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THE MINIMAL-FORM MODELLING OF MECHANISMS WITH FRICTION

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The paper deals with modelling and analysis of the effect of Coulomb-type friction on the dynamics of mechanisms (one-degree-of-freedom multibody systems), and presents a two-step process of converting the equations of motion for the friction-affected systems from a large set derived in terms of absolute variables to a minimal set in independent variables. The obtained governing equations consist of one dynamic equation, few kinematic equations, and algebraic relations for determination of the joint reactions as functions of the current state variables. The latter enable one to estimate the frictional effects that influence the system dynamics, and consequently to solve the governing equations (which formally are DAEs) as minimal-dimension ODEs. Two simple examples are provided to illustrate the frictional effects, conversion steps, and numerical simulation entanglements.

Key words: mechanisms dynamics, frictional effects, multibody systems

1. Introduction

Mechanisms are mechanical systems composed of possibly many rigid bodies interconnected by kinematic joints so that the resultant number of degrees of freedom of the system is relatively small, equal to the number of independent drivers. It is required that from mathematical modelling of such systems are the governing equations result (possibly in a minimal-form), applicable to both simulation of the mechanism motion under a given drive time-history and synthesis of the desired drive history for a specified (programmed) motion. The governing equations should also allow for the determination of joint reactions and, if needed, calculations of position, velocity and acceleration of any link (or point) during the simulated/programmed motion.

Numerous methods specialized in the dynamic analysis of mechanisms have been developed e.g. by Duffy (1980), Knapczyk and Lebiediew (1990), Morecki and Oderfeld (1987), Olędzki (1987), and Paul (1979). Most of them originate from d'Alembert's principle (equilibrium conditions of external and inertial forces and torques), and the mathematical language used is that of virtual work/displacement in the *physical* vector spaces relating particular links. The consequent analytical formulations as well as the graphical methods are then based on vector manipulations. Though this is a legitimate approach to the dynamic analysis of mechanisms, the powerful tool for investigation offered by computer methods (computer-aided analysis) has recently reoriented the way of formulating and solving the related problems resulting in what is recognized in the literature as the theory of *multibody systems*, see e.g. Nikravesh (1988), Roberson and Schwertassek (1988), and Schiehlen (1990). The present contribution attempts to apply the computer-oriented methods of multibody dynamics to the specific problems of mechanism analysis. Moreover, the Coulomb-type friction model including the stick-slip phenomenon in kinematic joints is considered. As this may cause problems in describing the mutual influence of the stick-slip state transitions in multiple frictional contacts for closed-loop systems with many degrees of freedom (Glocker and Pfeiffer, 1993), in the paper we focus the attention on only one-degree-of-freedom mechanisms. Also, for simplicity reasons, we confine ourselves to plane systems.

The proposed method of modelling friction-affected mechanism dynamics is divided into three stages. The starting point is the absolute variable formulation (Fig.1b) – the mechanism is first "exploded" to obtain a system of unbounded bodies, and then the constraints due to the kinematic joints are then reimposed. At this stage the frictional effects are also modelled. The formulation results in a large set of the differential-algebraic equations (DAEs) in terms of absolute coordinates and velocities of particular bodies and the *ideal* components of constraint reactions, often referred to as Lagrange's equations of the first kind. In the second step the system is converted into an open-loop (tree structure) system (Fig.1c) subject to the constraints of cut loops only. In result, using the joint coordinate method (Nikravesh, 1988, 1990; Roberson and Schwertassek, 1988; Wittenburg, 1977), a set of reduced-dimension governing DAEs is obtained. Finally, at the last stage, the remaining (closing) constraints are "eliminated" by using the coordinate partitioning (Wehage and Haug, 1982). The produced minimal-form governing equations are composed of one dynamic equation (often related to the driver motion), some kinematic equations, and algebraic relations for determination of the joint reactions as functions of the current state of motion. The equations are still DAEs

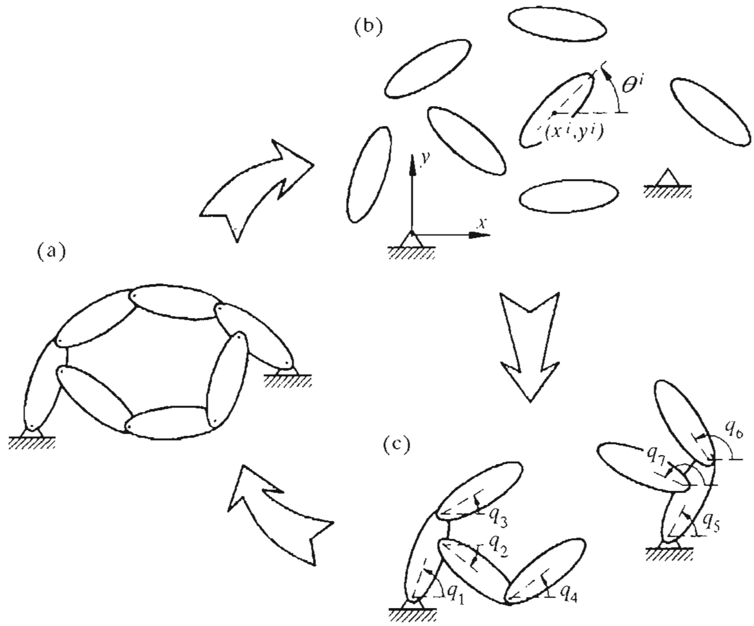

Fig. 1. Stages of modelling of friction-affected mechanisms

but the algorithm for solving them as ordinary differential equations (ODEs) is proposed. While the outlined approach to the minimal-form modelling of closed-loop multibody systems is rather a standard procedure for systems without friction (Nikravesh, 1990), its application to friction-affected systems has not been investigated thoroughly so far.

2. The absolute variable modelling of friction-affected mechanism dynamics

For a plane multibody system illustrated in Fig.1, the position of ith component body in an absolute (fixed) xyz coordinate system can be specified by $\boldsymbol{x}_i = [x, y, \theta]_i^\top$, $i = 1, ..., b$, where b is the number of bodies making up the system, and x_i, y_i and θ_i are the Cartesian translational coordinates of the origin (centre of mass) and the rotational coordinate of a body-fixed $(\xi\eta\zeta)_i$ coordinate system, respectively. Then, $\boldsymbol{x} = [(x, y, \theta)_1, ..., (x, y, \theta)_b]^\top = [x_1, ..., x_n]^\top$ is the global absolute coordinate vector, and $n = 3b$.

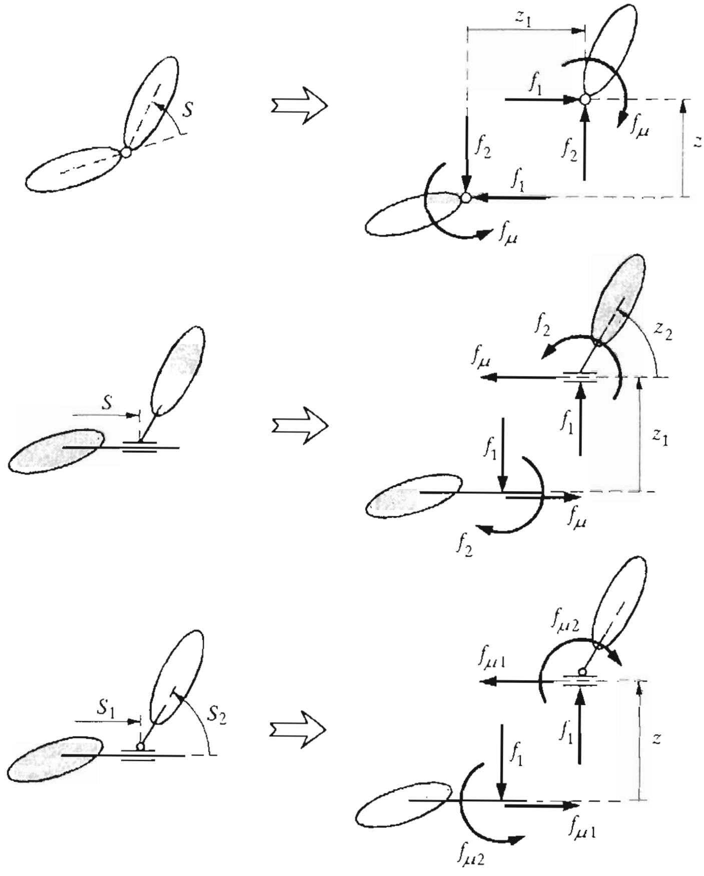

Fig. 2. Reactions in typical friction-affected plane joints

Assumed there are c constraint points due to the kinematic joints, let us introduce at each constraint point the local (translational/rotational) coordinates $\boldsymbol{z}_j$ and $\boldsymbol{s}_j$, $j = 1, ..., c$, that specify, respectively, the constrained and free directions of relative motion between the contacting bodies (the environment is regarded as *body* "0"). The total number of local coordinates at a constraint point is three. For typical joints of plane mechanisms, $\boldsymbol{z}_j$ and $\boldsymbol{s}_j$ are illustrated in Fig.2. The local coordinates can then be expressed in terms of the absolute coordinates $\boldsymbol{x}$, and for the jth constraint point the relations $\boldsymbol{z}_j = \boldsymbol{\Phi}_j(\boldsymbol{x})$ and $\boldsymbol{s}_j = \boldsymbol{\Psi}_j(\boldsymbol{x})$ will depend only on the coordinates of contacting bodies. For the whole system, the total vectors of the constrained and free coordinates are: $\boldsymbol{z} = [\boldsymbol{z}_1^\top, ..., \boldsymbol{z}_c^\top]^\top = [z_1, ..., z_m]^\top$ and $\boldsymbol{s} = [\boldsymbol{s}_1^\top, ..., \boldsymbol{s}_c^\top]^\top = [s_1, ..., s_l]^\top$, where

m is the number of constraints on the bodies $(m < n)$, and l is equal to the number $k = n - m$ of degrees of freedom for open-loop systems, while $l > k$ for closed-loop systems. Finally, we can write

$$
\begin{aligned}
z &= \boldsymbol{\Phi}(x) = \mathbf{0} \\
s &= \boldsymbol{\Psi}(x)
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
\boldsymbol{\Phi} &= [\boldsymbol{\Phi}_1^\top, ..., \boldsymbol{\Phi}_c^\top]^\top = [\phi_1, ..., \phi_m]^\top \\
\boldsymbol{\Psi} &= [\boldsymbol{\Psi}_1^\top, ..., \boldsymbol{\Psi}_c^\top]^\top = [\psi_1, ..., \psi_l]^\top
\end{aligned}
$$

Eqs (2.1) can be easily formulated analytically for all possible kinematic joints. The coordinates z express the physical directions jammed in particular joints, and the *ideal* reactions $\boldsymbol{f} = [f_1, ..., f_2]^\top$ of constraints $(2.1)_1$ are physical forces/moments pointed at z directions (see Fig.2). Introducing the velocity and acceleration forms of the constraint equation $(2.1)_1$,

$$
\begin{aligned}
\dot{\boldsymbol{\Phi}} &= \mathsf{C}\dot{x} = \mathbf{0} \\
\ddot{\boldsymbol{\Phi}} &= \mathsf{C}\ddot{x} + \dot{\mathsf{C}}\dot{x} = \mathbf{0}
\end{aligned}
\tag{2.2}
$$

which can be obtained analytically or using computer codes (Nikravesh, 1988; Roberson and Schwertassek, 1988; Schiehlen, 1990), the representation of the ideal constraint reactions in $\boldsymbol{x}$ directions is

$$
\boldsymbol{r}_0 = \mathsf{C}^\top \boldsymbol{f} \tag{2.3}
$$

where $\mathsf{C}(x) = \partial\boldsymbol{\Phi}/\partial\boldsymbol{x}$ is the $m \times n$ constraint Jacobian matrix. The n-vector $\boldsymbol{r}_0$ can be regarded as the representation of a generalized force of ideal reaction of the constraint $(2.1)_1$.

In the jth joint, the friction-induced forces/moments counteract the relative motion along $\boldsymbol{s}_j$ directions, and, in general, are non-linear functions of the relative motion velocity and ideal constraint reactions (Amstrong-Hélouvry et al., 1994; Dupont, 1993). For planar systems, the friction- induced moments and forces in rotational and translational joints, respectively, are usually modelled as follows (Dupont, 1993; Haug et al., 1986; Klepp, 1991; Wu et al., 1986)

$$
\begin{aligned}
f_{\mu r} &= -\mathrm{sgn}(\dot{s})\mu_r(\dot{s})d\sqrt{f_x^2 + f_y^2} \\
f_{\mu t} &= -\mathrm{sgn}(\dot{s})\mu_t(\dot{s})|f_n|
\end{aligned}
\tag{2.4}
$$

where

$\dot{s}$	–	relative rotational or translational velocity
μ_r, μ_t	–	respective coefficients of friction
d	–	bearing diameter
f_x, f_y	–	components of the resultant (ideal) reaction force in the bearing
f_n	–	normal force in the guidance.

For the joints illustrated in Fig.2 the respective friction moments/forces are

$$\begin{aligned} f_\mu &= -\operatorname{sgn}(\dot{s})\mu_r(\dot{s})d\sqrt{f_1^2 + f_2^2} \\ f_\mu &= -\operatorname{sgn}(\dot{s})\mu_t(\dot{s})|f_1| \\ f_{\mu 1} &= -\operatorname{sgn}(\dot{s}_1)\mu_t(\dot{s}_1)|f| \\ f_{\mu 2} &= -\operatorname{sgn}(\dot{s}_2)\mu_r(\dot{s}_2)d\sqrt{1 + \mu_t^2(\dot{s}_1)}|f| \end{aligned} \tag{2.5}$$

The friction forces/moments form an l-vector $\boldsymbol{f} = [f_{\mu 1}, ..., f_{\mu l}]^\top$, whose components correspond to the local coordinates $\boldsymbol{s}$ of relative motions (see Fig.2). Introducing the differentiated form of Eq $(2.1)_2$

$$\dot{\boldsymbol{s}} = \mathsf{C}_\mu \dot{\boldsymbol{x}} \tag{2.6}$$

which can be obtained analytically or using computer codes, the representation of frictional effects in the directions of absolute coordinates $\boldsymbol{x}$ is

$$\boldsymbol{r}_\mu = \mathsf{C}_\mu^\top \boldsymbol{f}_\mu \tag{2.7}$$

where $\mathsf{C}_\mu(\boldsymbol{x}) = \partial\boldsymbol{\Psi}/\partial\boldsymbol{x}$ is the $l \times n$ Jacobian matrix. Finally, the total generalized force of friction-affected constraints, represented in the directions of $\boldsymbol{x}$, can be written as

$$\boldsymbol{r}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{f}) = \boldsymbol{r}_0(\boldsymbol{x}, \boldsymbol{f}) + \boldsymbol{r}_\mu(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{f}) \equiv \mathsf{C}^\top(\boldsymbol{x})\boldsymbol{f} + \mathsf{C}_\mu^\top(\boldsymbol{x})\boldsymbol{f}_\mu(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{f}) \tag{2.8}$$

where $\boldsymbol{r}_0$ and $\boldsymbol{r}_\mu$ are, respectively, the ideal and non-ideal components of a generalized force of the reaction of friction-affected constraints. If some directions defined by $\boldsymbol{s}$ are friction-free, the corresponding entries of $\boldsymbol{f}_\mu$ equal to zero. More detailed discussion of geometrical aspects of frictional effects on multibody dynamics is provided in the next section.

The governing equations of motion of friction-affected multibody systems can now be written in terms of absolute variables in the following matrix form

$$\begin{aligned} \mathsf{M}\ddot{\boldsymbol{x}} &= \boldsymbol{h}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{\tau}) + \mathsf{C}^\top(\boldsymbol{x})\boldsymbol{f} + \mathsf{C}_\mu^\top(\boldsymbol{x})\boldsymbol{f}_\mu(\dot{\boldsymbol{x}}, \boldsymbol{f}) \\ \mathsf{C}(\boldsymbol{x})\ddot{\boldsymbol{x}} &= \boldsymbol{\xi}(\boldsymbol{x}, \dot{\boldsymbol{x}}) \end{aligned} \tag{2.9}$$

where $\mathbf{M} = \mathrm{diag}(\mathbf{M}_1, ..., \mathbf{M}_b) = \mathrm{const}$ is the $n \times n$ inertia matrix, $\mathbf{M}_i = \mathrm{diag}(m_i, m_i, J_i)$, m_i and J_i are the mass and inertial moments of the ith body, $\boldsymbol{h} = [\boldsymbol{h}_1^\top, ..., \boldsymbol{h}_b^\top]^\top$, and $\boldsymbol{h}_i = [F_{ix}, F_{iy}, M_i]^\top$, are the x and y components respectively of the external force and external force moment about the mass centre, and $\boldsymbol{\tau} = [\tau_1, ..., \tau_k]^\top$, $k = n - m$, are the driving moments/forces applied to the driving links. Eq $(2.9)_2$ represents the acceleration form $(2.2)_2$ of the constraint equation $(2.1)_1$, i.e. $\boldsymbol{\xi} = -\dot{\mathbf{C}}\dot{\boldsymbol{x}}$. The above governing equations form $2n + m$ DAEs in $2n$ differential variables $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ and m algebraic variables $\boldsymbol{f}$, and are commonly referred to as Lagrange's equations of the first kind.

The index of DAEs (2.9) is one. and many researches have preferred to solve them straighforward in the form they are, using the prediction-correction method or other specialized DAE solvers, see e.g. Blajer and Markiewicz (1995), Haug el al. (1986), Schiehlen (1990), and Wu et al. (1986). Though this is the simpliest and most direct approach to handling constrained motion, the algorithms that follow are often recognized as computationally inefficient (mainly due to large dimension of problems solved) and inaccurate (constraint violation). Moreover, according to (2.4) and (2.5), one deals with discontinuities in the right-hand sides of Eq $(2.9)_2$ when the components $\dot{\boldsymbol{s}}$ approach zero or change their signs. In particular, $\dot{\boldsymbol{s}}$ may also maintain zero for some periods, which is known as the stiction phenomenon. While these problems can usually be conquered for open-loop systems (Blajer and Markiewicz, 1995; Glocker and Pfeiffer, 1993; Haug et al., 1986; Wu et al., 1986), the explicit solution for friction-affected many-degree-of-freedom closed-loop systems, due to the stick-slip phenomena, is still upon consideration (Glocker and Pfeiffer, 1993; Klepp, 1991). These problems will be discussed in detail in Sections 4 and 5.

As the direct solution of DAEs (2.9) may cause certain difficulties, the other approach is to derive the equations of motion in terms of a minimum number of independent variables. Numerical integration of such equations is by far more efficient and accurate then that performed over equations expressed in terms of absolute variables. For the systems with *ideal* constraints (without friction), the produced governing equations are ODEs, and the number of dynamic equations is reduced to the number of degrees of freedom of the system. A variety of techniques of this type applicable to multibody systems of any structure have been demonstrated so far by e.g. Blajer (1994), Blajer et al. (1994), Nikravesh (1988,1990), Roberson and Schwertassek (1988), Schiehlen (1990), and Wehage and Haug (1982). However, the adaptation of these methods for friction-affected mechanisms is not quite evident (Amstrong-Hélouvry et al., 1994; Blajer and Markiewicz, 1995; Dupont, 1993; Glocker and Pfeiffer, 1993;

Klepp, 1991). The present paper is another contribution in this field. Prior to presentation of the proposed method, however, let us comment shortly on the nature of frictional effects on multibody dynamics, which is of crucial importance for further analysis.

3. The effect of friction on multibody dynamics

One of the main difficulties appearing in the minimal-form modelling of friction-affected systems emerges from the fact that the ideal ($\vec{r}_0$) and friction-induced ($\vec{r}_\mu$) components of the generalized constraint reaction ($\vec{r} = \vec{r}_0 + \vec{r}_\mu$) are not, in general, orthogonal to each other in the system configuration space. though the physical reaction forces/moments and the friction-induced forces/moments at individual joints are always orthogonal to each other in the local physical spaces. The non-orthogonality feature of $\vec{r}_0$ and $\vec{r}_\mu$ is illustrated in Fig.3; see also Blajer (1994), Blajer and Markiewicz (1995).

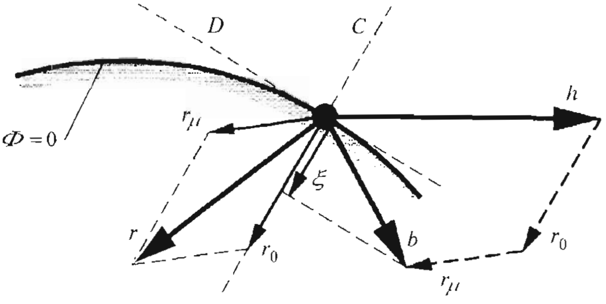

Fig. 3. Geometrical interpretation of the effect of friction on a multibody system

The system described in Section 2 can be regarded as a generalized particle in the n-space of configuration of unconstrained bodies, constrained to move on the manifold $\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{0}$. The dynamic equation $(2.9)_1$ can then be interpreted as $\vec{b} = \vec{h} + \vec{r}$, where $\vec{b}$ is the effective dynamic force vector represented in the directions of $\boldsymbol{x}$ by $\mathbf{M}\ddot{\boldsymbol{x}}$. An important observation is that the friction-induced (non-ideal) component $\vec{r}_\mu$ of the generalized reaction force $\vec{r} = \vec{r}_0 + \vec{r}_\mu$ of friction-affected constraints may be not tangent to the constraint manifold (see Fig.3), as it usually happens in simple dynamics problems (to which the present interpretation appeals). The generalized friction force $\vec{r}_\mu$ affects then the virtual motion along the manifold $\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{0}$ and simultaneously contributes to the balance of external and inertial forces in the orthogonal direction. The latter means that $\vec{r}_\mu$ ($\vec{r}_\mu = \mathbf{C}_\mu^\top \boldsymbol{f}_\mu$) influences $\vec{r}_0$ ($\vec{r}_0 = \mathbf{C}^\top \boldsymbol{f}$), and since $\boldsymbol{f}_\mu$ depends on $\boldsymbol{f}$, there is a mutual interdependence between the ideal and non-ideal constraint reaction components. In the language of mathematics, the nonorthogonality condition of $\vec{r}_\mu$ and $\vec{r}_0$ components of $\vec{r}$ reads $\vec{r}_0 \cdot \vec{r}_\mu \neq 0$, or more specifically (Blajer, 1994)

$$\boldsymbol{r}_0^\top \mathbf{M}^{-1} \boldsymbol{r}_\mu \neq 0 \qquad \Leftrightarrow \qquad \mathbf{C}\mathbf{M}^{-1}\mathbf{C}_\mu^\top \neq 0 \tag{3.1}$$

At a given position $\boldsymbol{x}$ on manifold $\boldsymbol{\Phi}(\boldsymbol{x}) = \mathbf{0}$. the m-dimensional orthogonal space $\mathcal{C}$ is defined by the constraint gradients represented as rows in the $m \times n$ constraint matrix $\mathbf{C}(\boldsymbol{x})$. The k-dimensional *tangent* space $\mathcal{D}$ can then be defined by an $n \times k$ maximal-rank matrix $\mathbf{D}(\boldsymbol{x})$ so that

$$\mathbf{D}^\top \mathbf{C}^\top = \mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{C}\mathbf{D} = \mathbf{0} \tag{3.2}$$

i.e. $\mathbf{D}$ is an orthogonal complement matrix to $\mathbf{C}$. Then, by projecting the dynamic equation $(2.9)_1$ into $\mathcal{D}$ and $\mathcal{C}$ (premultiplying it by $\left[\mathbf{D}, \mathbf{C}^\top \mathbf{M}^{-1}\right]^\top$) and using Eq $(2.9)_2$ we obtain (Blajer, 1994)

$$\begin{aligned} \mathbf{D}^\top \mathbf{M}\ddot{\boldsymbol{x}} &= \mathbf{D}^\top \boldsymbol{h} + \mathbf{D}^\top \mathbf{C}_\mu^\top \boldsymbol{f}_\mu \\ \boldsymbol{\xi} &= \mathbf{C}\mathbf{M}^{-1}\boldsymbol{h} + \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^\top \boldsymbol{f} + \mathbf{C}\mathbf{M}^{-1}\mathbf{C}_\mu^\top \boldsymbol{f}_\mu \end{aligned} \tag{3.3}$$

which represents $\vec{b}_0 = \vec{h}_D + \vec{r}_{\mu D}$ and $\vec{\xi} = \vec{h}_C + \vec{r}_0 + \vec{r}_{\mu C}$, i.e. the projection of $\vec{b} = \vec{h} + \vec{r}_0 + \vec{r}_\mu$ into $\mathcal{D}$ and $\mathcal{C}$, respectively (see Fig.3), and $\boldsymbol{\xi}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = -\dot{\mathbf{M}}\dot{\boldsymbol{x}}$ ($\vec{\xi} = \vec{b}_C$) is the constraint enforcement due to moving on the "curved" manifold $\boldsymbol{\Phi}(\boldsymbol{x}) = \mathbf{0}$. The friction forces/moments $\boldsymbol{f}_\mu$ (dependent on $\boldsymbol{f}$ and the state of motion) affect both the k dynamic equations $(3.3)_1$ and m algebraic equations $(3.3)_2$. As will be seen later on, the described non-linear coupling cause essential inconveniences in the modelling and analysis of the friction-affected systems. Note, that for a friction-free system ($\boldsymbol{f}_\mu = \mathbf{0}$), Eqs (3.3) uncouple and simplify to $\mathbf{D}^\top \mathbf{M}\ddot{\boldsymbol{x}} = \mathbf{D}^\top \boldsymbol{h}$ and $\boldsymbol{f} = (\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}\mathbf{M}^{-1}\boldsymbol{h} - \boldsymbol{\xi})$.

The other inconvenience of handling with systems with dry friction is possibility of the occurance of stiction. The phenomenon occurs when the relative velocity (sliding) at a particular joint goes to zero. Then, during stiction, the respective local coordinate of relative motion (included in $\boldsymbol{s}$) is locked up and should be regarded as the coordinate that temporarily specifies the constrained direction (should be moved to $\boldsymbol{z}$). Consequently, the corresponding stiction force/moment should be treated as a standard ideal constraint reaction, and therefore included into $\boldsymbol{f}$. The lock-up will continue as long as the reaction of additional stiction-induced constraint is less than a certain maximal magnitude, equal to a static coefficient of friction times the normal force. If the reaction approaches the static friction force magnitude, the stiction-induced constraint is deleted and sliding begins. For more details the reader is referred to e.g. Glocker and Pfeiffer (1993), Haug et al. (1986), Blajer and Markiewicz (1995), and Wu et al. (1986).

The described *stick-slip* problem (addition-deletion of the stiction-induced constraints) causes certain numerical problems in practice. During simulation, the relative velocities of sliding at joints must be constantly monitored in order to detect the instants the stiction begins. Then, for the periods of stiction, the system dynamic model described in Eqs (2.9) shold be modified adequately due to the change in s and z, and again the reaction of the stiction-induced constraint must be monitored in order to detect the instants of slip. Moreover, using the classical Coulomb friction model (Fig.4b), the problem of stiction/friction force discontinuity occurs, which causes impacts and must be very carefully handled during numerical simulation.

While the described difficulties in modelling of sticking-sliding state transitions and adequate changes in the structure of governing equations (2.9) (Blajer and Markiewicz, 1995) can be overcome for the open-loop systems, the situation becomes much more entangled for many-degree-systems with closed loops. Stiction occurring in more than one contact may lead to "over-constrained" systems, and the possible mutual influence of the state transitions in multiple frictional contacts may cause problems in detecting the sticking-sliding tendencies at particular joints, and uncertainties in the way of motion execution may appear (Glocker and Pfeiffer, 1993). Therefore, in this paper we limit ourselves to one-degree-of-freedom mechanism (closed-loop systems). The developed algorithm to handle the stick-slip phenomena is described in Section 5.

To illustrate the sticking-sliding discontinuity problems let us consider the following simple example, reported firstly by Dupont (1993). The two mass system shown in Fig.4a, moves in a horizontal plane (gravity is not included), and is enforced by F_1 and F_2 applied to the masses m_1 and m_2, respectively. Assumed that friction affects only the x-motion of mass m_1 within the slot, the motion in y-direction is governed by $(m_1 + m_2)\ddot{y} = F_2$, and $F_n = m_1|F_2|/(m_1 + m_2)$ is the value of normal force between the masses due to the inertial effects. If the mass m_1 slides with respect to the mass m_2, the friction force acting on the mass m_1 is then $F_\mu = -\text{sgn}(\dot{x})\mu_k F_n$, where μ_k is the coefficient of kinematic friction between the mass m_1 and the slot, and

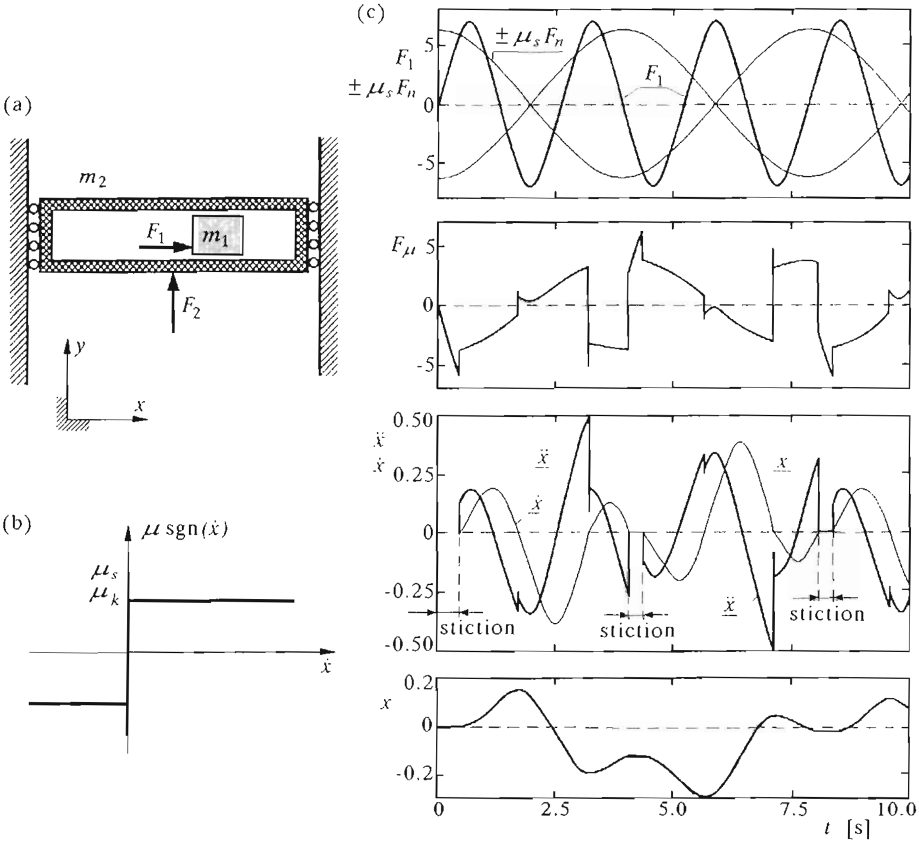

Fig. 4. Simple illustration of stick-slip phenomena

the motion of mass m_1 in x-direction is governed by

$$m_1\ddot{x} = F_1 - \text{sgn}(\dot{x})\mu_k \frac{m_1}{m_1+m_2}|F_2| \tag{3.4}$$

If the stiction occurs, the x-motion of mass m_1 is jammed, and (3.4) changes to

$$0 = F_1 - F_\mu \qquad \text{and} \qquad |F_\mu| \le \mu_s \frac{m_1}{m_1+m_2}|F_2| \tag{3.5}$$

where μ_s is the coefficient of static friction. Applying $F_1(t)$ and $F_2(t)$ appropriately, we can change from sliding to stiction (and vice versa) of mass m_1.

For the friction model as shown in Fig.4b, and the data: $m_1 = 10$, $m_2 = 30$, $F_1 = 7\sin(2.4t)$, $F_2 = 50\cos(0.8t)$, $\mu_s = 0.5$, $\mu_k = 0.3$, the simulation results are shown in Fig.4c. The discontinuities in acceleration and friction force occur each time the velocity $\dot{x}$ changes its direction. Two scenarios of the motion following these instants proceed. If at the instant $\dot{x} = 0$ the actual magnitude of F_1 surpasses the actual value of the static friction force $\mu_s F_n$, the stiction tendency is overcome and the motion continues with an abrupt reduced acceleration (due to the change in the direction of friction force). This scenario occurs at $t = 1.7$ s, 3.7 s, 5.7 s, 7.1 s, and 9.5 s. However, if at the instant $\dot{x} = 0$ the magnitude of F_1 is smaller than $\mu_s F_n$, the stiction cannot be passed and the mass m_1 is jammed in the slot. The stiction maintains as long as the magnitude of F_1 approaches the value $\mu_s F_n$ (for both the values are time-varying). This occurs for $t = 0.0 \div 0.5$ s, $4.1 \div 4.4$ s, and $8.0 \div 8.3$ s. During these periods the x-force between the masses is the stiction force, equal to F_1, and at the end of the stiction intervals the force magnitude jumps from the bigger value $\mu_s F_n$ to the smaller value $\mu_k F_n$.

4. The first-step reduction of equations of motion

As outlined in Introduction, the conversion of the motion equations (2.9) to a minimal set can be completed in two steps. In the first step, the system of "free" bodies is transformed to an open-loop (tree structure) system. To this end, the m constraints $(2.1)_1$ imposed on the system are divided into m_1 constraints due to the kinematic joints in the open-loop system and the remaining m_2 constraints (closing conditions), $m_1 + m_2 = m$. Accordingly, $(2.1)_1$ and (2.2) are

$$
\begin{matrix}
\boldsymbol{\Phi}_1(x) = 0 & & \mathsf{C}_1(x)\dot{x} = 0 & & \mathsf{C}_1(x)\ddot{x} = \boldsymbol{\xi}_1(x,\dot{\boldsymbol{x}}) \\
& \Rightarrow & & \Rightarrow & \\
\boldsymbol{\Phi}_2(x) = 0 & & \mathsf{C}_2(x)\dot{x} = 0 & & \mathsf{C}_2(x)\ddot{x} = \boldsymbol{\xi}_2(x,\dot{\boldsymbol{x}})
\end{matrix}
\qquad (4.1)
$$

where

$$
\begin{aligned}
\boldsymbol{\Phi} &= [\boldsymbol{\Phi}_1^\top, \boldsymbol{\Phi}_2^\top]^\top \\
\mathsf{C} &= [\mathsf{C}_1^\top, \mathsf{C}_2^\top]^\top \\
\boldsymbol{\xi} &= [\boldsymbol{\xi}_1^\top, \boldsymbol{\xi}_2^\top]^\top = [(-\dot{\mathsf{C}}_1\dot{\boldsymbol{x}})^\top, (-\dot{\mathsf{C}}_2\dot{\boldsymbol{x}})^\top]^\top
\end{aligned}
$$

The constraint partitioning is usually not unique, i.e. variant open-loop systems can be formed after cutting the closed loops in different places.

In order to formulate the equations of motion for a chosen open-loop system, the standard joint coordinate method (Nikravesh, 1988; Schiehlen, 1990) can be used. For such systems, $k_1 = n - m_1$ independent (joint) coordinates

$q = [q_1, ..., q_{k_1}]^\top$ can be defined (Fig.1c), and the explicit constraint equations $(4.1)_1$ can be replaced by their implicit forms

$$x = g(q) \quad \Rightarrow \quad \dot{x} = \mathsf{D}_1\dot{q} \quad \Rightarrow \quad \ddot{x} = \mathsf{D}_1\ddot{q} + \dot{\mathsf{D}}_1\dot{q} \tag{4.2}$$

where the $n \times k_1$ matrix $\mathsf{D}_1(x) = \partial g/\partial x$ is an orthogonal complement matrix to the $m_1 \times n$ constraint matrix C_1, i.e. $\mathsf{D}_1^\top \mathsf{C}_1^\top = \mathbf{0} \Leftrightarrow \mathsf{C}_1\mathsf{D}_1 = \mathbf{0}$. After substitution of Eq (4.2), the constraint equations $(4.1)_1$ are satisfied identically. For all typical joints, the relationships (4.2) can be easily obtained either analytically or by means of computerized symbolic manipulations.

Using the above definitions, the first-step reduction (with respect to constraints "1") of the dynamic equations $(2.9)_1$ yields to (Blajer, 1994; Nikravesh, 1990; Schiehlen, 1990)

$$\mathsf{D}_1^\top \mathsf{M}\mathsf{D}_1\ddot{q} = \mathsf{D}_1^\top(h - \mathsf{M}\dot{\mathsf{D}}_1\dot{q} + \mathsf{C}_2^\top f_2 + \mathsf{C}_\mu^\top f_\mu) \tag{4.3}$$

or in a symbolic form

$$\mathsf{M}'(q)\ddot{q} = h'(q, \dot{q}, \tau) + \mathsf{C}_2'^\top(q)f_2 + \mathsf{C}_\mu'^\top(q)f_\mu(q, \dot{q}, f) \tag{4.4}$$

where $\mathsf{M}' = \mathsf{D}_1^\top \mathsf{M}\mathsf{D}_1$ is the $k_1 \times k_1$ inertia matrix, $h' = \mathsf{D}_1^\top(h - \mathsf{M}\dot{\mathsf{D}}_1\dot{q})$ is of the dimension $k_1 \times 1$, the $m_2 \times k_1$ matrix C_2' of closing constraints can be obtained either as $\mathsf{C}_2' = \mathsf{C}_2[g(q)]\mathsf{D}_1$ or as $\mathsf{C}_2' = \partial\Phi_2'/\partial q$ for $\Phi_2'(q) = \Phi_2[g(q)]$, and, seemingly, $\mathsf{C}_\mu' = \mathsf{C}_\mu[g(q)]\mathsf{D}_1$ or $\mathsf{C}_\mu' = \partial\Psi_2'/\partial q$ for $\Psi_2'(q) = \Psi_2[g(q)]$. Eq (2.6) for determination of the relative motion velocities at particular joints can now also be expressed in q and $\dot{q}$

$$\dot{s} = \mathsf{C}_\mu[g(q)]\mathsf{D}_1(q)\dot{q} = \mathsf{C}_\mu'(q)\dot{q} \tag{4.5}$$

5. The final reduction and the minimal-form equations of motion

The open-loop system described in Section 4 is subject to m_2 constraints $\Phi_2'(q) = \mathbf{0}$ due to the closing conditions (closed loops). Thus, only $k = k_1 - m_2 = n - m$ from k_1 coordinates q are independent (for one-degree-of-freedom mechanisms we have $k = 1$). As it is usually difficult to introduce explicit relations between the dependent and independent coordinates, the variable partitioning is performed at the velocity level. The approach is usually referred to as the coordinate partitioning method, firstly introduced by Wehage and Haug (1982). Applied to the case in point, $\dot{q}$ are partitioned into k independent velocities u and m_2 dependent velocities v, denoted

$\dot{q} = [u^\top, v^\top]^\top$, and the partition should ensure that the first-order and second-order kinematic equations, respectively, of closing constraints are solvable with respect to v and $\dot{v}$. Namely, the partitioned forms of the higher-order constraint equations are

$$\begin{aligned} \dot{\Phi}_2' &\equiv \mathsf{C}_2'\dot{q} = \mathsf{U}u + \mathsf{V}v = 0 \\ \ddot{\Phi}_2' &\equiv \mathsf{C}_2'\ddot{q} - xi_2' = \mathsf{U}\dot{u} + \mathsf{V}\dot{v} = 0 \end{aligned} \tag{5.1}$$

where $\mathsf{C}_2' = [\mathsf{U}, \mathsf{V}]$. Solving Eqs (5.1) for v and $\dot{v}$, respectively, we have

$$\begin{aligned} \dot{q} &= \begin{bmatrix} \mathsf{I} \\ -\mathsf{V}^{-1}\mathsf{U} \end{bmatrix} u \equiv \mathsf{D}_2(q)u \\ \ddot{q} &= \begin{bmatrix} \mathsf{I} \\ -\mathsf{V}^{-1}\mathsf{U} \end{bmatrix} \dot{u} + \begin{bmatrix} \mathbf{0} \\ -\mathsf{V}^{-1}\xi_2' \end{bmatrix} \equiv \mathsf{D}_2(q)\dot{u} + \nu_2(q,u) \end{aligned} \tag{5.2}$$

where I is the $k \times k$ identity matrix. The dimension of D_2 is $k \times k$, and it is easy to show that $\mathsf{D}_2^\top \mathsf{C}_2'^\top = \mathbf{0} \Leftrightarrow \mathsf{C}_2'\mathsf{D}_2 = \mathbf{0}$. At a given position q, there is at least one such partition of $\dot{q}$ that $\det \mathsf{V} \neq 0$. In the case of mechanisms, it is usually reasonable to choose as independent coordinates those elements of q which relate the driving links. In a general case, and especially for many-degree-of-freedom systems, the projective criterion developed by Blajer et al. (1994) for the "best" choice of u may be useful.

Premultiplying Eq (4.3) by $\mathsf{D}_2^\top$ and substituting into Eqs (5.2), the dynamic equations can now be reduced to the number of k (one equation for one-degree-of-freedom mechanism), i.e.

$$\mathsf{D}_2^\top \mathsf{M}'\mathsf{D}_2\dot{u} = \mathsf{D}_2^\top(h' - \mathsf{M}'\nu_2) + \mathsf{D}_2^\top \mathsf{C}_\mu'^\top f_\mu \tag{5.3}$$

As $\operatorname{rank}(\mathsf{D}_2^\top \mathsf{M}'\mathsf{D}_2) = k = \max$, the above minimal-dimension equations of motion are solvable for the derivatives of k independent velocities u. However, as the equations are dependent on all joint coordinates q and joint reactions f (due to the frictional effects), the dynamic equations (5.3) should be completed by the kinematic equation $\dot{q} = \mathsf{D}_2 u$ defined by Eqs (5.2), and the algebraic equation $(3.3)_2$ for determination of f in terms of the current state values q and u. For the purpose of this formulation, Eq $(3.3)_2$ is manipulated to $d = \mathsf{CM}^{-1}h + \mathsf{CM}^{-1}\mathsf{C}^\top f + \mathsf{CM}^{-1}\mathsf{C}_\mu^\top f_\mu - \xi = 0$. The final minimal-form governing equations for friction-affected closed-loop multibody system can be obtained in the following symbolic form

$$\begin{aligned} \dot{q} &= \mathsf{D}_2(q)u \\ \mathsf{M}''(q)\dot{u} &= h''(q,u,\tau) + h''_\mu(q,u,f) \\ 0 &= d(q,u,\tau,f) \end{aligned} \tag{5.4}$$

where $\mathsf{M}'' = \mathsf{D}_2^\top \mathsf{M}' \mathsf{D}_2$ is the $k \times k$ inertia matrix, $h'' = \mathsf{D}_2^\top(h' - \mathsf{M}'\nu_2)$ is the k-vector ($k \times 1$ matrix) of external, centrifugal, Coriolis and or gyroscopic forces, respectively, projected into the directions of u, and $h''_\mu = \mathsf{D}_2^\top \mathsf{C}'_\mu{}^\top f_\mu$ expresses a similar projection of frictional forces. The above equations form $k_1 + k + m$ DAEs in $k_1 + k$ differential variables q and u, and m algebraic variables f. For one-degree-of-freedom mechanisms ($k = 1$), M'', h'' and h''_μ are coefficients and denote the reduced mass of the system and the reduced forces as mentioned, all with respect to the chosen independent velocity u. Eq (4.5) for determination of relative motion veloci- ties at particular joints can finally be transformed to

$$\dot{s} = \mathsf{C}'_\mu(q)\mathsf{D}_2(q)u = \mathsf{C}''_\mu(q)u \tag{5.5}$$

The advantage of using DAEs (5.4) is their small dimension, $k_1 + k + m = 2k + m_2 + m$ when compared to $2n + m$ of DAEs (2.9). For most mechanisms it is usually $k \ll n$, and the number m_2 of closing constraints is also rather small. Moreover, the solution is by assumption released from the problem of violation of constraints "1" and the tendency to violate constraints "2" by the solution $q(t)$ is considerably abated (Blajer et al., 1994). The main advantage is however that DAEs (5.4) can indirectly be solved as ODEs. Namely, at each instant of simulation, Eqs $(5.4)_1$ and $(5.4)_2$ can be regarded as $k_1 + k$ ODEs in q and u, while f required for calculation of h''_μ should be determined as the solution of m algebraic equations $(5.4)_3$ for the current state values q and u. The solution to Eqs (5.4) is then $q(t)$, $u(t)$ and $f(t)$. Using the solution and Eqs (5.2), the absolute positions, velocities and accelerations of each link can also be determined from Eq (4.2).

Let us now comment in detail on the problem of handling the stick-slip phenomenon. This will be done here for one-degree-of-freedom mechanisms ($k = 1$) and with reference to the minimal-form formulation (5.4). The following approach to detection of the sticking-sliding state transitions is proposed. At the considered instants of motion we have either $u = 0$ or $u \to 0$, and then, through Eq (5.5), $\dot{s} = \mathbf{0}$ or $\dot{s} \to \mathbf{0}$ - the velocities of sliding at all joints are equal to zero or simultaneously go to zero. In each case, taking $u = 0$ ($\dot{s} = \mathbf{0}$), from Eq $(5.4)_2$ we first determine $\dot{u}_{\mu=0}$ for $h''_\mu = 0$ (as if there was no friction in the system). Assumed $\dot{u}_{\mu=0} \neq 0$ (for $\dot{u}_{\mu=0} = 0$ stiction continues/begins by assumption), we thus estimate a tendency of motion, i.e. the intended direction of evolution of u for the "friction-free" system. From Eq (5.5) we then find the intended local sliding directions $\dot{\boldsymbol{s}}$, and having them we can orientate (in the opposite directions) the friction forces/moments $\boldsymbol{f}_\mu$. Setting the maximal values of the possible frictional forces, i.e. using the static friction coefficients μ_s in Eqs (2.4) and (2.5), from Eq $(5.4)_3$ we determine $\boldsymbol{f}$, and then from Eq $(5.4)_2$, $\dot{u}_{\mu=\mu_s}$. The signs of $\dot{u}_{\mu=0}$ and $\dot{u}_{\mu=\mu_s}$ are the same or opposite, the motion will start/continue or the stiction will continue/begin. Note that during the stiction determination of $\boldsymbol{f}$ from Eq $(5.4)_3$ is impossible as the values of $\boldsymbol{f}_\mu$ are not known (can range from zero to the maximal static friction force, and cannot be estimated explicitly). In other words, due to the additional friction-induced constraints, we deal with an "overconstrained" system.

6. Case studies

6.1. Example 1

In order to illustrate the minimal-form modelling of friction-affected mechanisms, let us consider the system shown in Fig.5. The absolute coordinates of the system are $\boldsymbol{x} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3, y_3, \theta_3]^\top$, and the matrices $\mathbf{M}$ and $\boldsymbol{h}$ introduced in $(2.9)_1$ are

$$
\begin{aligned}
\mathbf{M} &= \operatorname{diag}(m_1, m_1, J_1, m_2, m_2, J_2, m_3, m_3, J_3) \\
\boldsymbol{h} &= [0, -m_1 g, \tau, 0, -m_2 g, 0, 0, -m_3 g, 0]^\top
\end{aligned}
\tag{6.1}
$$

where m_i and J_i $(i = 1, 2, 3)$ are the link masses and the moments of inertia, g is the gravity acceleration, and τ is the torque applied to the link 1. All revolute joints are identical (of diameter d), and the respective coefficient of friction is μ_r. The sliding friction coefficient at the point O_4 is μ_t.

According to Eqs (2.1), the nine local coordinates $\boldsymbol{z}$ and the four local coordinates $\boldsymbol{s}$ can be expressed in terms of $\boldsymbol{x}$, and then the 8×9 matrix $\mathbf{C}$

and the 4×9 matrix C_μ can easily be derived. The respective formulae are

$$
z = \begin{bmatrix} x_1 - a_1\cos\theta_1 \\ y_1 - a_1\sin\theta_1 \\ x_2 - a_2\cos\theta_2 - x_1 - a_1\cos\theta_1 \\ y_2 - a_2\sin\theta_2 - y_1 - a_1\sin\theta_1 \\ x_3 - a_3\cos\theta_3 - x_2 - a_2\cos\theta_2 \\ y_3 - a_3\sin\theta_3 - y_2 - a_2\sin\theta_2 \\ y_3 + a_3\sin\theta_3 \\ \theta_3 + \pi/2 \end{bmatrix} \qquad s = \begin{bmatrix} \theta_1 \\ \theta_2 - \theta_1 \\ \theta_3 - \theta_2 \\ x_3 + a_3\cos\theta_3 \end{bmatrix}
\tag{6.2}
$$

$$
\mathsf{C} = \begin{bmatrix}
1 & 0 & a_1\sin\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -a_1\cos\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & a_1\sin\theta_1 & 1 & 0 & a_2\sin\theta_2 & 0 & 0 & 0 \\
0 & -1 & -a_1\cos\theta_1 & 0 & 1 & -a_2\cos\theta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & a_2\sin\theta_2 & 1 & 0 & a_3\sin\theta_3 \\
0 & 0 & 0 & 0 & -1 & -a_2\cos\theta_2 & 0 & 1 & -a_3\cos\theta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_3\cos\theta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\mathsf{C}_\mu = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -a_3\sin\theta_3
\end{bmatrix}
$$

The friction forces and moments $\boldsymbol{f}_\mu = [f_{\mu 1}, f_{\mu 2}, f_{\mu 3}, f_{\mu 4}]^\top$ can then be modelled as

$$
\boldsymbol{f}_\mu = \begin{bmatrix} -\mathrm{sgn}(\dot{s}_1)\mu_r d\sqrt{f_1^2 + f_2^2} \\ -\mathrm{sgn}(\dot{s}_2)\mu_r d\sqrt{f_3^2 + f_4^2} \\ -\mathrm{sgn}(\dot{s}_3)\mu_r d\sqrt{f_5^2 + f_6^2} \\ -\mathrm{sgn}(\dot{s}_4)\mu_t |f_7| \end{bmatrix}
\tag{6.3}
$$

where $\boldsymbol{f} = [f_1, \ldots, f_8]^\top$ are the ideal components of joint reaction forces/moments. Using Eqs (6.1) ÷ (6.3), the initial governing equations of the system can be formulated in the form of DAEs (2.9). The dimension of the DAEs is 26.

Let us now perform the first step-reduction of the initial DAEs. For the open-loop system as shown in Fig.5, the proposed partition of constraints $z = \boldsymbol{\Phi}(x) = \mathbf{0}$ defined by Eq $(6.2)_1$ is that the fifth and the sixth of

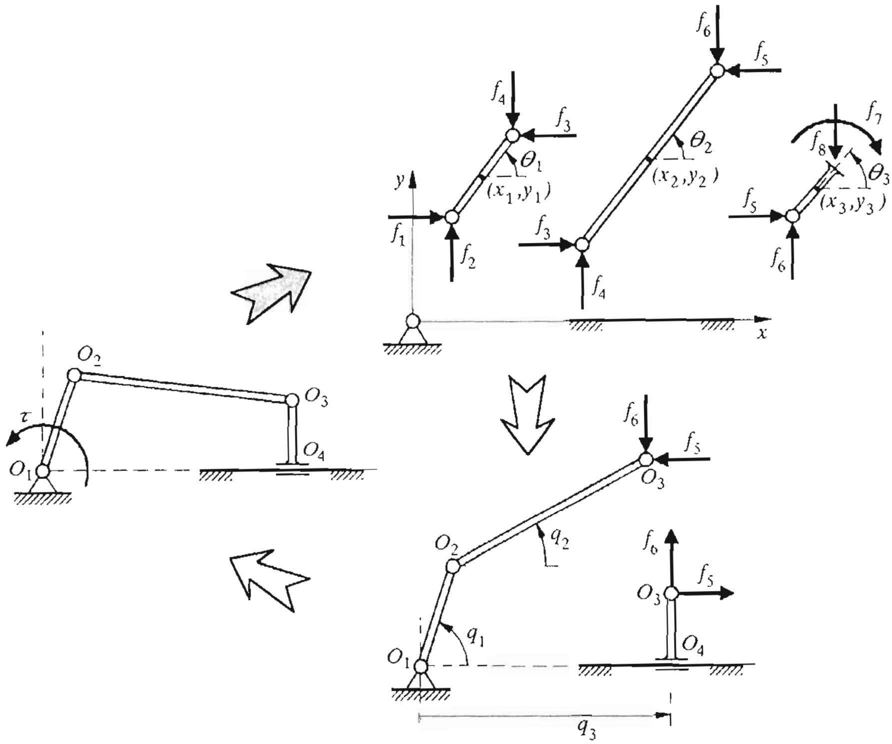

Fig. 5. Mechanism of Example 1

the constraints are treated as constraints "2". Then for $\boldsymbol{q} = [q_1, q_2, q_3]^\top$ as indicated, Eq (4.2) are defined by

$$\boldsymbol{x} = \begin{bmatrix} a_1 \cos q_1 \\ a_1 \sin q_1 \\ q_1 \\ 2a_1 \cos q_1 + a_2 \cos q_2 \\ 2a_1 \sin q_1 + a_2 \sin q_2 \\ q_2 \\ q_3 \\ a_3 \\ -\pi/2 \end{bmatrix} \qquad \mathsf{D}_1 = \begin{bmatrix} -a_1 \sin q_1 & 0 & 0 \\ a_1 \cos q_1 & 0 & 0 \\ 1 & 0 & 0 \\ -2a_1 \sin q_1 & -a_2 \sin q_2 & 0 \\ 2a_1 \cos q_1 & a_2 \cos q_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.4}$$

After formulating $\dot{\mathsf{D}}_1\dot{\boldsymbol{q}}$, the dynamic equations can then be rewritten in the form (4.3), where

$$\mathsf{M}' = \mathsf{D}_1^{\mathsf{T}}\mathsf{M}\mathsf{D}_1 = \begin{bmatrix} J_1 + (m_1 + 4m_2)a_1^2 & 2m_2 a_1 a_2 \cos(q_2 - q_1) & 0 \\ 2m_2 a_1 a_2 \cos(q_2 - q_1) & J_2 + m_2 a_2^2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$\boldsymbol{h}' = \mathsf{D}_1^{\mathsf{T}}(\boldsymbol{h} - \mathsf{M}\dot{\mathsf{D}}_1\dot{\boldsymbol{q}}) =$$

$$= \begin{bmatrix} \tau - (m_1 + 2m_2)ga_1 \cos q_1 + 2m_2 a_1 a_2 \dot{q}_2^2 \sin(q_2 - q_1) \\ -m_2 g a_2 \cos q_2 - 2m_2 a_1 a_2 \dot{q}_1^2 \sin(q_2 - q_1) \\ 0 \end{bmatrix} \tag{6.5}$$

$$\mathsf{C}_2'^{\mathsf{T}}\boldsymbol{f}_2 = \begin{bmatrix} 2a_1 \sin q_1 & -2a_1 \cos q_1 \\ 2a_2 \sin q_2 & -2a_2 \cos q_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_5 \\ f_6 \end{bmatrix}$$

$$\mathsf{C}_\mu'^{\mathsf{T}}\boldsymbol{f}_\mu = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{\mu 1} \\ f_{\mu 2} \\ f_{\mu 3} \\ f_{\mu 4} \end{bmatrix}$$

In order to perform the final reduction step described in Section 5, the constraints "2" must be expressed first in the joint coordinates $\boldsymbol{q}$ and their time-derivatives, i.e.

$$\boldsymbol{\Phi}_2'(\boldsymbol{q}) = \boldsymbol{0} \qquad \dot{\boldsymbol{\Phi}}_2' = \mathsf{C}_2'\dot{\boldsymbol{q}} = \begin{bmatrix} 2a_1 \sin q_1 & 2a_2 \sin q_2 & 1 \\ -2a_1 \cos q_1 & -2a_2 \cos q_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \boldsymbol{0} \tag{6.6}$$

$$\ddot{\boldsymbol{\Phi}}_2' = \mathsf{C}_2'\ddot{\boldsymbol{q}} - \boldsymbol{\xi}_2' \equiv \begin{bmatrix} 2a_1 \sin q_1 & 2a_2 \sin q_2 & 1 \\ -2a_1 \cos q_1 & -2a_2 \cos q_2 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} +$$

$$+ \begin{bmatrix} 2(a_1\dot{q}_1^2 \cos q_1 + a_2\dot{q}_2^2 \cos q_2) \\ 2(a_1\dot{q}_1^2 \sin q_1 + a_2\dot{q}_2^2 \sin q_2) \end{bmatrix} \boldsymbol{0}$$

Then, by choosing $\boldsymbol{u} = [\dot{q}_1]$, D_2 and $\boldsymbol{\nu}_2$ defined by Eqs (5.2) are

$$\mathbf{D}_2 = \begin{bmatrix} 1 \\ -\frac{a_1 \cos q_1}{a_2 \cos q_2} \\ 2a_1(\cos q_1 \tan q_2 - \sin q_1) \end{bmatrix}$$

$$\boldsymbol{\nu}_2 = \begin{bmatrix} 0 \\ \frac{a_1 u^2 \sin q_1}{a_2 \cos q_2}\left(1 + \frac{a_1 \cos q_1 \tan q_2}{a_2 \cos q_2 \tan q_1}\right) \\ -\frac{2a_1 u^2}{\cos q_2}\left(\cos(q_2 - q_1) + \frac{a_1 \cos^2 q_1}{a_2 \cos^2 q_2}\right) \end{bmatrix} \tag{6.7}$$

which enable one to formulate three kinematic equations $(5.4)_1$ and one dynamic equation $(5.4)_2$. Eight algebraic equations $(5.4)_3$ can then be derived according to Eq $(3.3)_2$ (the explicit form of the formulae is rather complicated, and will thus not be reported here). Finally, Eq (5.5) is

$$\dot{\boldsymbol{s}} = \begin{bmatrix} 1 \\ -1 - \frac{a_1 \cos q_1}{a_2 \cos q_2} \\ \frac{a_1 \cos q_1}{a_2 \cos q_2} \\ 2a_1(\cos q_1 \tan q_2 - \sin q_1) \end{bmatrix} u \tag{6.8}$$

The objective of the above example was to illustrate the process of converting the governing equations for a closed-loop friction-affected multibody system from a large set (of number 26) in absolute variables to a minimal set (of number 12) in independent variables. For demonstration purposes, the transformations have been done by symbolic manipulations. In practical applications and for more complex mechanisms, however, that may be very laborious (if applicable at all). Therefore, the conversion steps should rather be algorithmized in computer codes and performed numerically, what refers mainly to the final reduction step (by using the coordinate partitioning) and the formulation of algebraic equations $(5.4)_3$. The proposed symbolic-numerical method is then visibly computer-oriented.

Some results of numerical simulation of the mechanism motion are shown in Fig.6. Using the data: $m_1 = 3\,\mathrm{kg}$, $m_2 = 7\,\mathrm{kg}$, $m_3 = 2\,\mathrm{kg}$, $a_1 = 0.3\,\mathrm{m}$, $a_2 = 0.7\,\mathrm{m}$, $a_3 = 0.2\,\mathrm{m}$, and $J_i = m_i a_i^2/3$, the motion was performed for the friction-free mechanism (denoted $\mu = 0$) and the friction-affected mechanism (denoted μ), and the data relating the static (s) and kinetic (k) friction coefficients were: $\mu_{ts} = 0.3$, $\mu_{tk} = 0.2$, $\mu_{ts} = 0.15$, $\mu_{rk} = 0.1$, and $d = 0.05\,\mathrm{m}$. The graphs show a substantial difference between the two cases. Neglecting the frictional effects may thus impair the reliability of analysis of mechanism. As for the mechanism at hand the *stick-slip* phenomena are not clearly seen, let us consider the other example.

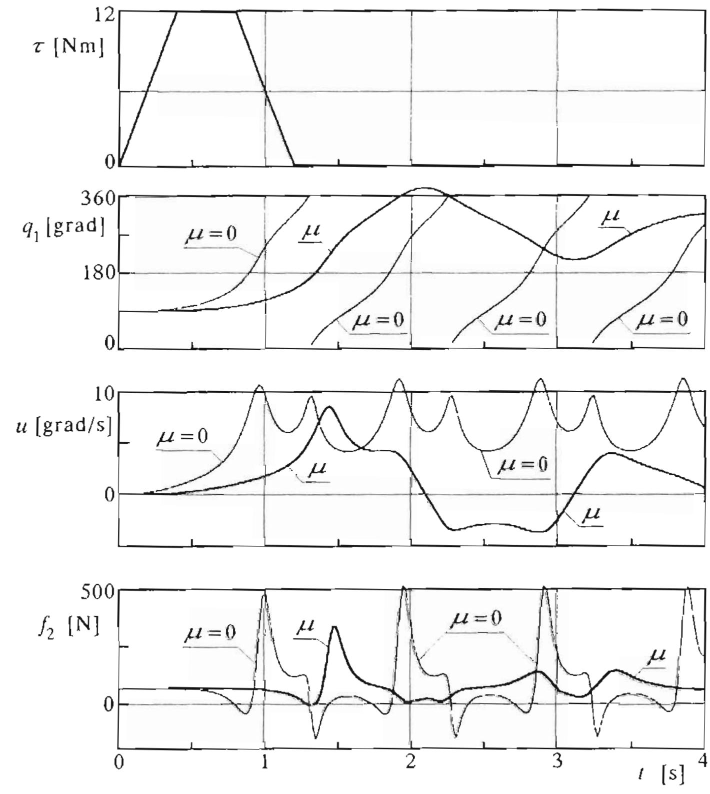

Fig. 6. Some simulation results of Example 1

6.2. Example 2

The considered slider-crank mechanism is shown in Fig.7. As the minimal-form modelling of the mechanism has already been reported by Blajer and Markiewicz (1995), here, we present only some numerical results. Using the same notation as in Example 1, the data were: $m_1 = 2.5\,\mathrm{kg}$, $m_2 = 4\,\mathrm{kg}$, $a_1 = 0.25\,\mathrm{m}$, $a_2 = 0.4\,\mathrm{m}$, $J_i = m_i a_i^2/3$, $\mu_{ts} = 0.3$, $\mu_{tk} = 0.2$, $\mu_{ts} = 0.15$, $\mu_{rk} = 0.1$, and $d = 0.05\,\mathrm{m}$, where "1" and "2" relates links OA and AB, respectively. The gravity acceleration is directed along Ox axis. The obtained numerical results are presented in Fig.8. The simulation starts from the unforced equilibrium (vertical position) and the mechanism is driven by a sinusoidally varying torque τ as seen. The motion that follows oscillates about the equilibrium position. Two scenarios occur when $u \to 0$ (here $u = \dot{q}_1$), according to the resultant of the driving torque and the gravitational force

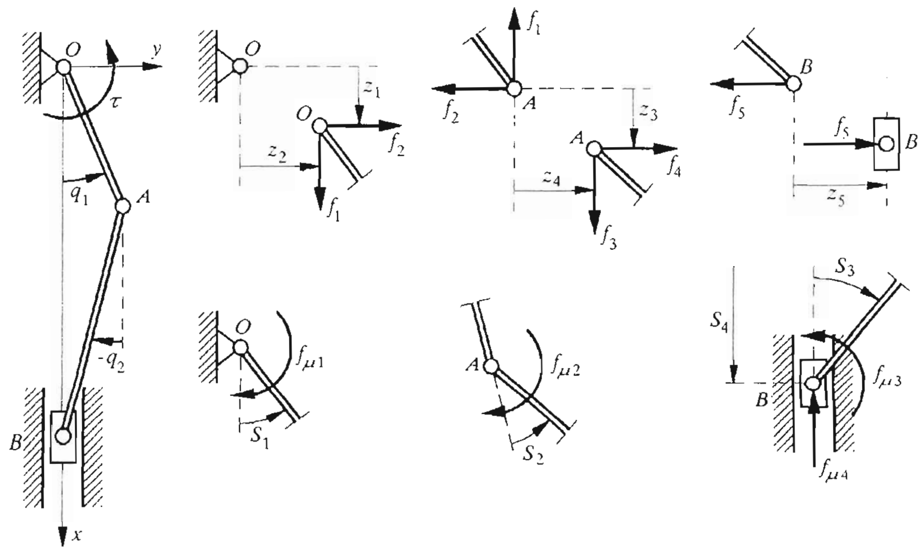

Fig. 7. Mechanism of Example 2

torque surpasses the frictional resistance in the system or not. In the former case, the motion continues (with the sign of u changed), and the observed impacts in the motion are due to the abrupt changes of the friction force directions. In the latter case the stiction begins and remains as long as the change in the driving torque will break the "stiction equilibrium". As said in Section 5, during stiction determination of the constraint reactions is impossible, which is represented in Fig.8 by breaks in the reported graph of f_1 joint force.

7. Conclusions

The proposed minimal-form modelling of friction-affected mechanisms may be advantageous for many reasons. It provides one with an automatic and simple approach to modelling of frictional effects in absolute variables, and then converting the arising large set of DAEs (2.9) to the minimal-form governing equations (5.4). As compared to the initial DAEs, the final formulation assures improved efficiency and precision of numerical simulation. It is especially useful in handling the stick-slip phenomena. A more physical insight into the problems solved can also be gained. Time-variations of the mechanism state

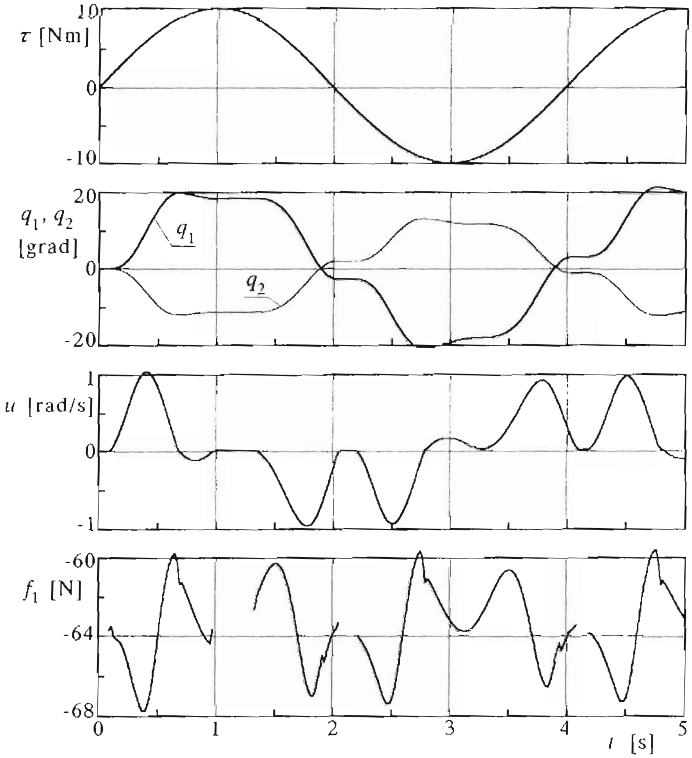

Fig. 8. Some simulation results of Example 2

variables and the joint reactions (if necessary, the position, velocity and acceleration of any link/point as well) can be obtained by numerical simulation. The minimal-form governing equations can also be conveniently used to synthesize the required driving torque in a specified (programmed) mechanism motion. Namely, assumed the program of motion is defined by a specified driver motion, $\boldsymbol{q}_u = \boldsymbol{q}_u^s(t)$, from the specification we have directly $\boldsymbol{u}^s(t) = \dot{\boldsymbol{q}}_u^s(t)$ and $\dot{\boldsymbol{u}}^s(t) = \ddot{\boldsymbol{q}}_u^s(t)$, and then using $\boldsymbol{\Phi}_2'(\boldsymbol{q}) = \boldsymbol{0}$ we can solve for the program variations of the other joint coordinates, $\boldsymbol{q}_v = \boldsymbol{q}_v^s(t)$. Applying $\boldsymbol{u}^s(t)$, $\dot{\boldsymbol{u}}^s(t)$, and $\boldsymbol{q}^s(t) = [\boldsymbol{q}_u^{s\top}(t), \boldsymbol{q}_v^{s\top}(t)]^\top$, the required $\tau^s(t)$ as well as $\boldsymbol{f}^s(t)$ can be determined as a solution of Eqs $(5.4)_2$ and $(5.4)_3$ for a given t, which are now $k+m$ algebraic equations in τ and $\boldsymbol{f}$.

The described conversion steps leading to the minimal-form formulation (5.4) can be performed analytically or numerically. Usually, the first-step of conversion can conveniently be done analytically, i.e. by pencil and paper or by computerized symbolic manipulations, and the final reduction step as well as the formulation of algebraic equations $(5.4)_3$ should rather be gained by means of numerical methods. While a similar conversion procedure for friction-free multibody systems has already been described e.g. by Nikravesh (1990), its application to friction-affected mechanisms is new.

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Modelowanie mechanizmów z tarciem w postaci równań o minimalnym wymiarze

Streszczenie

Praca podejmuje zagadnienie modelowania dynamiki mechanizmów z uwzględnieniem ewentualnego tarcia suchego w połączeniach (ograniczono się do układów płaskich o jednym stopniu swobody). Modelowanie efektów tarcia realizowane jest w zmiennych absolutnych poszczególnych członów W zmiennych tych formułowane są też wyjściowe równania ruchu – nieliniowe równania różniczkowo-algebraiczne (RRA) o maksymalnym wymiarze. Prezentowana jest następnie dwukrokowa metoda redukcji tych równań do możliwie najmniejszego wymiaru. W pierwszym kroku wykorzystywana jest metoda zmiennych złączowych, w drugim – metoda podziału zmiennych. Otrzymane finalne równania ruchu składają się z pojedynczego dynamicznego równania różniczkowego zredukowanego do ruchu członu napędzającego, kilku równań kinematycznych uzależniających pochodne po czasie współrzędnych złączowych od prędkości członu napędzającego oraz równań algebraicznych (w liczbie równej liczbie więzów połączeń kinematycznych) dla wyznaczania reakcji w połączeniach. Efekty tarcia suchego reprezentowane są w równaniu dynamicznym oraz równaniach algebraicznych, sprzęgając je ze sobą. Rozwiązanie tych ostatnich ze względu na wartości reakcji w połączeniach dla aktualnego stanu ruchu jest postawą dla określenia reprezentacji sił tarcia w równaniu dynamicznym i jego efektywnego całkowania numerycznego. Tym samym, finalne minimalno-wymiarowe równania ruchu, stanowiące formalnie układ RRA, rozwiązywane mogą być pośrednio jako równania różniczkowe zwyczajne (RRZ). Prezentowane są dwa przykłady prostych mechanizmów płaskich dla zilustrowania omówionych etapów modelowania mechanizmów z tarciem oraz złożonych zagadnień symulacji numerycznej takich układów. Trudności związane są przede wszystkim z algorytmizacją zjawisk przy przechodzeniu od tarcia statycznego do kinetycznego i odwrotnie.

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