JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
4, **37**, 1999

RANDOM VORTEX METHOD APPROACH TO AXISYMMETRIC JET IN A LARGE TANK

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In the paper we give an extention of the Vortex Blob Method to cover axisymmetric flows of viscous liquid. We also present a simple simulation of a jet flowing into a half-space large tank.
This example shows that the method works and that a more reliable numerical simulation can be performed.

Key words: viscous jet flow, stochastic method, axisymmetric vortex blob

1. Introduction

The aim of this paper is to show that the stochastic Vortex Blob Method (VBM) can be used in simulation of symmetric flows of viscous liquid. We consider the flow in the half-space large tank what seems to be a very simple geometric and kinematic problem. The liquid flows into the tank through the bottom orifice. The initial-boundary problem is solved using the Lagrangian vortex method. This method seems to be an extention of the well known 2D VBM. When applying and extending this method two steps are required.

First, one has to design an axisymmetric vorticity carrier. Then, it is to necessary to formulate the appropriate Neumann problem. Both steps are described in this paper. First, a brief formulation of the VBM is presented.

2. Essence of vortex blob method

The axisymmetric flow of viscous liquid is represented by the following set of differential equations

$$\begin{aligned}
&\omega = \frac{\partial V_z}{\partial r} - \frac{\partial V_r}{\partial z} \\
&\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial r}\Big[\omega\Big(V_r - \frac{\nu}{r}\Big)\Big] + \frac{\partial}{\partial z}(\omega V_z) = \nu\Big(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2}\Big) \\
&\frac{\partial (rV_r)}{\partial r} + \frac{\partial (rV_z)}{\partial z} = 0
\end{aligned} \tag{2.1}$$

First of these equations defines the vorticity ω, the second one is the vorticity transport equation and the last one expresses mass conservation. The symbols V_r and V_z denote radial and axial components of velocity, respectively.

The vorticity equation $(2.1)_1$ is written here in the form consistent with the Planck-Fokker-Kołmogorow (P-F-K) equation (cf Gardiner, 1989) for the probability density function $p(t,x,y|0,x_0,y_0)$

$$\frac{\partial p}{\partial t} + \frac{\partial (ap)}{\partial x} + \frac{\partial (bp)}{\partial y} = D\Delta p \qquad D = \text{const} > 0$$

This equation represented the probability density formed by a family of stochastic processes. This family, denoted by $x(t;x_0,y_0)$, $y(t;x_0,y_0)$, results from Ito's stochastic differential equations

$$dx = a dt + \sqrt{2D}\, dW_x \qquad dy = b dt + \sqrt{2D}\, dW_y$$

with the initial conditions $x|_0 = x_0$, $y|_0 = y_0$.

Probability of the transition $(x_0, y_0) \rightarrow (x, y)$ defines the function $p(t,x,y|0,x_0,y_0)$ given as follows

$$P(x,y \in A) = \int\limits_A p(t,\xi,\eta|0,x_0,y_0)\; d\xi d\eta$$

Comparing the P-F-K and the vorticity equation, we write

$$\begin{aligned}
&dr = \Big(V_r - \frac{\nu}{r}\Big)\, dt + \sqrt{2\nu}\, dW_r \\
&dz = V_z\, dt + \sqrt{2\nu}\, dW_z \\
&r|_0 = r_0 \qquad z|_0 = z_0
\end{aligned} \tag{2.2}$$

In the above dW_i denotes the increments of the Wiener processes. The idea of the VBM is the following: the system of small vorticity carriers, centers at

(r_k, z_k) of which move according to equations (2.2), constitute the vorticity field.

This field is defined as

$$\omega(t,r,z) = \int p(t,r,z|0,r_0,z_0)\omega_0(r_0,z_0)\, dr_0 dz_0$$

and solves the system (2.1).

Each vortex blob is a small object carrying the vorticity contribution

$$\omega_k = \Gamma_k \chi(|r-r_k|,|z-z_k|)$$

where χ denotes an arbitrary function which vanishes outside a small neighborhood of the point (r_k, z_k). It moves according to the solutions of Eqs (2.2).

The total vorticity is expressed as the sum

$$\omega = \sum_{(k)} \omega_k \tag{2.3}$$

The details of this method were presented in our previous papers, e.g. Styczek (1987), Modrzewska and Styczek (1991), Błażewicz and Styczek (1993), Styczek et al. (1994).

Now, we show the generalisation of this method for an axisymmetric flow. First, we write the velocity field $\vec{V}$ as the sum

$$\vec{V} = \vec{V}_J + \vec{V}_0 + \vec{V}_n + \vec{V}_A \tag{2.4}$$

where $\vec{V}_J$ and $\vec{V}_A$ are potential fields, $\vec{V}_0$ and $\vec{V}_n$ are induced by the blobs already present in the flow domain. We consider two classes of blobs. The first class named "old" was introduced before and still remains under consideration. The second one, i.e. "new" blobs are created at every time step and cancels any violation of the boundary conditions. They appear at choosen points located close to the boundary of the flow region. Since the initial locations of these blobs are known, only their vorticity charges should be found. This and determination of the velocity are essential problems of the method.

Let us express the tangential and normal velocity components calculated on the boundary

$$\begin{aligned} V^t &= V_J^t + V_0^t + V_n^t + V_A^t \\ V^n &= V_J^n + V_0^n + V_n^n + V_A^n \end{aligned} \tag{2.5}$$

Both the boundary values V^t and V^n are given and known. They are equal to zero in the case of rigid, motionless wall. Since $\vec{V}_A$ is a potential field and vanishes at infinity, there exists a linear (integral) operator L, such that (Styczek, 1987)

$$V_A^t = LV_A^n \tag{2.6}$$

Substituting Eq (2.6) into Eqs (2.5) one obtains the boundary equation for V_n^t and V_n^n, both depending on the boundary value of vorticity. This equation allows one to find the boundary value of vorticity - or its discrete approximation given by the new class of blobs. We write the boundary equation in the explicit form

$$V^t - V_J^t - V_0^t + LV_0^n = V_n^t - LV_n^n \tag{2.7}$$

The terms V^n and V_J^n are assumed to be equivalent

$$V^n = V_J^n = \frac{\partial \varPhi_J}{\partial n} = V_0^n \qquad \Delta\varPhi_0 = 0$$

which means that a Neumann problem is to be solved. Eq (2.7) is a singular integral equation. It can be solved in the mean sense.

We integrate Eq (2.7) with respect to the arc-length coordinate over the interval (s_i, s_{i+1}).

It can be easily seen that

$$\int\limits_{s_i}^{s_{i+1}} LV^n\, ds = \int\limits_{s_i}^{s_{i+1}} \frac{\partial \varPhi}{\partial s}\, ds = \delta_i^{i+1}\varPhi$$

Making use of this expression we write

$$\int\limits_{s_i}^{s_{i+1}} (V^t - V_0^t)\, ds - \delta_i^{i+1}\varPhi_J + \delta_i^{i+1}\varPhi_0 = \sum_{(k)} \Gamma_k \Big\{ \int\limits_{s_i}^{s_{i+1}} T(s,k)\, ds - \delta_i^{i+1}\varPhi(k) \Big\} \tag{2.8}$$

The potential $\varPhi_0$ is a result of the term V_0^n, i.e. solution of

$$\frac{\partial \varPhi_0}{\partial n} = V_0^n \qquad \Delta\varPhi_0 = 0$$

and $T(s,k)$ denotes the tangential component of velocity induced by a blob located at the point (r_k, z_k). Similarly, the potential $\varPhi(k)$ denotes the harmonic function corresponding to the normal component given by the same blob

$$\frac{\partial \varPhi(k)}{\partial n} = N(s,k) \qquad \Delta\varPhi(k) = 0$$

The symbol $N(s,k)$ represents the normal component induced by kth new blob. Clearly, its charge is unitary, since its charge Γ_k has been incorporated into the system (2.8).

Note that the system (2.8) is a set of linear algebraic equations for $\{\Gamma_k\}$. It is required that the number of new blobs and the integration intervals be equal. It also means that new blobs are located "over" the corresponding segments of the integration.

Summing up, two problems have to be solved. The first problem is to construct an axisymmetric vortex blob. The second one is to formulate a solver for the Neumann problem. Having solved both, we will be able to find the new class of blobs at any time displace, then, find a new class of blobs, an so on.

3. Axisymmetric vortex blob

Let ω be a given function. Introducing the velocity components as

$$V_r = -\frac{1}{r}\frac{\partial}{\partial z}(r\Phi_*) \qquad\qquad V_z = \frac{1}{r}\frac{\partial}{\partial r}(r\Phi_*)$$

one obtains the following equation

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \Phi_*}{\partial r}\Big) + \frac{\partial^2 \Phi_*}{\partial z^2} - \frac{\Phi_*}{r^2} = -\omega(9) \tag{3.1}$$

This equation can be solved using the Hankel integral transform H_k, with $k=1$

$$\Phi_* = \int_0^\infty \widehat{\Phi}(z,s) s J_1(rs)\, ds = H_1(\widehat{\Phi})$$

The transform $\widehat{\Phi}$ satisfies an ordinary differential equation

$$\frac{d^2\widehat{\Phi}}{dz^2} - s^2\widehat{\Phi} = -\widehat{\omega} \tag{3.2}$$

where $\widehat{\omega}$ is the transform of ω. The formal solution of Eq (3.2) is

$$\widehat{\Phi} = \int_{-\infty}^{\infty} G(z,\zeta)\widehat{\omega}(s,\zeta)\, d\zeta$$

The fundamental solution of Eq (3.2) is well known

$$G(z,\zeta) = \frac{1}{2s}\begin{cases} \exp[s(z-\zeta)] & \text{for} \quad \zeta \geqslant z \\ \exp[-s(z-\zeta)] & \text{for} \quad \zeta \leqslant z \end{cases}$$

It is convenient to assume that vorticity has the form

$$\dot{\omega}(r,z) = \omega_i(r) z_i(z) \tag{3.3}$$

with

$$\omega_i(r) = \begin{cases} r & \text{for} \quad R_i < r < R_{i+1} \\ 0 & \text{for} \quad r < R_1 \ \cup \ R_{i+1} < r \end{cases}$$

$$z_i(z) = \begin{cases} 1 & \text{for} \quad z_i < z < z_{i+1} \\ 0 & \text{for} \quad z < z_1 \ \cup \ z_{i+1} < z \end{cases}$$

We note that the vorticity ω in the form assumed above is conserved along the trajectory if the liquid is inviscid. This fact is related to the known equation (cf Batchelor, 1967)

$$\frac{d}{dt}\Big(\frac{\omega}{r}\Big) = \frac{d}{dt}(\text{const}) = 0$$

The Hankel transform of $\omega_i(r)$ can be calculated in an elementary way. It is

$$\widehat{\omega}_i = \frac{1}{s}\Big[R_{i+1}^2 J_2(R_{i+1}s) - R_i^2 J_2(R_i s)\Big]$$

and the integral with respect to z can also be simply evaluated. It involves exponential and hyperbolic functions (cf Hedar, 1998). The velocity is regular everywhere and quickly vanishes at infinity. Unfortunately, both formulae for V_r and V_z contain single integrals.

Nonetheless, both integrants (involving the Bessel functions J_1 and J_2) vanish relatively fast while r gets large.

4. Potential boundary problem

It is impossible to find the explicit formulas for potential, unless the shape of the region is known. We will consider the axisymmetric harmonic problem for $z > 0$

$$\Delta\phi = 0 \qquad\qquad \frac{\partial \phi}{\partial z}\Big|_{z=0} = f(r) \tag{4.1}$$

First, we solve this problem for $f(r) = 1$, $r \leqslant a$ and $f(r) = 0$, $r > a$.

Expressing solution of the Laplace equation with the use of the transform $H_0(\Phi)$ we get

$$\Phi = -a \int_0^\infty \frac{1}{s} J_0(rs) J_1(as) \mathrm{e}^{-sz} \, ds \tag{4.2}$$

It is also the solution of potential problem corresponding to $V_z\Big|_{z=0}$ equal to 1 or 0 for $r < a$ or $r > a$, respectively. This means, that Φ_J is given by Eq (4.2).

Differentiating the above formula, we obtain components of $\vec{V}_J$. Now, we are able to solve the problem (4.1).

First, we consider an auxillary problem

$$f(r) = F(r,\zeta) = \begin{cases} 1 & \text{for} \quad \zeta < r < \zeta + \Delta r \\ 0 & \text{for} \quad r < \zeta \ \vee \ \zeta + \Delta r < r \end{cases}$$

The Hankel transform of F is

$$\widehat{F} = \frac{1}{s}\Big[(\zeta + \Delta r) J_1\Big(s(\zeta + \Delta r)\Big) - \zeta J_1(s\zeta)\Big]$$

In this case the transform of an arbitrary function f can be expressed as an infinite sum

$$\widehat{f}(s) = -\frac{1}{s} \sum_{(n)} \zeta_n J_1(\zeta_n s)[f(\zeta_{n+1}) - f(\zeta_n)]$$

or, employing the Stiltjes integral

$$\widehat{f}(s) = -\frac{1}{s} \int_0^\infty \zeta J_1(\zeta s) \, df(\zeta) \tag{4.3}$$

The above formulas allow one to find the potential as

$$\Phi(r,z) = -\int_0^\infty J_0(rs) \widehat{f}(s) \mathrm{e}^{-sz} \, ds \tag{4.4}$$

where $\widehat{f}$ has been found before. Note, that formulas for $\widehat{f}$ and Φ give Φ_J while $f(r)$ was given previously. In this case indeed $df = -\delta(\xi - a)\, d\xi$ and Eq (4.3) yields Φ_J.

5. Implementation

As it was mentioned before, the initial-boundary problem is given by the set of equations (2.1) and the initial-boundary data

$$\vec{V}\Big|_{t=0} = 0 \qquad V_r\Big|_{z=0} = 0 \qquad V_z\Big|_{z=0} = \begin{cases} 1 & \text{if } \ r < a \\ 0 & \text{if } \ r > 0 \end{cases}$$

The Neumann problem solver is adapted to such a geometry. The new blobs are introduced close to the plane $z = 0$. Each of them overlaps the boundary. It can be argued that rectangular cross-section of a blob does not form the best possible shape, but it allows for separation of variables which considerably simplifies the induction formulas.

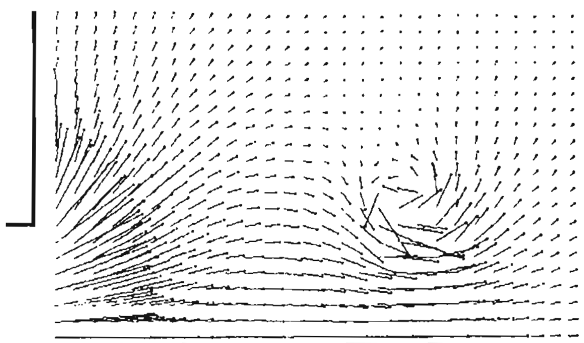

Fig. 1. Instantanous flow, $\mathrm{Re} = 10^5$

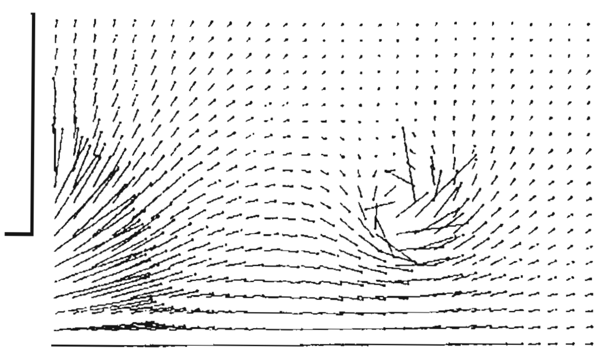

Fig. 2. Instantanous flow, $\mathrm{Re} = 10^4$

An ordinary PC Pentium 200 MHz computer was used. One time-step took approximately 200 s when the number of blobs was close to $2 \cdot 10^4$. The geometry was constant in time and some elements of the code were executed only once. The viscosity used in calculations ensures that the Reynolds number UD/ν is in the range $10^4 \div 10^5$.

This value allows one to compare the axial component of velocity with the empirical one given by formulas from Schlichting (1979). The Fig.3 shows that the agreement is rather qualitative.

One cannot expect that the calculated field should be very reliable since the number of computational elements is fairly low.

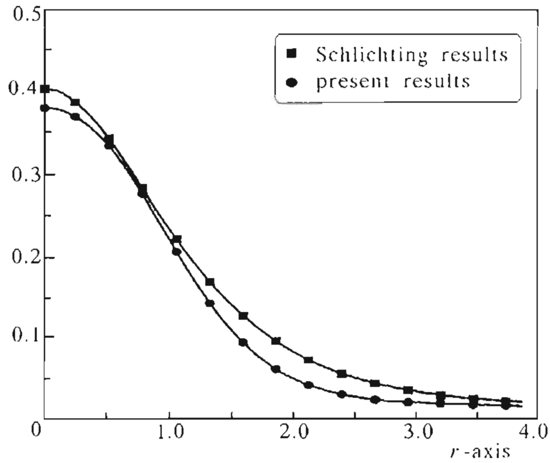

Fig. 3. Axial component of the velocity (mean value) $z = 1.5D$, $\mathrm{Re} = 10^5$

The motion starts instantaneously at $t = 0$. This impulsive "switch-on" causes large and intensive vortex structure advected downstream. Similar phenomenon was observed in the case of plane jet considered before (cf Modrzewska and Styczek, 1991).

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Zastosowanie stochastycznej metody wirowej do symulacji zatopionej osiowosymetrycznej strugi

Streszczenie

W pracy podano rozszerzenie metody "Vortex Blob" na ruch osiowosymetryczny i przedstawiono symulację strugi płynu lepkiego wpływajacego do pólograniczonego obszaru.

Przykład pokazuje możliwości, choć nie jest wyliczony perfekcyjnie.

Manuscript received April 28, 1999; accepted for print June 8, 1999