JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
4, **37**, 1999

POST-BUCKLING ANALYSIS OF ELASTIC-PLASTIC PLATES BASING ON THE TSAI-WU CRITERION

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This work deals with the analysis of post-buckling state in rectangular plates subject to uniaxial in-plane compression within the elastic and elasto-plastic range. The problems of initial out-of-flatness and different geometry of rectangular plate are considered. The analysis is carried out on the basis of non-linear theory of plates involving plasticity. It is assumed that the yield values in tension and compression tests of a plate material are different. Thus, the Tsai-Wu Criterion is applied. The solution is obtained an the analytical-numerical way using the Prandtl-Reuss equations. As a result of numerical calculations the load-shortening curves for the considered plates are obtained.

Key words: nonlinear stability, elastic-plastic plates

Notation

a, b	–	plate width and length, respectively
C	–	yield limit in the compression test
E	–	Young modulus
f	–	amplitude of total deflection
h	–	plate thickness
M_x, M_y, M_{xy}	–	sectional bending moments
N_x, N_y, N_{xy}	–	sectional forces
u, v	–	in-plane displacements
S	–	plate shortening
S_Y	–	shortening corresponding to the yield limit
T	–	yield limit in the tension test

w, w_0	–	total and initial deflection, respectively
$\varepsilon_{xm}, \varepsilon_{ym}, \gamma_{xym}$	–	membrane strains
$\varepsilon_{xb}, \varepsilon_{yb}, \gamma_{xyb}$	–	bending strains
Φ	–	Airy's stress function
λ	–	scalar parameter in the Prandtl-Reuss equations
ν	–	Poisson ratio
σ_{av}	–	average compressive stress corresponding to the shortening S
σ_{cr}	–	critical stress
σ_u	–	ultimate stress – load carrying capacity
$\sigma_{xm}, \sigma_{ym}, \tau_{xym}$	–	membrane stresses
$\sigma_x, \sigma_y, \tau_{xy}$	–	total stresses

1. Introduction

Thin plates are used in manufacturing of different structures: columns and girders of closed and open cross-sections, panels, etc. Frequently, the load carrying capacity of such structures subject to the increasing loads is determined by their stability in the elastic or inelastic range. Depending on the loading, shape and dimensions of a structure the stability loss may occur in a different way: e.g. global buckling of a whole structure, local buckling of components, coupled buckling, etc.

In the case of local buckling the first step in understanding the behaviour of a thin-walled plate structure is to find a complete solution for an individual plate element.

The load carrying capacity of thin plates subject to the in-plane loading can be determined by many approximate relations (e.g. Rhodes and Harvey, 1971) or more precisely by analysing the stresses and strains in the elastic range and then in the elasto-plastic range. As a result of such analysis the curves showing the relationship between the load and shortening of a plate (i.e., L-S curves) are obtained and the maximum load carrying capacity can be found out (in Fig.1 a typical curve for a perfect plate subject to the axial compression is presented). These "precise" analyses of individual plate or structure behaviour have been conducted in purely numerical (e.g. Rondal and Maquoi, 1985) or analytical-numerical way (among others: Graves-Smith, 1971; Little, 1982; Grądzki and Kowal-Michalska, 1988). In each case the material characteristic has to be known in the whole range of loading. For more comprehensive survey of the literature on this problem, see Bradfield (1982).

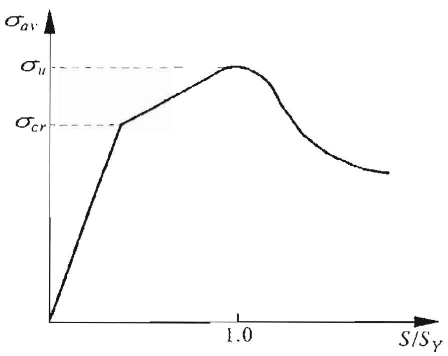

Fig. 1. Typical load versus shortening curve for a perfect plate

It should be noted that geometrical parameters, material characteristics and initial imperfections (out-of-flatness and residual stresses) exert a great influence on the character of L-S curves and should be taken into account, e.g. Grądzki and Kowal-Michalska (1991).

When the behaviour of a whole plate structure is considered the problem becomes more complicated because the conditions of co-operation of plate components should be incorporated (Grądzki, 1998).

Most of the works mentioned above dealt with isotropic materials. Modern materials, as composites are in general anisotropic but some of them (e.g. fibrous composites) can be treated as the orthotropic ones and then the appropriate relations for orthotropic plates have to be employed in the elastic and elasto-plastic ranges (Kowal-Michalska, 1995).

To evaluate the load carrying capacity of a plate the yield criterion has to be assumed. For isotropic materials usually the Huber-Mises Yield Criterion is applied. If orthotropic plates are analysed Hill's Criterion (Hill, 1950) is used. In all these criteria it is assumed that the material uniaxial stress-strain curves obtained by tensile and compressive tests overlap. Meanwhile, it is well known that many materials do not fulfil this condition. There are such traditional materials as concrete, cast iron, wood the compressive strength of which is much greater than the tensile one. On the other hand, modern materials used in reinforced composites as kevlar, glass fibres, etc. are much stronger in tension than in compression. To determine the ultimate strength of structural members made of such materials Tsai and Wu (1971) proposed a modification of Hill's Yield Criterion.

In the present paper the analysis of post-buckling state of thin plates made from the materials isotropic in the elastic range but with different tensile and compressive yield is carried out. The considerations are based on the nonlinear theory of plates. In the elasto-plastic range the stress-strain relations are described by the Prandt-Reuss equations where the infinitesimal increments are replaced by the finite ones. Thus the problem can be solved only in an iterative way and the solution is reached by an analytical-numerical method.

2. Structural problem

The rectangular plates subject to the in-plane compression (Fig.2) are considered.

The following boundary conditions are imposed on the plate edges (Table 1):

- Loaded edges are simply supported and remain straight and parallel during loading
- Unloaded edges can be simply supported or clamped,there are no membrane stresses what means that the plate can be treated as a wall of a box structure.

Table 1. Boundary conditions

Loaded edges $y = \pm b/2$	Unloaded edges $x = \pm a/2$	
$w - w_0 = 0$	$w - w_0 = 0$	
$M_y = 0$	simply supported $M_x = 0$	clamped $\partial w/\partial x = 0$
$v = -Sb/2$ $N_{xy} = 0$ $\int\limits_{-a/2}^{a/2} N_y \, dx = -\sigma_{av} a h$	$N_{xy} = 0$ $N_x = 0$	

The assumed uniaxial stress-strain characteristic of the plate material is shown in Fig.3. In the elastic range the material is isotropic but with different tensile and compressive yield limit values.

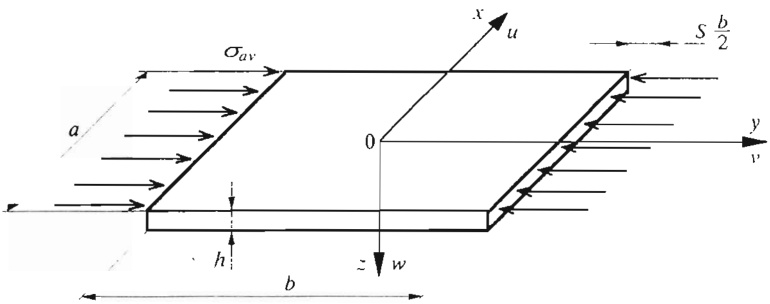

Fig. 2. Geometry and loading of the plate

3. Governing equations in the elastic range

In the investigations the relationships of nonlinear theory of plates are applied. For a plate with a initial out-of-flatness w_0 the geometrical relationships are assumed as follows

$$
\begin{aligned}
\varepsilon_{xm} &= \frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2 - \frac{1}{2}\Big(\frac{\partial w_0}{\partial x}\Big)^2 \\
\varepsilon_{ym} &= \frac{\partial v}{\partial y} + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2 - \frac{1}{2}\Big(\frac{\partial w_0}{\partial y}\Big)^2 \\
\gamma_{xym} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \\
\varepsilon_{xb} &= z\kappa_x \qquad \varepsilon_{yb} = z\kappa_y \qquad \gamma_{xyb} = 2z\kappa_{xy} \\
\kappa_x &= -\frac{\partial^2(w-w_0)}{\partial x^2} \qquad \kappa_y = -\frac{\partial^2(w-w_0)}{\partial y^2} \qquad \kappa_{xy} = -\frac{\partial^2(w-w_0)}{\partial x \partial y}
\end{aligned}
\tag{3.1}
$$

Taking into account the physical relations between strains and stresses (Hooke's law), the compatibility equation and introducing Airy's stress function in a form

$$
\begin{aligned}
\sigma_{xm} &= \frac{N_x}{h} = \frac{\partial^2 \varPhi}{\partial x^2} \qquad \sigma_{ym} = \frac{N_y}{h} = \frac{\partial^2 \varPhi}{\partial y^2} \\
\tau_{xym} &= \frac{N_{xy}}{h} = -\frac{\partial^2 \varPhi}{\partial x \partial y}
\end{aligned}
\tag{3.2}
$$

the well known von Karman equation is obtained

$$E \times \Big(\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4}\Big) = \\ = \Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2 - \Big(\frac{\partial^2 w_0}{\partial x \partial y}\Big)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} \tag{3.3}$$

The total deflection function w fulfilling kinematic boundary conditions (Table 1) is assumed in a classical form:
— for simply supported plates with unloaded edges

$$w = f \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \tag{3.4}$$

— for clamped plates with unloaded edges

$$w = f \cos^2\frac{\pi x}{a} \sin\frac{2\pi y}{b} \tag{3.5}$$

Further it is assumed that the initial deflection of a plate has the identical form as the total deflection (with the amplitude f_0).

Substituting the functions (3.4) or (3.5) into Karman's equation and taking into account the boundary conditions, Airy's stress function is found out. Next on the basis of Hooke's law and Eqs (3.1) the strains and displacement functions are determined in the elastic range.

4. Constitutive relations in the elasto-plastic range

Considering the post-buckling state in the elasto-plastic range the following assumptions are made:

- Material of the characteristic shown in Fig.3 meets the Tsai-Wu Criterion
- All assumptions of nonlinear theory of plates still hold
- Forms of the displacement functions are the same in the elastic and elasto-plastic ranges
- According to the plastic flow theory the stress-strain relations are described by the Prandtl-Reuss equations.

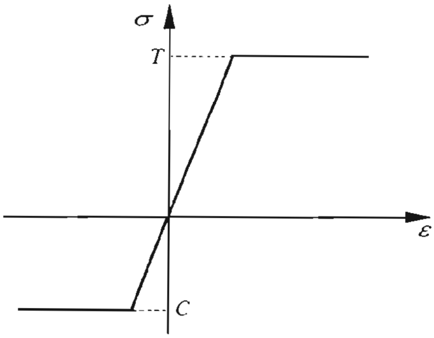

Fig. 3. Material tensile and compressive characteristics

The elasto-plastic analysis is conducted on the basis of Tsai-Wu Yield Criterion that is formulated for the considered problem as follows

$$a_1(\sigma_x + \sigma_y) + \sigma_x^2 + \sigma_y^2 - a_{12}\sigma_x\sigma_y + 3a_3\tau_{xy}^2 - a_0 = F \tag{4.1}$$

where

$$a_0 = CT \qquad a_1 = C - T$$

$$a_{12} = \frac{2CT}{C^2 + T^2} \qquad a_3 = \frac{2}{3}\frac{C^2 + CT + T^2}{C^2 + T^2}$$

T and C – yield limits in tension and compression tests, respectively.

It is assumed that after yielding differential increments of strain $d\varepsilon$ are a total of elastic and plastic components $d\varepsilon^e$ and $d\varepsilon^p$.

The plastic components $d\varepsilon^p$ are described by the Prandtl-Reuss equations

$$d\varepsilon_{ij}^p = \lambda\frac{\partial F}{\partial\sigma_{ij}} \qquad (i,j = x,y) \tag{4.2}$$

where λ is a positive definite scalar.

In practical calculations the infinitesimal increments of strains and stresses have to be replaced by the finite ones, so the resulting expressions for elasto-plastic stress increments are

$$\begin{aligned}
\Delta\sigma_x &= \frac{E}{1-\nu^2}[\Delta\varepsilon_x + \nu\Delta\varepsilon_y - \lambda(S_x + \nu S_y)] \\
\Delta\sigma_y &= \frac{E}{1-\nu^2}[\Delta\varepsilon_y + \nu\Delta\varepsilon_x - \lambda(S_y + \nu S_x)] \\
\Delta\tau_{xy} &= \frac{E}{2(1+\nu)}(\Delta\gamma_{xy} - 2\lambda\tau_{xy})
\end{aligned} \tag{4.3}$$

where

$$\lambda = \frac{(S_x + \nu S_y)\Delta\varepsilon_x + (S_y + \nu S_x)\Delta\varepsilon_y + a_3(1-\nu)\tau_{xy}\Delta\gamma_{xy}}{S_x^2 + S_y^2 + 2\nu S_x S_y + 2a_3^2(1-\nu)\tau_{xy}^2}$$

$$S_x = \frac{1}{3}(a_1 + 2\sigma_x - a_{12}\sigma_y) \tag{4.4}$$

$$S_y = \frac{1}{3}(a_1 + 2\sigma_y - a_{12}\sigma_x)$$

5. Method of solution

The iterative approach to the problem is employed using a combination of analytical and numerical solutions based on the Rayleigh-Ritz variational method involving the relations of flow theory of plasticity (Prandtl-Reuss equations). It was proved by Graves-Smith (1971) that the variational method could be applied to the elasto-plastic plates undergoing finite deflections. This method has been used in many works.

The Rayleigh-Ritz variational method requires evaluation of the increment of total plate energy (5.1). The symmetry of the problem allows us to restrict the considerations to the quadrant of plate, shown in Fig.2. The elastic and plastic energy increments are evaluated in a numerical way. In order to accomplish this, discretisation of the plate is performed. The volume V of the plate is divided into appropriate cubicoids. The values of the energy increments, calculated in each cubicoid, are summed for the whole plate.

It should be emphasized that during the analytical-numerical solution the response of the plate to the increment of shortening S (Fig.2) is searched for.

The increment of potential energy in the elastic and elasto-plastic range has the form

$$\Delta W = \int\limits_V \Big[\Big(\sigma_x + \frac{1}{2}\Delta\sigma_x\Big)\Delta\varepsilon_x + \Big(\sigma_y + \frac{1}{2}\Delta\sigma_y\Big)\Delta\varepsilon_y + \Big(\tau_{xy} + \frac{1}{2}\Delta\tau_{xy}\Big)\Delta\gamma_{xy}\Big]\, dxdydz \tag{5.1}$$

where

$\sigma_x, \sigma_y, \tau_{xy}$ – stresses before the shortening increment ΔS is applied

$\Delta\sigma, \Delta\varepsilon$ – stress and strain increments produced by ΔS.

Next, numerical minimisation of the functional (5.1) is performed versus the independent deflection parameters.

In each step of calculations the following aspects are taken into account:

- *Occurrence of active, passive and neutral processes, respectively*
- Reduction of stresses to the actual yield surface, strayed from it because finite increments are used.

The average stress corresponding to the load applied to the plate is obtained numerically, using the Virtual Work Principle

$$\sigma_{av} = \frac{1}{abh}\frac{\partial(\Delta W)}{\partial S} \tag{5.2}$$

6. Results of numerical calculations

As a result of numerical calculations the L-S curves have been obtained (as $\sigma^* = \sigma_{av}/(200\,\text{MPa})$ versus $S^* = SE/(200\,\text{MPa})$) for different geometrical imperfections and of plate geometries. For all the plates considered, the material properties are assumed as follows: $E = 2.05 \cdot 10^5$ MPa, $\nu = 0.3$.

For chosen examples the distribution of plasticity regions in the plate during loading have also been found out.

The influence of compressive yield limit C on the collapse behaviour of almost flat plates $(f_0 = 0.001h)$ of geometrical parameter $a/h = 80$ and $a/h = 55$ is analysed, for values of C varying from 100 MPa to 400 MPa (Fig.4 and Fig.5). It is assumed that the plates are simply supported along all edges. In all cases the increase of the compressive yield value causes the increase of load carrying capacity.

The influence of various tensile yield values on L-S curves is presented in Fig.6. It can be clearly seen that the variation of tensile yield value exerts no influence on the ultimate load in all the considered cases so the ultimate load of plates subject to compression depends mainly on the compressive yield value.

In Fig.7 the results obtained for simply supported plates for different values of geometrical parameter a/h and initial imperfection amplitudes, are shown. It can be observed that the occurrence of initial imperfections changes the character of L-S curves. The curves for almost flat plates reach maximum and the curves for plates with large initial imperfections lie below. For a rather thick plate $(a/h = 55,\ f_0 = 0.001h)$ the curve has distinctive maximum and for a thinner one $(a/h = 80,\ f_0 = 0.001h)$ the "plateau" appears.

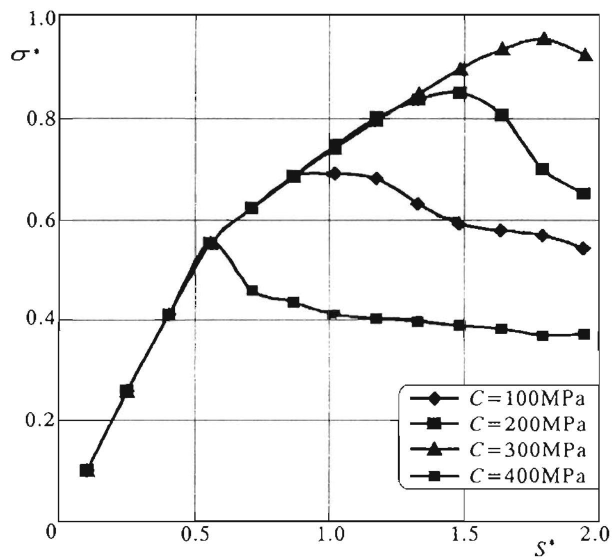

Fig. 4. Load-shortening curves for simply supported plates for different compressive yield values, $a/h = 80$, $f_0 = 0.01h$, $T = 200\,\mathrm{MPa}$

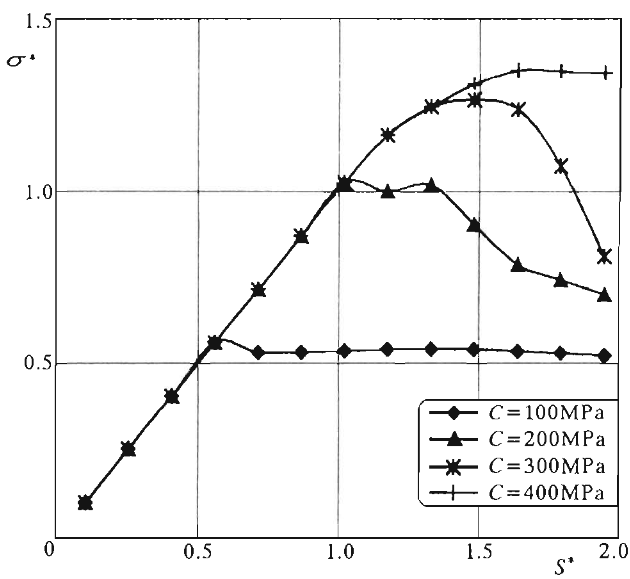

Fig. 5. Load-shortening curves for simply supported plates for different compressive yield values, $a/h = 55$, $f_0 = 0.01h$, $T = 200\,\mathrm{MPa}$

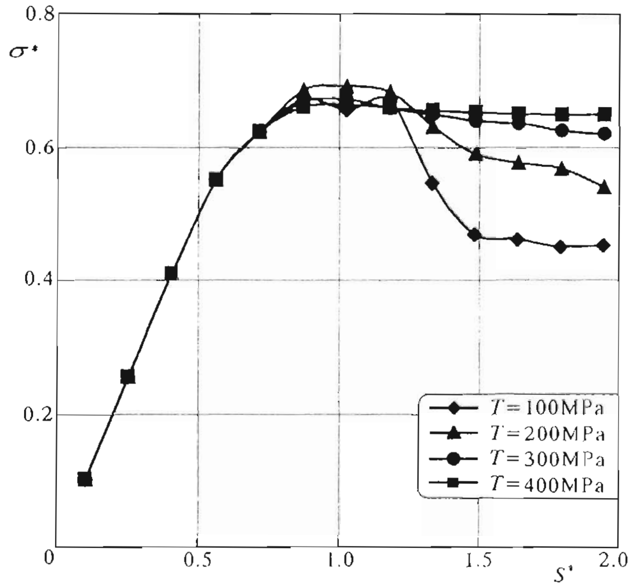

Fig. 6. Load-shortening curves for simply supported plates for different tensile yield values, $a/h = 80$, $f_0 = 0.01h$, $C = 200\,\mathrm{MPa}$

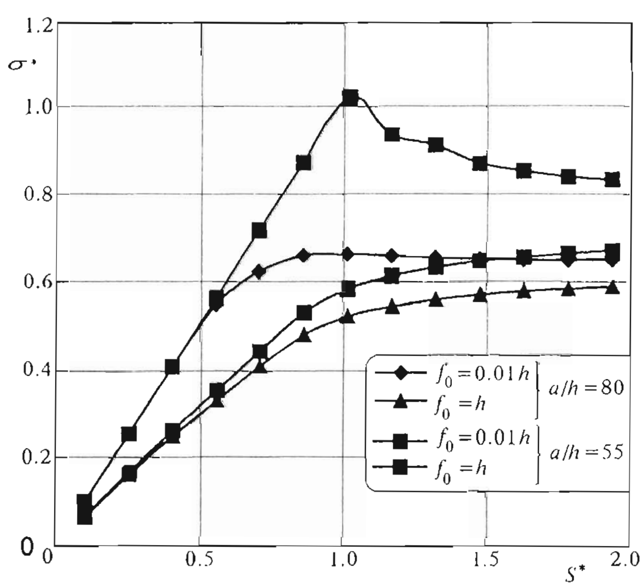

Fig. 7. Influence of initial imperfections on the load-shortening curves for simply supported plates, $T = 400\,\mathrm{MPa}$, $C = 200\,\mathrm{MPa}$

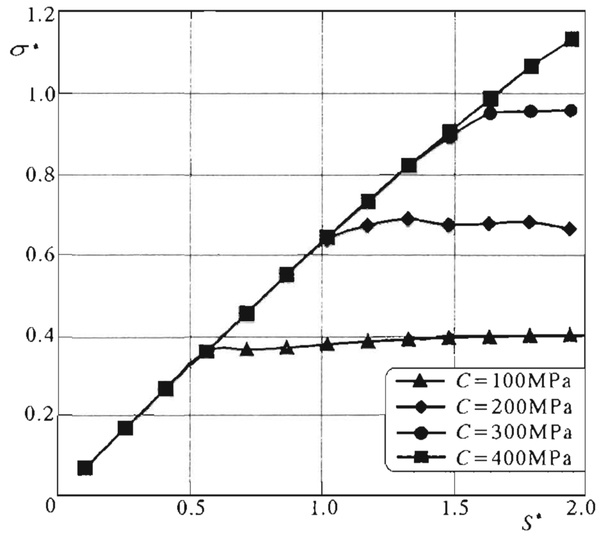

Fig. 8. Load-shortening curves for the plates with the unloaded edges clamped for different compressive yield values, $a/h = 80$, $f_0 = 0.5h$, $T = 200\,\mathrm{MPa}$

Fig.8 shows the influence of tensile and compressive yield values, respectively, on the L-S curves for plates with the unloaded edges clamped. The increase in compressive yield value causes the increase in maximum load values. For this kind of boundary conditions the variation of compressive yield value exert a slight influence on the character of load-shortening curves.

Fig.9 shows the influence of compressive yield limit C on the L-S curves for plates with the unloaded edges clamped for the two values of $a/h = 55$ and $a/h = 80$. The behaviour of considered plates is similar to the simply supported plates (see Fig.4 and Fig.5) and the remarks about the influence of C can be repeated here.

Comparison of the L-S curves for simply supported plates and plates with clamped unloaded edges, for various values of compressive yield limit is presented in Fig.10. For the plates of $C = 100\,\mathrm{MPa}$ differences in load carrying capacity are small and maximum load is the same. For larger values of C ($C = 300\,\mathrm{MPa}$ and $C = 400\,\mathrm{MPa}$) the differences appear for simply supported plates. The clamping of unloaded edges causes the increase in load carrying capacity of the considered plates.

Distribution of elastic and plastic regions is determined for each of the considered plates. For example, in Fig.11 spread of plastic and elastic zones

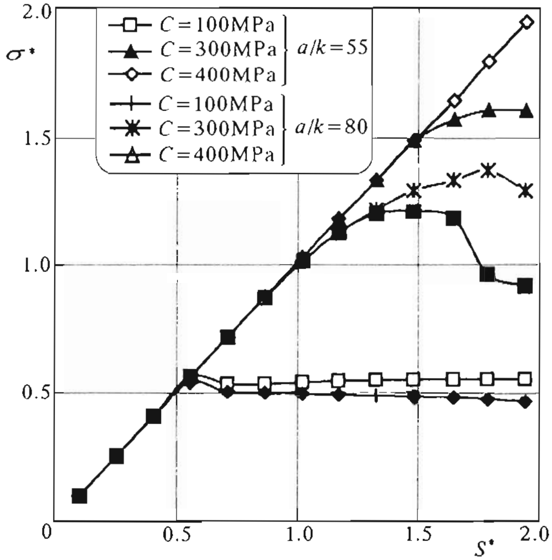

Fig. 9. Load-shortening curves for the plates with the unloaded edges clamped for different tensile yield values, $T = 200\,\mathrm{MPa}$, $f_0 = 0.01h$

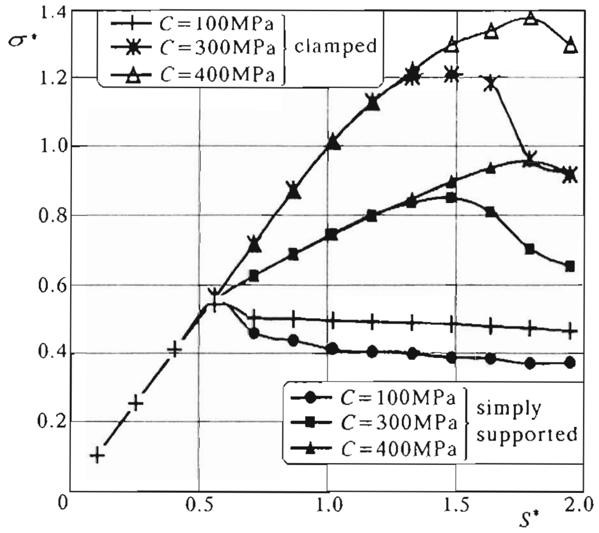

Fig. 10. Observed differences in the character of L-S curves caused by clamping of the unloaded edges, $a/h = 80$, $f_0 = 0.01h$, $T = 200\,\mathrm{MPa}$

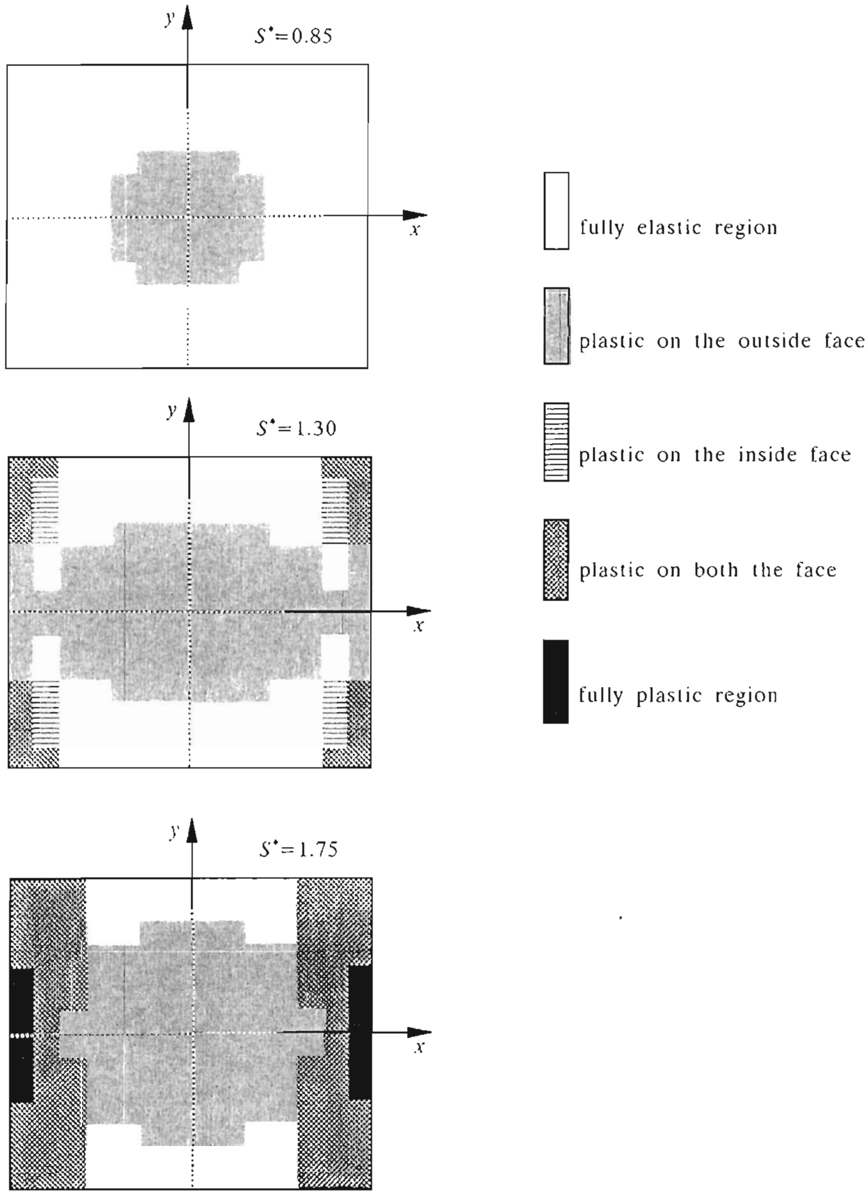

Fig. 11. Spread of elastic and plastic regions in a simply supported plate, $a/h = 80$, $f_0 = 0.01h$, $T = 400\,\mathrm{MPa}$, $C = 200\,\mathrm{MPa}$

for simply supported plate of $a/h = 80$ is presented, for three values of shortening S^*. For this plate, the plastic zones appear in the middle of a plate but for higher values of shortening the regions along the unloaded edges become fully plastic.

7. Final remarks

The method presented above makes it possible to obtain the load-shortening curves for uniaxially compressed plates from a material with different tensile and compressive yield limit values.

As it could be expected variation of the compressive yield limit values exerts a great influence on the value of ultimate load and on the character of L-S curves while the variation of tensile yield limit value exerts no influence on the maximum load magnitude and only slightly changes the character of L-S curves in the elasto-plastic range. Thus, when only the ultimate load of the compressed plate is to be found it can be assumed that the value of tensile yield limit is the same as the compressive one and the problem may be considered on the basis of Huber-Mises Yield Criterion.

It should be emphasized that these considerations should be treated as a first step into the analysis of orthotropic plates with different material characteristics obtained in tension and compression test.

Acknowledgements

The work has been financially supported by the State Committee for Scientific Research (KBN) under grant No. PB 251/T07/97/12.

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Analiza stanu zakrytycznego sprężysto-plastycznych płyt na podstawie kryterium Tsai-Wu

Streszczenie

W pracy przeprowadzono analizę stanu zakrytycznego w obszarze sprężystym i sprężysto-plastycznym prostokątnych płyt poddanych ściskaniu. Rozważano płyty wstępnie wygięte, o różnej geometrii, wykonane z materiału o różnych wartościach granic plastyczności na ściskanie i rozciąganie. Badania prowadzono w oparciu o nieliniową teorię cienkich płyt z uwzględnieniem kryterium plastyczności Tsai-Wu. Rozwiązanie uzyskano na drodze analityczno-numerycznej stosując równania Prandtla-Reussa w postaci przyrostowej. Wyniki obliczeń numerycznych przedstawiono na wykresach.

Manuscript received March 3, 1999; accepted for print June 8, 1999