JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, 36, 1998

SPATIAL PERTURBATIONS IN THE SOLIDIFICATION PROCESS DUE TO MOLD DELAMINATIONS

VOLODYMYR J. PAUK

MAŁGORZATA WOŹNIAK

Department of Geotechnical and Structure Engineering, Łódź University of Technology

The solidification process involving small periodical perturbations due to the system of mold delaminations is considered. Imperfections at the mold wall are modelled by non-ideal thermal contact conditions. It leads to the small perturbation in the solid-liquid interface and in the temperature distribution in the solid phase. With the help of approximate technique, known as the heat-balance integral method, the problem is reduced to the sequence of ordinary differential equations in amplitudes of the Fourier series of the solidification front. The shape of solid-liquid interfaces is shown.

Key words: solidification, mold delamination

1. Introduction

The heat transfer problem involving the phase change, known as the Stefan problem is met in the modelling of the solidification and melting processes. This class of problems stimulated considerable practical interest because the applications (ice formation, metallurgical progresses, geophysical, astrophysical, chemical problems etc.) are of greatest engineering importance. The wide bibliography on solidification and thawing problems may be found in the monograph by Lunardini (1991).

In this paper we consider the two-dimensional non-stationary heat transfer problem in the half-space with a change of phases involving delaminations between the solid phase and mold. These imperfections are distributed periodically and yield small periodical perturbations in the temperature field and in the solid-liquid moving interface.

With the help of approximate technique, known as the heat-balance integral method, the problem is reduced to the sequence of ordinary differential equations in the macro-thickness of solidified layer and amplitudes of perturbations in the solid-liquid interface. The analytical solutions are obtained neglecting squares of small quantities.

The perturbations in the phase change problem produced by the inhomogeneities of mold boundary conditions were previously studied by Mullins and Sekerka (1964), Nai-Yi Li and Barber (1989), Howarth (1990, 1993).

2. Formulation of the problem

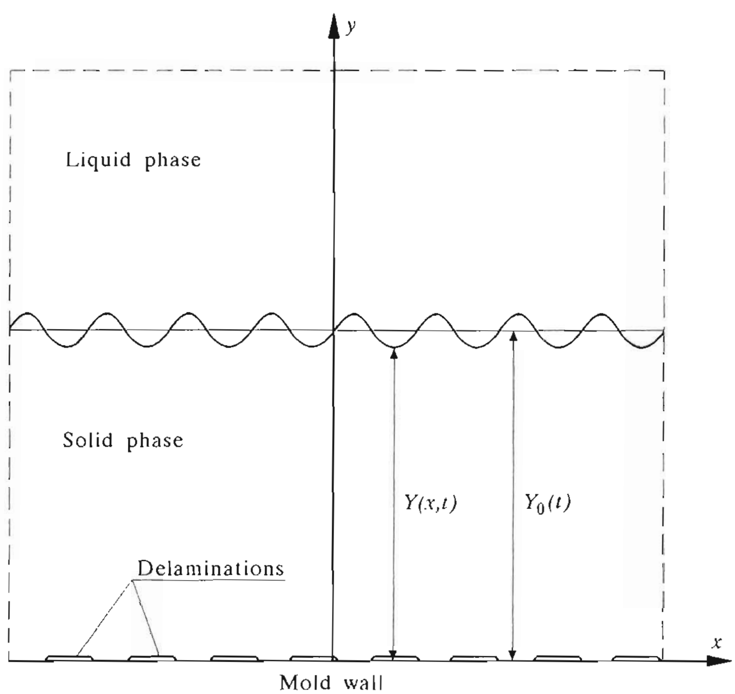

Fig. 1.

Consider a solidification process at the plane wall $y = 0$. Let us assume that the half-space $y > 0$ is initially filled with a liquid of the melting temperature T_m. The mold temperature is constant and equal to $\overline{T}_0$. Since $\overline{T}_0 < T_m$ the solidification process strarts at the mold. For instants $t > 0$ the solid phases occupies the region $0 < y < Y(x,t)$ where the solidification front $Y(x,t)$ is unknown and moves in the y-direction. It is assumed that the thermal contact between the solid and mold wall is non-ideal because a system of delaminations appears on the line $y = 0$. These boundary imperfections are situated periodicaly with the period l. The problem will be idealized on the assumption that the temperature of the liquid phase is always equal to T_m and that all material properties of the solid remain unchanged.

The problem fomulated above is reduced to solving of the heat conduction equation in the temperature $T(x,y,t)$ of the solid phase

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} &T(x,0,t) - \overline{T}_0 = r(x,t)\frac{\partial T(x,0,t)}{\partial y} \\ &T(x,Y,t) = T_m \\ &\frac{\partial T(x,Y,t)}{\partial n} = \frac{\rho L}{K}\frac{\partial Y}{\partial t} \end{aligned} \tag{2.2}$$

where k, K, ρ are the themal diffusivity, conductivity and mass density of the solid, respectively and n is the normal to the solidification line. The following l-periodical function of thermal resistance $r(x,t)$ can be assumed

$$r(x,t) = \begin{cases} r_0 = \text{const} & |x| < \frac{l_0}{2} \\ 0 & \frac{l_0}{2} < |x| < \frac{l}{2} \end{cases} \tag{2.3}$$

where l_0 is the length of isolated delamination.

We shall consider small imperfections in the thermal contact between the solid and mold wall. Introducing the small parameter

$$\delta = \frac{\max_{0<x<l} |T(x,0,t) - \overline{T}_0|}{\overline{T}_0} \ll 1 \tag{2.4}$$

we can write $r(x,t)$, $r_0 \in O(\delta)$.

The formulation of the problem should be completed by the initial conditions

$$T(x,y,0) = T_m \qquad\qquad Y(x,0) = 0 \tag{2.5}$$

which describe the situation before the solidification process.

Let us assume that small periodical imperfections in the thermal contact at the line $y = 0$ yield small periodical perturbations in the temperature field and solid-liquid front. We shall present the unknown functions $T(x,y,t)$, $Y(x,t)$ as follows

$$T(x,y,t) = T_0(y,t) + \widetilde{T}(x,y,t) \qquad Y(x,t) = Y_0(t) + \widetilde{Y}(x,t) \tag{2.6}$$

where the terms $T_0(y,t)$ and $Y_0(t)$ can be treated as the field of macrotemperature in the solid and the macrolocation of solidification line, respectively; the functions $\widetilde{T}(x,y,t)$ and $\widetilde{Y}(x,t)$ represent small periodical oscilations of the above functions, i.e.

$$\widetilde{T}(x,y,t) \ll T_0(y,t) \qquad \widetilde{Y}(x,t) \ll Y_0(t)$$

After expanding the functions $\widetilde{T}(x,y,t)$, $\widetilde{Y}(x,t)$ into the Fourier cosine series, Eqs (2.6) can be transformed

$$\begin{aligned} T(x,y,t) &= T_0(y,t) + \sum_{j=1}^{\infty} T_j(y,t)\cos(x\omega_j) \\ Y(x,t) &= Y_0(t) + \sum_{j=1}^{\infty} Y_j(t)\cos(x\omega_j) \end{aligned} \tag{2.7}$$

where $T_j, Y_j \in O(\delta)$ are unknown small amplitudes of perturbations in the temperature field and solidification line, respectively, and $\omega_j = 2\pi j/l$ are the frequencies of these oscilations. Here and farther on the index j runs over $1, 2, 3, \dots$

3. Method of solution

For the solution to the problem formulated above the heat-balance integral method proposed by Goodman (1958) is to be used. The first main idea of this approximate technique consists in solving the heat conduction equation (2.1) in the averaged form. This form is called the heat-balance integral equation. It is obtained by integration of the heat conduction equation (2.1) across the solidified layer. In result we obtain

$$\frac{1}{k}\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial T(x,Y,t)}{\partial y} - \frac{\partial T(x,0,t)}{\partial y} \tag{3.1}$$

where the new fuction

$$\Psi(x,t) = \Theta(x,t) - T_m Y(x,t) \tag{3.2}$$

is introduced and

$$\Theta(x,t) = \int\limits_0^{Y(x,t)} T(x,y,t)\,dy \tag{3.3}$$

is the mean temperature in the solid layer.

The second hypothesis of the heat-balance integral method postulates the distribution of temperature across the solidified layer. Let us assume that the unknown functions $T_0(y,t)$ and $T_j(y,t)$ can be presented by the following second-order polinomials of y

$$\begin{aligned} T_0(y,t) &= a_0(t) + b_0(t)[y - Y_0(t)] + c_0(t)[y - Y_0(t)]^2 \\ T_j(y,t) &= a_j(t) + b_j(t)[y - Y_0(t)] + c_j(t)[y - Y_0(t)]^2 \end{aligned} \tag{3.4}$$

where the time-dependent coefficients a_0, b_0, c_0 and a_j, b_j, c_j can be found from boundary conditions (2.2).

Substituting into the boundary condition $(2.2)_1$ Eq $(2.7)_1$ and the Fourier cosine expansion of the function (2.3) and separating in the obtained formula the uniform and periodic terms we obtain the sequence of equations

$$T_0(0,t) = \overline{T}_0 \qquad\qquad T_j(0,t) = \frac{\partial T_0(0,t)}{\partial y} r_j(t) \tag{3.5}$$

Here, $r_j(t)$ are the amplitudes of Fourier series of the thermal resistance function $r(x,t)$. The conditions (3.5) are obtained neglecting values of an order $O(\delta^2)$.

The boundary condition $(2.2)_2$ is formulated on the slightly folded solidification front $Y(x,t)$. Expanding the temperature $T(x,y,t)$ on the line $Y(x,t)$ into a Taylor series in the vicinity of the surface $Y_0(t)$ and neglecting the terms $O(\delta^2)$ we obtain

$$\begin{aligned} T(x,Y,t) &= T_0(Y,t) + \sum_{j=1}^{\infty} T_j(Y,t)\cos(x\omega_j) = \\ &= T_0(Y_0,t) + \frac{\partial T_0(Y_0,t)}{\partial y} \sum_{j=1}^{\infty} Y_j(t)\cos(x\omega_j) + \sum_{j=1}^{\infty} T_j(Y_0,t)\cos(x\omega_j) \end{aligned} \tag{3.6}$$

Satisfying the boundary conditions $(2.2)_2$ with the help of Eq (3.6) and equating the terms of equal frequencies ω_j we arrive at the equations

$$T_0(Y_0,t) = T_m \qquad\qquad T_j(Y_0,t) + Y_j(t)\frac{\partial T_0(Y_0,t)}{\partial y} = 0 \tag{3.7}$$

Before using the heat-balance integral method the boundary condition $(2.2)_3$ must be transformed to the form (see Goodman, 1959)

$$\Big[\frac{\partial T}{\partial n}\Big]^2 = -kA\nabla^2 T \qquad \text{for} \qquad y = Y(x,t) \tag{3.8}$$

Applying to this condition the procedure similar to that used above we obtain the sequence of conditions on the line $y = Y_0(t)$

$$\begin{aligned} &\Big[\frac{\partial T_0(Y_0,t)}{\partial y}\Big]^2 = -kA\frac{\partial^2 T_0(Y_0,t)}{\partial y^2} \\ &2\frac{\partial T_0(Y_0,t)}{\partial y}\Big[\frac{\partial T_j(Y_0,t)}{\partial y} + Y_j(t)\frac{\partial^2 T_0(Y_0,t)}{\partial y^2}\Big] = \\ &= -kA\Big[\frac{\partial^2 T_j(Y_0,t)}{\partial y^2} + Y_j(t)\frac{\partial^3 T_0(Y_0,t)}{\partial y^3} - \omega_j^2 T_j(Y_0,t)\Big] \end{aligned} \tag{3.9}$$

Satisfying the boundary conditions (3.5), (3.7), (3.9) with the help of Eqs (3.4) we arrive at the system of algebraic equations in functions $a_0(t)$, $b_0(t)$, $c_0(t)$ and $a_j(t)$, $b_j(t)$, $c_j(t)$ which has the solution

$$\begin{aligned} a_0(t) &= T_m & b_0(t) &= -\frac{kA\kappa}{Y_0(t)} \\ c_0(t) &= -\frac{kA(2\kappa+\mu)}{2Y_0^2(t)} & a_j(t) &= \frac{kA\kappa Y_j(t)}{Y_0(t)} \end{aligned}$$
$$\begin{aligned} b_j(t) &= \frac{1}{\kappa-1}\frac{kA}{Y_0^2(t)}\Big\{(\kappa+\mu)r_j(t) + \kappa Y_j(t)\Big[2\kappa - 1 + \mu - \frac{1}{2}\omega_j^2 Y_0^2(t)\Big]\Big\} \\ c_j(t) &= \frac{\kappa}{\kappa-1}\frac{kA}{Y_0^2(t)}\Big\{(\kappa+\mu)r_j(t) + Y_j(t)\Big[\kappa + \mu - \frac{1}{2}\omega_j^2 Y_0^2(t)\Big]\Big\} \end{aligned} \tag{3.10}$$

where the denotations

$$\kappa = 1 - \sqrt{1+\mu} \qquad\qquad \mu = 2\frac{T_m - \overline{T}_0}{kA}$$

are introduced.

The solwing procedure includes the following steps:

(i) calculation of the temperature $T(x,y,t)$ in the solid phase using the formulae $(2.7)_1$, (3.4);

(ii) determination of the mean temperature $\Theta(x,t)$ using Eq (3.3);

(iii) calculation of the function $\Psi(x,t)$ defined by Eq (3.2);

(iv) substitution the functions $\Psi(x,t)$ and $T(x,y,t)$ into the heat-balance integral (3.1) and separation periodic and uniform terms in the obtained equation.

In result of the described procedure we arrive at the sequence of differential equations

$$Y_0'(t)Y_0(t) = kH_0 \qquad\qquad Y_j'(t) + P_j(t)Y_j(t) = R_j(t) \tag{3.11}$$

where

$$\begin{aligned}
H_0 &= 6\frac{\mu+1-\sqrt{\mu+1}}{\mu+5+\sqrt{\mu+1}}\\
P_j(t) &= \frac{k\kappa}{Y_0^2(t)}\,\frac{\frac{1}{2}\omega_j^4Y_0^4(t)+[H_0+2\kappa-\mu]\omega_j^2Y_0^2(t)-6(\mu+1)}{\frac{\kappa}{2}\omega_j^2Y_0^2(t)+2\kappa^2-9\kappa-\kappa\mu+6}\\
R_j(t) &= \frac{\kappa+\mu}{Y_0^2(t)}\,\frac{[(12\kappa-6-H_0)-(2\kappa-3)\omega_j^2Y_0^2(t)]kr_j(t)-(2\kappa-3)r_j'(t)}{\frac{\kappa}{2}\omega_j^2Y_0^2(t)+2\kappa^2-9\kappa-\kappa\mu+6}
\end{aligned} \tag{3.12}$$

These equations must be solved together with the initial conditions

$$Y_0(t) = 0 \qquad\qquad Y_j(t) = 0 \tag{3.13}$$

which are obtained Eqs $(2.5)_2$, $(2.7)_2$.

The macrothickness of the solid layer $Y_0(t)$ which satisfies Eqs $(3.11)_1$, $(3.13)_1$ can be presented by the formula

$$Y_0(t) = \sqrt{2kH_0t} \tag{3.14}$$

The solutions to the problems $(3.11)_2$, $(3.13)_2$ can be obtained in the following form

$$Y_j(t) = \int\limits_0^t R_j(\zeta)\exp\Big[\int\limits_t^{\zeta} P_j(s)\;ds\Big]\;d\zeta \tag{3.15}$$

Introducing the dimensionless functions

$$h_0(\tau) = \omega Y_0(t) \qquad\qquad h_j(\tau) = \omega Y_j(t) \qquad\qquad \omega = \frac{2\pi}{l} \tag{3.16}$$

the obtained solutions can be rewritten as

$$\begin{aligned}
h_0(\tau) &= \sqrt{2H_0\tau}\\
h_j(\tau) &= \int\limits_0^{\tau} R_j^*(\xi)\exp\Big[\int\limits_{\tau}^{\xi} P_j^*(s)\;ds\Big]\;d\xi
\end{aligned} \tag{3.17}$$

where $\tau = k\omega^2 t$ is the dimensionless time coordinate and

$$P_j^*(\tau) = \frac{\kappa}{h_0^2(\tau)} \frac{\frac{1}{2}j^4 h_0^4(\tau) + [H_0 + 2\kappa - \mu] j^2 h_0^2(\tau) - 6(\mu + 1)}{\frac{\kappa}{2} j^2 h_0^2(\tau) + 2\kappa^2 - 9\kappa - \kappa\mu + 6}$$

$$R_j^*(\tau) = \frac{\kappa + \mu}{h_0^2(\tau)} \cdot$$

$$\cdot \frac{[12\kappa - 6 + (2\kappa - 4)H_0 - (2\kappa - 3) j^2 h_0^2(\tau)] r_j^*(\tau) - (2\kappa - 3) r_j'^* h_0^2(\tau)}{\frac{\kappa}{2} j^2 h_0^2(\tau) + 2\kappa^2 - 9\kappa - \kappa\mu + 6}$$

Taking into account Eq (2.3) the dimensionless amplitudes of Fourier series of the thermal resistance $r_j^*(\tau) = \omega r_j(t)$ can be represented in the following form

$$r_j^*(\tau) = \frac{2\omega r_0}{\pi j} \sin(\pi j \lambda) \tag{3.18}$$

where $\lambda = l_0/l$ is a parameter of the problem.

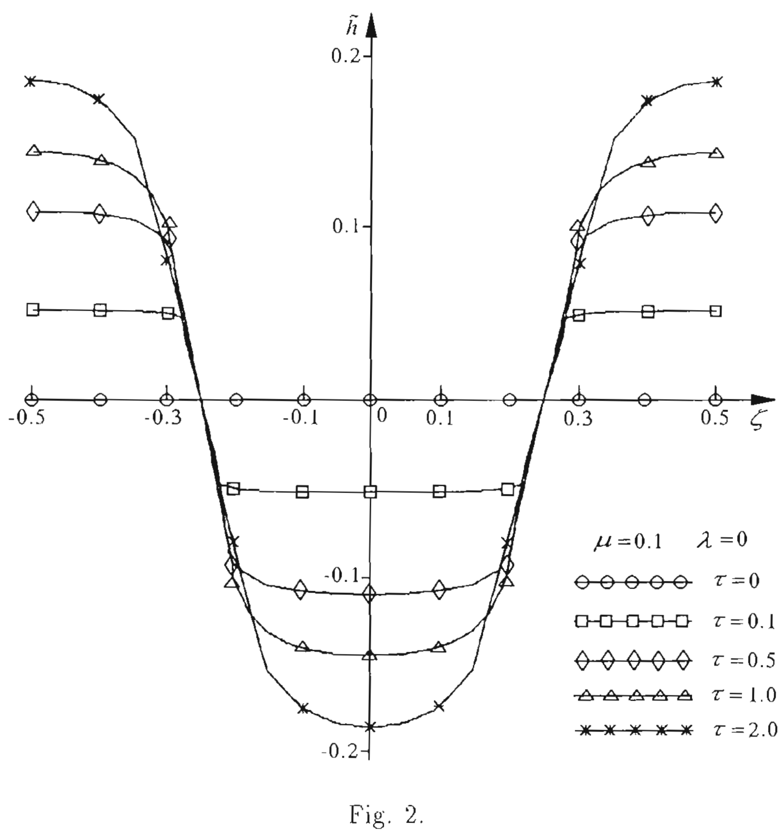

Fig. 2.

4. Results

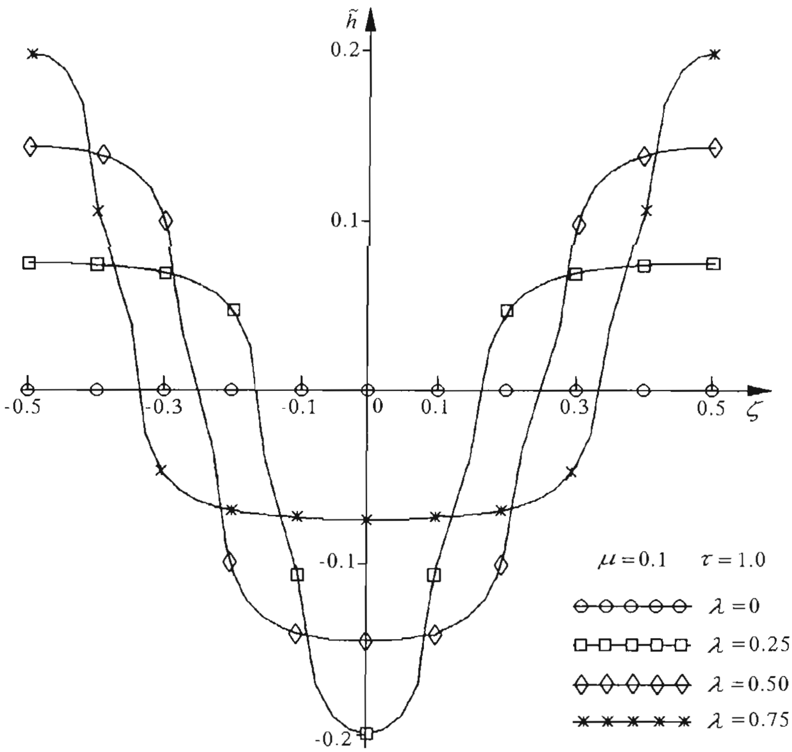

Fig. 3.

The attention in the numerical analysis will be concentrated on the shape of solid-liquid interface. Let us introduce the dimensionless function

$$\widetilde{h}(\zeta,\tau) = \frac{1}{r_0}\widetilde{Y}(x,t) \qquad \zeta = \frac{x}{l} \tag{4.1}$$

which describes oscilations in the solidification line. In accordance with Eq $(2.7)_2$ this function was calculated as the sum

$$\widetilde{h}(\zeta,\tau) = \sum_{n=1}^{N} h_n(\tau)\cos(2\pi n\zeta) \tag{4.2}$$

where the amplitudes $h_n(\tau)$, $n = 1,...,N$ are determinated from Eqs $(3.17)_2$ and the number N is chosen depending on the values of input parameters.

Fig.2 presents the shape of moving interface $\widetilde{h}(\zeta,\tau)$ whithin the limits of one period $|\zeta| \leq 1/2$. These results were obtained for fixed values of the parameters λ, μ and for some values of the dimensionless time τ. It is clear that perturbations of the solidification boundary increase with time.

The effect of the delamination length l_0, i.e. parameter λ, on the shape of the solidification line is shown in Fig.3 for the fixed values of time and coefficient μ. The considerable changes in the perturbation shape are observed.

References

1. GOODMAN T.R., 1958, The Heat-Balance Integral and its Application to Problems Involving a Change of Phase, *Trans. ASME*, **80**, 335-342
2. HOWARTH J.A., 1990, The Stefan Problem for a Slightly Wavy Wall, *Mech. Res. Comm.*, **17**, 25-31
3. HOWARTH J.A., 1993, The Stefan Problem with Slightly Varying Wall Temperature, *Mech. Res. Comm.*, **20**, 59-64
4. LUNARDINI V.J., 1991, *Heat Transfer with Freezing and Thawing*, Elsevier Publishers, Amsterdam
5. MULLINS W.W., SEKERKA R.F., 1964, Stability of a Planar Inerface During Solidification of a Dilute Binary Alloy, *J. of Appl. Phys.*, **35**, 444-451
6. NAI-YI LI, BARBER J.R., 1989, Sinusoidal Perturbation Solutions for Planar Solidification, *Int. J. of Heat and Mass Transfer*, **32**, 935-941

Zaburzenia przestrzenne procesów krzepnięcia wywołane delaminacjami na ściance formy

Streszczenie

W pracy rozpatruje się zagadnienie dwuwymiarowego niestacjonarnego przeplywu ciepla w pólprzestrzeni ze zmianami fazowymi. Uwzględniono zaburzenia przestrzenne wywołane delaminacjami pomiędzy fazą stalą a ścianką formy. Imperfekcje mają rozkład periodyczny i powodują male periodyczne zaburzenia zarówno w polu temperatury jak i w ruchomej powierzchni rozgraniczającej fazy stalą i cieklą.

W oparciu o przybliżoną metodę znaną jako metodę calek bilansu ciepla, zagadnienie sprowadza się do ciągu równań różniczkowych zwyczajnych dla makro-grubości krzepnącej warstwy oraz amplitud zaburzeń frontu krzepnięcia. Rozwiązania analityczne otrzymano pominąwszy kwadraty pewnych malych wielkości. Na wykresach przedstawiono zmiany zaburzeń w czasie dla różnych wartości wyjściowych parametrów.

Manuscript received November 5, 1997; accepted for print December 16, 1997