One of the most robust and mathematically sound is the equilibrated implicit residual one, Ainsworth and Oden (1993). Similar techniques can be applied when using the BEM approach (cf Demkowicz et al., 1989, 1991) where also the residuum of integral equation can be used to measure the numerical solution error.

In the nonelastic problems an additional estimate of the error is necessary for nonelastic strain tensor. This field, approximated by triangular elements, is the input data for the BEM analysis. The standard interpolation error estimation based on the Taylor series (Becker et al., 1981) seems to be useful here. Let ε_{ij}^{*h} be a linear interpolant of nonelastic strain, suppose that ε_{ij}^{*} has bounded the second derivatives and let $e_{ij} = \varepsilon_{ij}^{*} - \varepsilon_{ij}^{*h}$ be the interpolation error function. Following the method proposed by Becker et al. (1981) for 1D problem, the error estimate for an arbitrary triangle K assumes the form

$$\|e_{ij}\|_{\infty} \leq Ch^2 \tag{3.6}$$

C being a constant independent of the element size h but proportional to maximum of the second derivatives of the plastic strain, i.e to $d^2\varepsilon_{ij}^{*}$, where

$$d^2 f = \frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2}h_x^2 + 2\frac{\partial^2 f}{\partial x \partial y}h_x h_y + \frac{\partial^2 f}{\partial y^2}h_y^2\Big) \tag{3.7}$$

and h_x, h_y are distances from the closest node of traingle in the x and y directions, respectively. The general conclusion from the above analysis is that, wherever the second derivatives of nonelastic strain are high, the interior mesh should be refined or enriched there.

3.4. Minimization

Eq (3.5) can be used together with discrete form of Eq (2.10) to obtain the following relation

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \mathsf{F}(\boldsymbol{x})\boldsymbol{\epsilon}^* \tag{3.8}$$

which expresses the stress tensor at any point $\boldsymbol{x}$ in terms of the nodal values of the nonelastic strain. Eq (3.8) is substituted into the functional (2.14) resulting in the following function

$$\mathsf{I}(\boldsymbol{\epsilon}^*) = (\boldsymbol{\epsilon}^*)^{\top}\mathsf{G}\boldsymbol{\epsilon}^* \tag{3.9}$$

Minimization of the last function in the bounded domain (2.15), gives the nodal values of the nonelastic strain tensor. The minimization is done by a package of optimization routines. Now, the residual displacement, strain and stress fields can also be computed.

4. Numerical results

Solution to the benchmark cylinder problem as well as to a real engineering one, i.e. a railroad rail subject to service condition load are discussed in this section.

A thick walled cylinder under cyclic internal pressure was solved as a plane strain problem. The main objective of this test was validation of the proposed algorithm and the computer code. The advantage of the axisymmetry was not taken into account *a priori*, however only a sector of the cross section was analyzed with about 100 scalar degrees of freedom. The numerical solution differed from the analytical one by less than 2% in L_∞ norm for residual stresses.

The following parameters were assumed for the analysis of residual states in a railroad rail $E = 207\,\mathrm{GPa}$, $\nu = 0.3$, $\sigma_y = 480\,\mathrm{MPa}$, $P = 147\,\mathrm{kN}$ (maximum of the assumed cyclic load). The vertical force P, modeling contact, was distributed biparabolically on a $1.9\,\mathrm{cm} \times 1.2\,\mathrm{cm}$ patch at the rail head top surface.

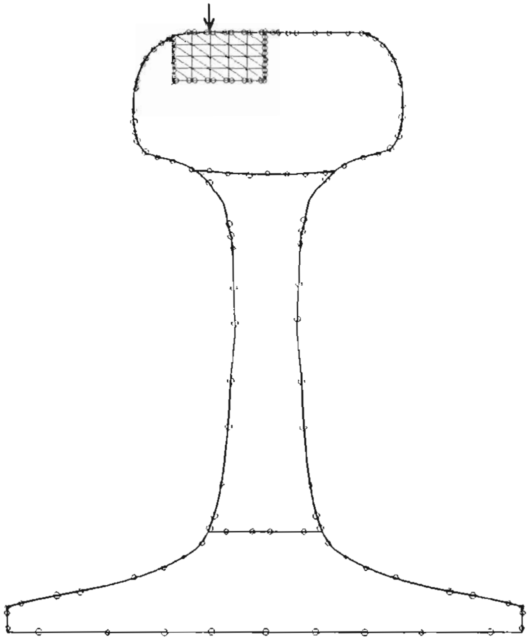

Fig. 1. BEM discretization of the railroad rail cross-section

The boundary element discretization, with about 340 scalar degrees of freedom is shown in Fig.1 and the corresponding result in Fig.2. BEM was one

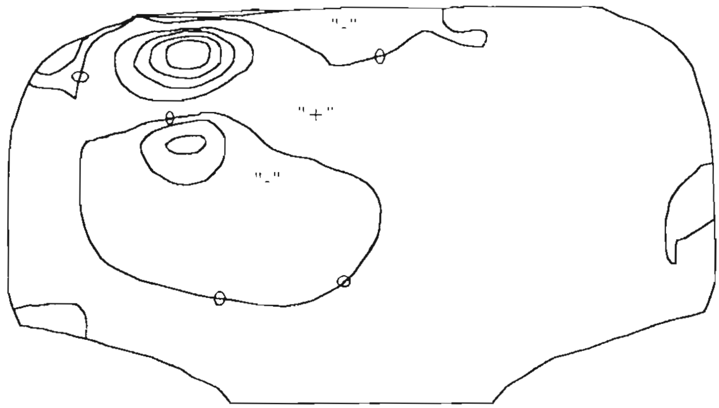

Fig. 2. Longitudinal residual stress in the rail head computed using the BEM approach

of the methods, which were used for discretization of the problem of residual stress estimation. Opposite to FEM and the FDM, it required incorporation of kinematic quantities to the formulation (the selfequilibrated stresses are expressed in terms of the nonelatic strain). Later on such approach become one of the key points of generalization of the model onto a material with kinematic hardening and had also to be applied in both the FEM and FDM discretizations. Extreme values of the residual stress components are compared with those obtained by FEM (after Holowiński and Orkisz, 1992) and the FDM (after Orkisz and Pazdanowski, 1995) in Table 1. For much smaller number of degrees of freedom BEM gave results similar to the other methods.

Table 1. Comparison between extreme residual stress components obtained by different numerical methods for the rail

Stress Component		Method (number of degrees of freedom)		
		BEM (340)	FEM (3200)	FDM (3000)
σ_{xx}	min	−211	−205	−221
	max	117	125	156
σ_{yy}	min	−125	−94	−108
	max	54	52	77
σ_{zz}	min	−200	−171	−160
	max	49	48	66
σ_{xy}	min	−74	−63	−83
	max	57	62	62

5. Final remarks

An application of the BEM approach to the analysis of residual stresses in railroad rails was the main objective of this work. This method has been used first to find a solution to the benchmark problem, i.e. a thick walled cylinder subject to cyclic internal pressure. The results obtained using BEM were compared with the analytical ones showing a good agreement.

The method was also successfully applied to the preliminary analysis of residual states in railroad rails and validated their engineering value. The BEM approach proved to be an efficient method of analysis of residual stresses. The following conclusions have been drawn:

- a few degrees of freedom were enough to obtain reasonable results.
- the BEM algorithm is more complicated than the FEM or FDM ones,
- the efficiency of the method is better for small plastic zones (such a situation exists in real rail problems).

Some further research work should be undertaken on the shakedown analysis by BEM. Error estimation and adaptation are the most important tasks, since they would increase efficiency of the solution and reliability of the results.

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Zastosowanie metod elementów brzegowych do rozwiązywania zagadnień typu "shakedown"

Streszczenie

W artykule opisano model numeryczny służący do rozwiązywania zagadnień dostosowania się cial sprężysto-plastycznych do obciążeń cyklicznych. Zastosowano pośrednią i bezpośrednią wersję metody elementów brzegowych. Wyniki obliczeń numerycznych porównano z otrzymanymi przy pomocy metod elementów i różnic skończonych. Ich zgodność potwierdza skuteczność i poprawność zastosowanego podejścia. Jako dalsze usprawnienie algorytmu zaproponowano włączenie do algorytmu oszacowania *a posteriori* błędu obliczeń oraz adaptację siatki.

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