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A MODEL OF THERMOELASTIC DYNAMIC CONTACT IN CONDITIONS OF FRICTIONAL HEAT AND WEAR

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We have constructed and investigated a model of thermoelastic contact of solids with inertia forces taken into account, in conditions of frictional heat and mechanical wear. It has been assumed that a layer is in a relative motion with respect to a wall. During the motion the gap between the layer and the wall changes due to heat expansion of the layer. The influence of material constants, velocity and parameters of the model on the contact characteristics have been investigated. The solution to the problem has been reduced to a system of the Volterra-Hammerstein non-linear integral equations, which in turn has been solved by means of the developed algorithm. The numerical solutions have been presented in diagrams.

Key words: dynamic contact problem, friction heat, wear, Volterra-Hammerstein integral equations

1. Introduction

The inertia forces play an important role in the motion of solids with sliding contact. As a result of sliding the friction forces appear, consequently the moving solid deccelarates and we can speak of braking action. The friction forces are accompanied by the frictional heat, which in the cases of two-sided constraints lead to additional frictional contact forces, additional heat generation and increase of wear. On the other hand such additional friction reduces the relative speed of solids and, in turn, the amount of generated frictional heat. Then the temperature of the solids decreases and the relative speed can increase again generating the increase in the emission of frictional heat. It is important to know the variation of the thermo-elastic characteristics, i.e. pressure, temperature, velocity of the relative motion, and wear. In the initial stage of motion they can be monotonic, generally it is not the case.

The purpose of the paper is to construct a dynamical model of thermoelastic contact of solids in the conditions of frictional heat generation, wear, and confinement of the solid volume due to the thermal expansion. The characteristic features of such a model are discussed. A similar problem has been already discussed by Pyryev (1994), however without wear taken into account. So far, the problems either with uniform velocity without inertia forces taken into account were considered (see for example Alexandrov et al., 1990; Yevtushenko and Pyryev, 1997) or with constant pressure applied during the whole process of braking (Chichinadze et al., 1979).

We shall assume that the contact region and the geometry of solids are such that we are in position to approximate problem by a 1D model. The application of 1D models let us discuss the problem analytically and draw interesting conclusions (Alexandrov, 1990; Chichinadze, 1979; Lee and Barber, 1993; Olesiak et al., 1997; Pyryev et al., 1995). The theoretical investigation of the braking processes is complicated even in 1D approximation and, as we shall see, it leads to non-linear boundary value problems.

In this paper the boundary value problem of the quasistationary thermoelastic problem with non-linear boundary conditions has been reduced to Voltera's type equations.

2. Statement of the problem

Let an elastic, heat conducting layer of thickness L and heat conductivity K be rigidly fixed along one (bottom) of its boundaries (Fig.1). The other (uppon) boundary is deformed by a rigid, thermally isolated half-infinite wall up to the value of $u = u_0\phi_u(t)$. The solid body has mass M and moves under the action of force $F = F_0\phi_F(t)$ in the direction of z axis. The quantities L and M are treated as the weight functions, i.e. they are taken with respect to the unit of the contact surface. The initial velocity of the plate at $t = 0$ is equal to v_0. The friction force $F_{fr} = fp$ arises in the contact region between the layer and the moving solid and consequently a heat production and wear take place leading to an increase in temperature. The heat exchange between the layer and surrounding space, obeying Newton's law, takes place through the lower bounding plane.

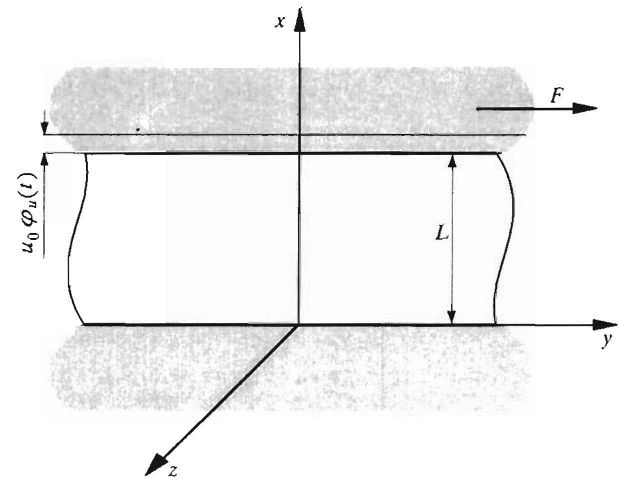

Fig. 1. Relative motion of solids with heat of friction and wear

Our task is to determine field of temperature $\theta(x,t)$, fields of displacements $u(x,t)$, $w(x,t)$ along x and z axes, respectively. The value of the layer wear is $u^w(t)$ while the speed of the solid is equal to $v(t)$.

From the mathematical standpoint the problem can be reduced to the solution to the system of differential equations of quasistatic uncoupled theory of elasticity

$$
\frac{\partial}{\partial x}\Big[\frac{\partial}{\partial x}u(x,t) - \alpha\frac{1+\nu}{1-\nu}\theta(x,t)\Big] = 0 \qquad \frac{\partial^2}{\partial x^2}w(x,t) = 0
$$
$$
\frac{\partial^2}{\partial x^2}\theta(x,t) = \frac{1}{k}\frac{\partial}{\partial t}\theta(x,t) \qquad x \in (0,L) \tag{2.1}
$$

and equations of the half-infinite plate motion

$$
M\frac{d}{dt}v(t) = F_0\phi_F(t) - fp(t) \tag{2.2}
$$

We have the following mechanical boundary conditions

$$
u(0,t) = w(0,t) = 0 \qquad \sigma_{xz}(L,t) = fp(t)
$$
$$
u(L,t) = -u_0\phi_u(t) + u^w(t) \tag{2.3}
$$

the thermal boundary conditions

$$
K\frac{\partial}{\partial x}\theta(0,t) = \alpha^\theta\theta(0,t) \qquad K\frac{\partial}{\partial x}\theta(L,t) = fv(t)p(t) \tag{2.4}
$$

and the initial conditions

$$\theta(x,0) = 0 \qquad v(0) = v_0 \qquad x \in (0,L) \tag{2.5}$$

The use is made of a model of abrasive wear (Archard, 1959; Goryacheva et al., 1988) for which

$$u^w(t) = K_u \int_0^t v(\xi)p(\xi)\, d\xi \tag{2.6}$$

We assume that the wear resistance coefficient K_u is constant.

The considered problem has a meaningful solution for time $t < t_i$. Time t_i is defined for the time of contact such that the contact pressure $p(t) = -\sigma_{xx}(0,t) = -\sigma_{xx}(L,t)$ and the solid velocity $v(t)$ are non-negative. From physical conditions it is necessary to set limitations on stresses and temperature under which the thermoelastic model of the layer makes sense (wear of the layer cannot exceed the resource of wearing Alexandrov, 1990).

From the Duhamel-Neumann relations we find the normal component of the stress tensor for the layer

$$\sigma_{xx} = \frac{E}{1-2\nu}\Big[\frac{1-\nu}{1+\nu}\frac{\partial u}{\partial x} - \alpha\theta\Big] \qquad \sigma_{xz} = \frac{E}{1+\nu}\frac{\partial w}{\partial x} \tag{2.7}$$

where

E	–	Young modulus
ν	–	Poisson ratio
k	–	coefficient of thermal diffusivity
α	–	coefficient of the linear thermal expansion
f	–	coefficient of friction
$1/\alpha^\theta$	–	thermal resistance.

The contact pressure p and displacement w are determined from Eq (2.1) by use of Eqs (2.7) and the boundary conditions (2.3)

$$\begin{aligned} p(t) &= \frac{\widetilde{E}}{L}\Big[u_0\phi_u(t) - u^w(t) + \widetilde{\alpha}\int_0^L \theta(\xi,t)\, d\xi\Big] \\ w(x,t) &= \frac{1+\nu}{E} f p(t) x \end{aligned} \tag{2.8}$$

where

$$\widetilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad \widetilde{\alpha} = \alpha\frac{1+\nu}{1-\nu}$$

In order to reduce the system of equations to a dimensionless form we introduce the following quantities

$$
\begin{array}{lllll}
x' = \frac{x}{L} & \tau = \frac{t}{t_*} & \tau_i = \frac{t_i}{t_*} & u' = \frac{u}{u_0} & v' = \frac{v}{v_*} \\
v_0' = \frac{v_0}{v_*} & u^{w\prime} = \frac{u^w}{u_0} & \theta' = \frac{\theta}{\theta_*} & p' = \frac{p}{p_*} & \\
H = \frac{E\alpha k}{K(1-\nu)} & \Omega = \frac{H(1-\nu)}{(1-2\nu)} & \widetilde{\Omega} = f\Omega & \mathrm{Bi} = \frac{L\alpha^\theta}{K} & \\
f_0 = \frac{F_0}{p_*} & a = \frac{L^2\widetilde{E}u_0}{Mk^2} & k_u = \widetilde{E}K_u & \xi = \frac{k_u}{\widetilde{\Omega}} &
\end{array}
$$

and the characteristic parameters

$$
t_* = \frac{L^2}{k} \qquad v_* = \frac{k}{L} \qquad p_* = \frac{\widetilde{E}u_0}{L} \qquad \theta_* = \frac{u_0}{\widetilde{\alpha}L}
$$

As a result, omitting the primes we arrive at the boundary value problem of heat conduction equation with the following non-linear boundary conditions

$$
\frac{\partial^2}{\partial x^2}\theta(x,\tau) = \frac{\partial}{\partial \tau}\theta(x,\tau) \qquad x \in (0,1) \qquad \tau \in (0,\tau_i) \tag{2.9}
$$

$$
\frac{\partial}{\partial x}\theta(0,\tau) = \mathrm{Bi}\theta(0,\tau) \qquad \frac{\partial}{\partial x}\theta(1,\tau) = \widetilde{\Omega}v(\tau)p(\tau) \quad \tau \in (0,\tau_i) \tag{2.10}
$$

$$
\theta(x,0) = 0 \qquad x \in (0,1) \tag{2.11}
$$

The pressure and the speed, respectively, can be written in the form

$$
p(\tau) = \phi_u(\tau) - u^w(\tau) + \int\limits_0^1 \theta(\eta,\tau)\,d\eta \tag{2.12}
$$

$$
u^w(\tau) = k_u \int\limits_0^\tau p(\eta)v(\eta)\,d\eta \tag{2.13}
$$

$$
v(\tau) = v_0 + a\Big[S_F(\tau) - S_{fr}(\tau)\Big] \tag{2.14}
$$

where the pulling force impulse $S_F(\tau)$ and the friction force impulse $S_f(\tau)$ for the time t are defined by the formulae

$$
S_F(\tau) = f_0 \int\limits_0^\tau \phi_F(\xi)\,d\xi \qquad S_{fr}(\tau) = f\int\limits_0^\tau p(\xi)\,d\xi \tag{2.15}
$$

The representation (2.13) has a physical meaning for $v(\tau) > 0$ while inequality $v(\tau) < 0$ serves as the condition of the nonexistence of the frictional heat source.

Let us note that in the case when the speed of motion is constant the problem becomes a linear one and can be solved by using one of the known methods.

3. Solution to the problem

The solution of the boundary value problem (2.9) ÷ (2.13) can be obtained by the use of the Laplace integral transforms with respect to time τ, denoted by a bar over the pertinent function

$$\Big\{\overline{\theta}(x,s),\overline{v}(s),\overline{p}(s),\overline{u}^w(s),\overline{\phi}_u(s),\overline{\phi}_F(s),\overline{q}(s)\Big\} =$$
$$= \int\limits_0^\infty \Big\{\theta(x,\tau),v(\tau),p(\tau),u^w(\tau),\phi_u(\tau),\phi_F(\tau),q(\tau)\Big\}\exp(s\tau)\;d\tau$$

In the space of transforms the system of equations (2.9), (2.10) takes the following form

$$\frac{d^2}{dx^2}\overline{\theta}(x,s) = s\overline{\theta}(x,s) \qquad x \in (0,1) \tag{3.1}$$

$$\frac{d}{dx}\overline{\theta} = \begin{cases} \mathrm{Bi}\overline{\theta} & \text{for} \quad x=0 \\ \Omega[\widetilde{v}\overline{p}(s)+\overline{q}(s)] & \text{for} \quad x=1 \end{cases} \tag{3.2}$$

where

$$q(\tau) = [v(\tau)-\widetilde{v}]p(\tau) \tag{3.3}$$

Solution to Eq (3.1) is known to be

$$\overline{\theta}(x,s) = A(s)\sinh\sqrt{s}x + B(s)\cosh\sqrt{s}x$$

Upon determining parameters $A(s)$ and $B(s)$ we find from the boundary conditions (3.2)

$$\begin{aligned} &\overline{\theta}(x,s) = \overline{Q}(s)s\overline{g}(x,s) \qquad \overline{p}(s) = \overline{\phi}_u(s) + \overline{Q}(s)s^2\overline{G}(s) \\ &\overline{v}(s) = \frac{v_0}{s} + a\Big[\frac{1}{s}(f_0\overline{\phi}_F(s) - f\overline{\phi}_u(s)) - f\overline{Q}(s)s\overline{G}(s)\Big] \end{aligned} \tag{3.4}$$

where

$$\overline{u}^w(s) = \overline{Q}(s)s\overline{I}(s) \qquad \Delta_1(s) = \mathrm{Bi}\cosh\sqrt{s} + \sqrt{s}\sinh\sqrt{s}$$

$$\overline{g}(x,s) = \frac{\Delta_3(x,s)}{\Delta(s)} \qquad \Delta_2(s) = \frac{1}{s}\mathrm{Bi}(\cosh\sqrt{s} - 1) + \frac{1}{\sqrt{s}}\sinh\sqrt{s}$$

$$\overline{G}(s) = \frac{\Delta_4(s)}{s^2\Delta(s)} \qquad \Delta_3(x,s) = \frac{1}{\sqrt{s}}\mathrm{Bi}\sinh\sqrt{s}x + \cosh\sqrt{s}x$$

$$\overline{I}(s) = \frac{\xi\Delta_1(s)}{s\Delta(s)} \qquad \Delta_4(s) = s\Delta_2(s) - \xi\Delta_1(s)$$

and

$$\Delta(s) = s\Delta_1(s) - \widetilde{\Omega}\widetilde{v}\Delta_4(s) \qquad \overline{Q}(s) = \widetilde{\Omega}[\widetilde{v}\overline{\phi}_u(s) + \overline{q}(s)] \tag{3.5}$$

In order to find the inverse transforms in the space of originals we make use of the residue and convolution theorems Carslaw and Jaeger (1964). In the result we obtain the solution to the boundary value problem (2.9) ÷ (2.13) with unknown parameters under the integral sign

$$\begin{aligned}
\theta(x,\tau) &= \psi_\theta(x,\tau) + \widetilde{\Omega}q(\tau) * \frac{d}{d\tau}g(x,\tau)\\
p(\tau) &= \psi_p(\tau) + \widetilde{\Omega}q(\tau) * \frac{d^2}{d\tau^2}G(\tau)\\
v(\tau) &= \psi_v(\tau) - af\widetilde{\Omega}q(\tau) * \frac{d}{d\tau}G(\tau)\\
u^w(\tau) &= \psi_u(\tau) + \widetilde{\Omega}q(\tau) * \frac{d}{d\tau}I(\tau)
\end{aligned} \tag{3.6}$$

where

$$\psi_\theta(x,\tau) = \widetilde{\Omega}\widetilde{v}\phi_u(\tau) * \frac{d}{d\tau}g(x,\tau)$$

$$\psi_p(\tau) = \phi_u(\tau) + \widetilde{\Omega}\widetilde{v}\phi_u(\tau) * \frac{d^2}{d\tau^2}G(\tau)$$

$$\psi_v(\tau) = v_0 + a\Big\{[f_0\phi_F(\tau) - f\phi_u(\tau)] * H(\tau) - f\widetilde{\Omega}\widetilde{v}\phi_u(\tau) * \frac{d}{d\tau}G(\tau)\Big\}$$

$$\psi_u(\tau) = \widetilde{\Omega}\widetilde{v}\phi_u(\tau) * \frac{d}{d\tau}I(\tau)$$

$$g(x,\tau) = \sum_{m=1}^{\infty}\frac{\Delta_3(x,s_m)}{\Delta'(s_m)}\exp(s_m\tau)$$

$$
\begin{aligned}
G(\tau) &= -\frac{1}{\widetilde{\Omega}\widetilde{v}}\tau + \frac{1}{\xi\widetilde{\Omega}^2\widetilde{v}^2} + \sum_{m=1}^{\infty} \frac{\Delta_4(s_m)}{s_m^2\Delta'(s_m)} \exp(s_m\tau) \\
I(\tau) &= \frac{1}{\widetilde{\Omega}\widetilde{v}} + \sum_{m=1}^{\infty} \frac{\xi\Delta_1(s_m)}{s_m\Delta'(s_m)} \exp(s_m\tau) \qquad (3.7) \\
\Delta'(s_m) &= \frac{1}{2}\Big\{ s_m\Big[(\mathrm{Bi}+1)S_m + C_m\Big] - \widetilde{v}\widetilde{\Omega}\Big[C_m - S_m + \mathrm{Bi}(S_m + 2C_m^0)\Big] + \\
&\quad +2\Big[\Delta_1(s_m) - \widetilde{\Omega}\widetilde{v}\Delta_2(s_m)\Big] + \xi\widetilde{\Omega}\widetilde{v}\Big[(1+\mathrm{Bi})S_m + C_m\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1(s_m) &= \mathrm{Bi}C_m + s_m S_m & \Delta_2(s_m) &= S_m - \mathrm{Bi}C_m^0 \\
\Delta_3(x,s_m) &= \mathrm{Bi}S_m^x + C_m^x & \Delta_4(s_m) &= s_m\Delta_2(s_m) - \xi\Delta_1(s_m) \\
S_m &= \sinh\sqrt{s_m} & S_m^x &= \sinh(x\sqrt{s_m}) \\
C_m &= \cosh\sqrt{s_m} & C_m^0 &= (1 - C_m)s_m^{-1} \\
C_m^x &= \cosh(x\sqrt{s_m})
\end{aligned}
$$

Here $*$ denotes the convolution integral of two functions with respect to time

$$
\phi(\tau) * q(\tau) = \int_0^{\tau} \phi(\xi)q(\tau - \xi)\, d\xi
$$

$H(\tau) = 0$ for $\tau < 0$, $H(\tau) = 1$ for $\tau > 0$; $\widetilde{v}$ is a special parameter, Pyryev (1994), such that it enables us to improve the results of numerical analysis for greater times and the increase of the step with respect to time; s_m denote the roots of the characteristic equations $\Delta(s_m) = 0$, $m = 1, 2, \ldots$. Their behaviour depends on the parameters of the model and dimensionless velocity $\widetilde{v}$, and is analysed in Appendix A. It turns out that the properties of the roots s_1 and s_2 change when $\widetilde{v}$ is equal to $\widetilde{v}_0$, $\widetilde{v}_m$, $(m = 1, 2, 3, \ldots)$. Eqs $(3.6)_{2,3}$ constitute a system of nonlinear Volterra-Hammerstein integral equations of the second kind and convolution type with respect to $p(\tau)$ and $v(\tau)$ (cf Verlan' and Sizikov, 1986)

4. Characteristic properties of the solution

In the sequel we shall discuss the properties of thermo-elastic contact during accelerating and braking processes in the case of the constant value, in time, of the walls temperature and a constant force applied to the layer, i.e.

$$
\phi_u(\tau) = H(\tau) \qquad\qquad \phi_F(\tau) = H(\tau)
$$

Depending on the initial parameters we obtain different states of the system.

4.1. Motion of the layer with constant velocity

In the case when the velocity $v(\tau) = \widetilde{v}$ is constant, function $q(\tau)$ vanishes and the problem becomes a linear one and the solution is much simpler.

Parameter $\xi = KK_u(1-\nu)/[f\alpha k(1+\nu)]$ characterises the wear and thermal expansion.

In the case of wear absence $\xi = 0$ the contact pressure and temperature approach the steady-state regime when the speed is smaller than its critical value of $\widetilde{v}_0$

$$p_c = \frac{\widetilde{v}_0}{\widetilde{v}_0 - \widetilde{v}} \qquad\qquad \theta_c(x) = \frac{x\mathrm{Bi}+1}{\mathrm{Bi}} p_c \widetilde{\Omega}\widetilde{v} \tag{4.1}$$

Then the heat production and emission are in equilibrium. For $\widetilde{v}$ approaching its critical value $\widetilde{v}_0$ the time which is necessary to reach the steady-state regime increases.

For speeds greater than the critical value $\widetilde{v} > \widetilde{v}_0$ there will be an exponential increase of the temperature and contact pressure. The system has no time to cool down. The frictional thermoelastic instability takes place, i.e even for the smallest external disturbance of the system (in our case – compression of the layer) the exponential increase of the temperature and contact pressure occurs.

For $0 < \xi < 1$, i.e. in the case when the rate of the thermal expansion dominates over the rate of wear $\widetilde{v} \leq v_2$ the contact time $t = \infty$, and the contact characteristics tend, with time, to their steady-state values $p_c = 0$, $\theta_c(x) = 0$, $u_w = 1$. For $\widetilde{v}$ close to v_2 the time necessary to reach the steady state increases. In the case when $v_2 < \widetilde{v} < v_3$ the contact time is limited. The minimum contact time arises for speeds $\widetilde{v} \approx (v_2 + v_3)/2$, i.e. when $\mathrm{Im}s_1$ reaches its maximum value. If $\widetilde{v}$ tends to v_3 the maximum value of the contact charachteristics increases. In the case when $\widetilde{v}$ is greater than v_3 the frictional thermoelastic instability occurs, i.e the contact characteristics increase exponentially $\exp(s_1\tau)$.

In the case $\xi \geq 1$ i.e. when the wear rate is greater than the rate of thermal expansion and $\widetilde{v} \leq v_2$ the contact characteristics tend, with time, to the steady-state solution to the problem (steady-state contact pressure $p_c = 0$, steady-state temperature $\theta_c(x) = 0$, steady-state wear rate $u^w = 1$). In the case $\widetilde{v} \geq v_2$ the time of contact is limited but formally the steady-state solution still exists. Increase of the sliding speed results in the decrease of the contact time.

For $\xi \geq 1$ the contact pressure tends always monotonously to zero in contrast to the case $0 < \xi < 1$ when it has the maximum value.

Thus for $\mathrm{Bi} \in [0,\infty)$, $\xi \in [0,1)$, $\tilde{v} \in [v_3,\infty)$ the frictional thermoelastic instability occurs, i.e. contact characteristics behave exponentially as $\exp(s_1\tau)$. Such a kind of instability is observed in frictional thermoelastic contact under assumption of constant relative displacements of bodies Morov (1985), Pyryev (1994), Pyryev et al. (1995), Yevtushenko and Pyryev (1997).

4.2. Behaviour of the contact characteristics at the initial moment of time

From analytical properties of solution (3.4) we obtain the following asymptotics of the thermo-elastic contact characteristics for small values of time, $\tau \to 0$

$$
\begin{aligned}
&\theta(1,\tau) = v_0\tilde{\Omega}2\sqrt{\frac{\tau}{\pi}} + O(\sqrt{\tau^3}) \qquad\qquad p(\tau) = 1 + v_0\tilde{\Omega}(1-\xi)\tau + O(\tau^2)\\
&u^w(\tau) = v_0\tilde{\Omega}\xi\tau + O(\tau^2) \qquad\qquad (4.2)\\
&v(\tau) = v_0 + a(f_0 - f)\tau - \frac{1}{2}afv_0\tilde{\Omega}(1-\xi)\tau^2 + O(\tau^3)
\end{aligned}
$$

5. Numerical solution and conclusions

A numerical solution to the system of non-linear Eqs (3.6) has been obtained by using the method of finite differences of dimensionless time τ. The corresponding numerical scheme is given in Appendix B.

The numerical analysis of the solution to the problem was carried out for the case of steel layer ($\alpha = 14\cdot 10^{-6}\,^{\circ}\mathrm{C}^{-1}$, $K = 21\,\mathrm{W/(m^{\circ}C)}$, $k = 5.9\cdot 10^{-6}\,\mathrm{m^2/s}$, $\nu = 0.3$, $E = 190\cdot 10^9\,\mathrm{Pa}$) for $L = 3\cdot 10^{-2}\,\mathrm{m}$, $u_0 = 1\cdot 10^{-6}\,\mathrm{m}$. In Pyryev (1994) for the values $v_0 = 2.54$, $f = 0.01$, $f_0 = 0.12$, $\mathrm{Bi} = 28.6$, $a = 1.1\cdot 10^2$ and in conditions of absence of the wear it was shown that characteristics of the contact look like damping oscillations when they are entering its stationary regime (in particular the steady-state value of pressure $p_c = 11.7$, steady-state speed $v_c = 91.5$, steady-state temperature $\theta_c = 20.8$). This is shown by dashed curves in Fig.2a,b,c. In Fig.2 we have shown the behaviour of the contact pressure, sliding speed of the plate, contact temperature and the value of the wear during the acceleretion of the plate for the same values but with wear taken into account. The wear leads to the limitation of the interaction time $p(\tau_i) = 0$. The speed increases to the value $v(\tau_i) = v_0 + a[S_F(\tau_i) - S_{fr}(\tau_i)]$. In the case of very small values of

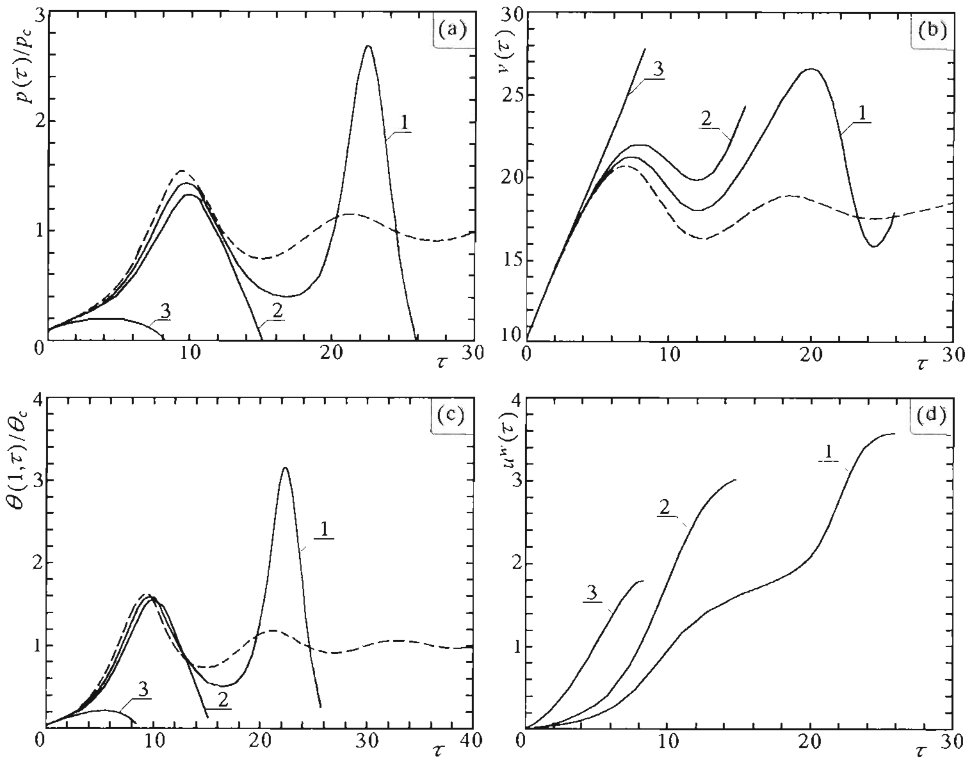

Fig. 2. Time variation of: (a) the relative contact pressure, (b) the relative velocity of the solid in motion, (c) the temperature in the contact plane, (d) the dimensionless wear; 1 – $\xi = 0.68 \cdot 10^{-2}$, 2 – $\xi = 0.14 \cdot 10^{-1}$, 3 – $\xi = 0.68 \cdot 10^{-1}$

the wear-resistance coefficient (curves 1) intensity of the wear, contact pressure, temperature and the relative speed have oscillatory character. With the growth of the wear-resistance coefficient such behaviour changes, and besides the time τ_i of interaction and the wear value $u^w(\tau_i)$ decrease.

The proposed model and the obtained solution let us study the behaviour of the contact characteristics in the process of braking under conditions of absence of the pulling force ($F = 0$). The numerical analysis was performed for the values $v_0 = 1\,\mathrm{m/s}$, $\mathrm{Bi} = 1.43$, $f = 0.1$ and for different values of coefficients a and ξ which characterise the mass of the plate and relation between the value of wear resistance and heat expansion, respectively. We have assumed the following values of the dimensionless parameters: $t_* = 153\,\mathrm{s}$, $v_* = 1.97 \cdot 10^{-4}\,\mathrm{m/s}$, $p_* = 8.53 \cdot 10^6\,\mathrm{Pa}$, $\theta_* = 1.28°\mathrm{C}$. In Fig.3 we have shown the time evolution of the relative motion speed, contact pressure, contact

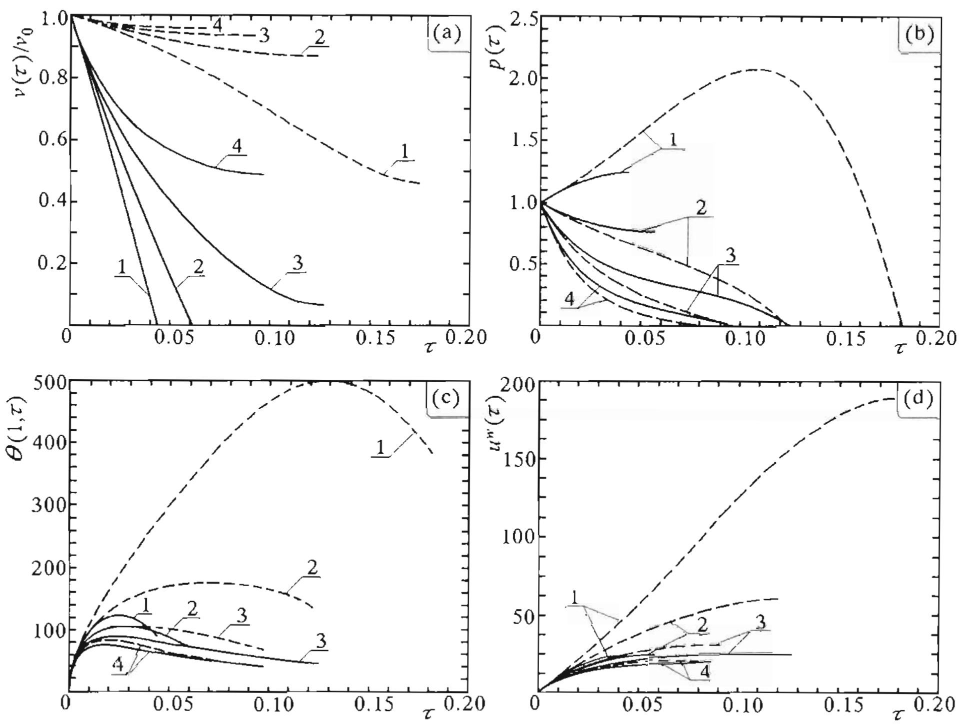

Fig. 3. The dimensionless velocity (a), the contact pressure (b), the contact temperature (c) and wear (d) during the process of braking; 1 – $\xi = 0.99$, 2 – $\xi = 1.01$, 3 – $\xi = 1.03$, 4 – $\xi = 1.05$, continues curves correspond to $\alpha = 10^6$, dashed ones to $\alpha = 10^5$

temperature and the wear rate. As a result of the braking process the moving solid can come to the stop $v(\tau_i) = 0$ (continuous curves 1,2). In this case an increase in parameter ξ leads to an increase in the braking time τ_i ($\tau_i = \tau_1 = 0.0045$ for $\xi = 0.99$, $\tau_i = \tau_2 = 0.0062$ for $\xi = 1.01$) and the wear value ($u^w(\tau_1) = 23.91$, $u^w(\tau_2) = 24.39$). Contact pressure decreases at the moment of stopping ($p(\tau_1) = 1.24$, $p(\tau_2) = 0.75$) when the contact temperature reaches its maximum value. An increase in the wear resistance coefficient or the mass of the moving solid makes the contact between two solids disappear $p(\tau_i) = 0$, plate doesn't stop and the speed reduces to the value $v(\tau_i) = v_0 - aS_f(\tau_i)$ (continuous curves 1,2 and dashed curves 1÷4 shown in Fig.3). In this case the increase in parameter ξ leads to the decrease in the stopping time, the value of wear, ($u^w(\tau_i) = 189.23$, 60.44, 30.82, 20.23, for $\xi = 0.99$, 1.01, 1.03, 1.05 and $a = 10^5$; and $u^w(\tau_i) = 24.78$, 19.35, for $\xi = 1.03$, 1.05 and $a = 10^6$), the maximum value of the contact temperature and value of the speed reduction $v_0 - v(\tau_i)$. A similar effect can be observed also when the value of parameter a decreases.

Appendix A

In this Appendix the behaviour of the characteristic equation $\Delta(s) = 0$ roots $s_m, m = 1, 2, 3, ...$ in the complex plane of the Laplace transform complex parameter s is considered.

The characteristic function $\Delta(s)$ is given by Eq (3.5). The numerical analysis of the characteristic equation has the following features: let us note that $\text{Im}s_m = 0$, $\text{Re}s_m < 0$ for $m = 3, 4, ...$ Roots s_1 and s_2 lie in the right or in the left half of the complex plane s depending on the entering parameters of the problem. For $\xi \geq 1$ they always lie in the left half of the complex plane, besides for $\widetilde{v} < v_2$ or $\widetilde{v} > v_3 > 0$ they are negative and for $v_2 < \widetilde{v} < v_3$ they are complex-conjugate. For $0 < \xi < 1$ the roots for $\widetilde{v} < v_2$ are negative and for $v_2 < \widetilde{v} < v_1$ they are complex-conjugate with a negative real part, for $v_1 < \widetilde{v} < v_3$ - they are complex-conjugate with a positive real part and for $v_3 < \widetilde{v}$ - then roots are positive.

In the case of wear absence ($\xi = 0$) the roots of the characteristic equation lie on the real axis ($\text{Im}s_m = 0, m = 1, 2, 3, ...$) besides for $\widetilde{v} < \widetilde{v}_0$ — $\text{Re}s_m < 0$, $m = 1, 2, 3, ...$ and for $\widetilde{v} > \widetilde{v}_0$ — $\text{Re}s_1 > 0$, $\text{Re}s_m < 0$, $m = 2, 3, ...$

Thus, the properties of the roots change when the dimensionless speed $\widetilde{v}$ is equal to v_m.

Let us write the approximate expressions for s_1 and s_2 by expanding the characteristic function $\Delta(s)$ into series in the neighbourhood of the origin.

For small values of s roots s_1 and s_2 can be written in explicite form

$$s_{1,2} = -\widetilde{\alpha} \pm \widetilde{\omega} \tag{A.1}$$

$$\widetilde{\alpha} = \frac{b}{2\widetilde{a}} \qquad \widetilde{\omega} = |\widetilde{\alpha}|\sqrt{1-h} \qquad h = \frac{4\widetilde{a}c}{b^2} \qquad \text{Bi}_4 = \frac{1+\text{Bi}/4}{6(1+\text{Bi}/2)}$$

$$\widetilde{a} = \frac{\widetilde{v}_1}{\text{Bi}_1} - \widetilde{v}\text{Bi}_4 \qquad b = \widetilde{v}_1 - \widetilde{v} \qquad c = \widetilde{\Omega}\widetilde{v}\widetilde{v}_1\xi$$

$$\widetilde{v}_1 = \frac{\widetilde{v}_0}{1-\xi} \qquad \widetilde{v}_0 = \frac{\text{Bi}_1}{\widetilde{\Omega}} \qquad \text{Bi}_1 = \frac{\text{Bi}}{1+\text{Bi}/2}$$

Eqs (3.7) let us to write the approximate expressions for v_m, $m = 1,2,3$, i.e. $v_m \approx \widetilde{v}_m$, $m = 1,2,3$, where

$$\widetilde{v}_m = \widetilde{v}_1 \frac{1 + 2\Omega_3(1 \pm \sqrt{1 - \mathrm{Bi}_2/\Omega_3})}{1 + 4\Omega_3 \mathrm{Bi}_1 \mathrm{Bi}_4} \qquad m = 3,2 \tag{A.2}$$

and

$$\Omega_3 = \frac{\xi}{1-\xi} \qquad \mathrm{Bi}_2 = \frac{1 + 5\mathrm{Bi}/6 + 5\mathrm{Bi}^2/24}{(1 + \mathrm{Bi}/2)^2}$$

The properties of the roots s_1 and s_2 permit us to predict the time behaviour of functions $g(x,\tau)$, $G(\tau)$, $I(\tau)$ which are included in the solution (3.6). The value of $\pi/\mathrm{Im}s_1$ corresponds to semiperiod of oscillations of functions $g(x,\tau)$, $G(\tau)$, $I(\tau)$ and the time of contact τ_i can be combined with it: if $\mathrm{Im}s_1$ increases then time of contact decreases. The value $\mathrm{Re}s_1 > 0$ describes the character of increasing of contact characteristics according to the exponential law $\exp(\mathrm{Re}s_1)$ and their extremal quantities.

Appendix B

For the numerical analysis of integral equations (3.6) the quadrature method with taking into account the functions behaviour is applied (cf Verlan' et al., 1986). To numerical solution of the non-linear integral equations (3.6) we apply the method proposed by Pyryev (1994) with the behaviour of functions taken into account

$$g(1,\tau) = 2\sqrt{\frac{\tau}{\pi}}\Big[1 + \frac{2}{3}(1-\xi)\widetilde{\Omega}\widetilde{v}\tau + O(\tau^2)\Big]$$
$$G(\tau) = \frac{1}{2}(1-\xi)\tau^2 + O(\tau^3) \qquad I(\tau) = \xi\tau + O(\tau^2) \qquad \tau \to 0 \tag{B.1}$$

Integrals on the right-hand sides of integral equations (3.6) have been calculated using the trapezium formula and in the case of Eq $(3.6)_1$ by the formula

$$\int\limits_0^{\tau_i} q(\xi)\frac{d}{d\tau_i}g(1,\tau_i-\xi)\,d\xi = (1-\delta_{i1})\Delta\tau\sum_{n=0}^{i-1}{}'' q(\tau_n)\frac{d}{d\tau_i}g(1,\tau_i-\tau_n) +$$
$$+\frac{2}{3}\sqrt{\frac{\Delta\tau}{\pi}}[2q(\tau_i) + q(\tau_{i-1})] \tag{B.2}$$

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{i-1} < \tau_i = \tau \qquad \Delta\tau = \tau_n - \tau_{n+1})$$

Finally we obtain the following formulae

$$
\begin{aligned}
v_i &= \psi_{vi} - af\widetilde{\Omega}\Delta\tau \sum_{m=0}^{i-1}{}' q_m \frac{d}{d\tau}G_{i-m} \\
p_i &= \frac{1}{1-\frac{1}{2}\widetilde{\Omega}(1-\xi)\Delta\tau[v_iH(v_i)-\widetilde{v}]}\Big(\psi_{pi} + \widetilde{\Omega}\Delta\tau\sum_{m=0}^{i-1}{}' q_m\frac{d^2}{d\tau^2}G_{i-m}\Big) \\
q_i &= p_i[v_iH(v_i)-\widetilde{v}] \qquad\qquad (B.3) \\
\theta_i &= \psi_{\theta i} + \widetilde{\Omega}\Big[\frac{2}{3}\sqrt{\frac{\Delta\tau}{\pi}}(2q_i+q_{i-1}) + (1-\delta_{i1})\Delta\tau\sum_{m=0}^{i-1}{}'' q_m\frac{d}{d\tau}g_{i-m}\Big] \\
u_i &= \psi_{ui} + \frac{1}{2}\Delta\tau\xi\widetilde{\Omega}q_i + \widetilde{\Omega}\Delta\tau\sum_{m=0}^{i-1}{}'' q_m\frac{d}{d\tau}I_{i-m} \qquad i=1,2,3,\dots
\end{aligned}
$$

where

$$
\begin{array}{llll}
v_i = v(i\Delta\tau) & p_i = p(i\Delta\tau) & \theta_i = \theta(1,i\Delta\tau) & u_i = u^w(i\Delta\tau) \\
\psi_{vi} = \psi_v(i\Delta\tau) & \psi_{pi} = \psi_p(i\Delta\tau) & \psi_{\theta i} = \psi_\theta(1,ih) & \psi_{ui} = \psi_u(i\Delta\tau)
\end{array}
$$

$$
\frac{d}{d\tau}g_i = \frac{d}{d\tau}g(1,i\Delta\tau) \qquad \frac{d}{d\tau}G_i = \frac{d}{d\tau}G(i\Delta\tau)
$$

$$
\frac{d^2}{d\tau^2}G_i = \frac{d^2}{d\tau^2}G(i\Delta\tau) \qquad \frac{d}{d\tau}I_i = \frac{d}{d\tau}I(i\Delta\tau)
$$

$\delta_{im} = 1$ when $i = m$, $\delta_i = 0$ when $i \neq m$.

Eq (B.3) exist when $\Delta\tau \neq 2/[(1-\xi)\widetilde{\Omega}(v_i-\widetilde{v})]$. The prime $(\cdot)'$ means that the first term in the sum Eqs (B.2), (B.3) is taken with the multiplier 1/2 while the double prime $(\cdot)''$ means that both the first and the last terms in the sum Eqs (B.2), (B.3) are taken with the factor 1/2.

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Model dynamicznego kontaktowego zagadnienia termosprężystości przy uwzględnieniu ciepła tarcia i zużycia materiału

Streszczenie

Sformułowaliśmy i zbadaliśmy pewien model kontaktu ciał sprężystych przy uwzględnieniu sił bezwładności w warunkach wydzielania ciepła tarcia i mechanicznego zużycia materiału. W rozpatrywanym zagadnieniu blok materiału i sprężysta warstwa znajdują się w ruchu względnym. Odstęp między poruszającymi się ciałami zmienia się w skutek rozszerzalności cieplnej warstwy. Zbadany został wpływ stałych materiałowych, prędkości względnej i parametrów modelu na charakterystyki kontaktu. Rozwiązanie układu równań różniczkowych cząstkowych, opisujących rozpatrywane zagadnienie, zostało zredukowane do układu nieliniowych równań całkowych Volterry-Hammersteina. Rozwiązanie liczbowe otrzymaliśmy z pomocą zaproponowanego algorytmu. Zostały one przedstawione w postaci wykresów.

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