JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **37**, 1999

SUPERCRITICAL STABILITY AND BIFURCATIONS IN AXIALLY MOVING WEB

KRZYSZTOF MARYNOWSKI

Department of Machines Dynamics, Technical University of Łódź
e-mail: kmarynow@ck-sg.p.lodz.pl

Dynamic stability and bifurcations in axially moving web have been investigated. To analyse supercritical dynamic behaviour of the thin web the beam model is considered. A general velocity proportional damping force is added to the non-linear governing equation. Approximate solution of the partial differential equation of motion is obtained using the Galerkin method. The investigation procedure follows that derived from the Hopf bifurcation theory by Iooss and Joseph and consists in seeking approximate periodic solutions of non-linear equations of the web motion in a parametric form. The moving web may encounter divergent or flutter instability at supercritical transport speeds. The attention is focused on free vibrations in the neighbourhood of some points on the stability boundary in the flutter region of the linearized system. The Hopf bifurcation kind (sub- and supercritical) has been investigated at these points.

Key words: moving web, damping, dynamic stability, bifurcation

Notation

b	–	width of the web
b_e, b_i	–	external and internal damping coefficients, respectively
c	–	axial transport speed
D	–	flexural stiffness of the plate
E	–	Young modulus of the web along x axis
h, l	–	thickness and length of the web, respectively
m	–	number of the natural frequency
M_x, M_y, M_{xy}	–	bending moment resultants
N_x, N_y, N_{xy}	–	in-plane stress resultants
q	–	transverse loading

R	–	axial tension
t	–	time
w	–	transverse displacement of the web middle surface
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	–	strain tensor components for the middle surface of the web
β_1, β_2	–	dimensionless damping coefficients
κ	–	equivalent stiffness of the rolls support structure
ν	–	Poisson ratio of the plate
ρ	–	mass density of the plate
σ	–	real part of eigenvalue
ω	–	natural frequency of the plate (imaginary part of eigenvalue).

1. Introduction

Axially moving materials one can find in industry as band saw blades, power transmission belts, magnetic tapes and paper webs. Excessive vibrations of moving webs increase defects and can lead to failure of the web. The analysis of vibration and dynamic stability in such systems is very important for design of manufacturing devices.

A lot of earlier works in this field were focused on dynamic investigations of string-like and beam-like axially moving systems (e.g. Wickert and Mote, 1988 (rev), 1990). In the case of a two-dimensional axially moving thin web, the exact dynamic solutions satisfying the non-linear coupled equations governing the web motion, probably cannot be determined in a closed form. Recent works have analysed the equilibrium displacement, stress distribution (Lin and Mote, 1995), wrinkling phenomenon (Lin and Mote, 1996), stability of axially moving isotropic plate (Lin, 1997) and dynamic behaviour of axially moving orthotropic plate (Marynowski and Kołakowski, 1999).

The aim of this paper is to analyse the dynamic stability and bifurcations in an axially moving web. The web motion is damped by a general velocity proportional damping force. To analyse the supercritical dynamic behaviour of thin web the beam model is considered. An approximate solution of the governing partial differential equation is obtained using the Galerkin method. The investigation procedure follows that derived from the Hopf bifurcation theory by Iooss and Joseph (1980). It consists in seeking the approximate periodic solutions of non-linear equations of the web motion in a parametric form using the Fredholm alternative. The moving web may encounter divergent or flutter instability at supercritical transport speeds. The attention is focused on free vibrations in the neighbourhood of some points on the stability boundary in the flutter instability region of the linearized system. The Hopf bifurcation character (sub- and supercritical) is investigated at these points.

2. Mathematical model of the moving web system

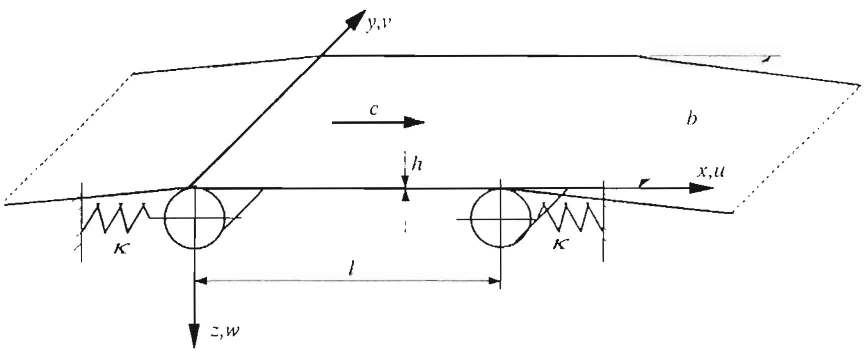

Fig. 1. Model of the axially moving web

A long elastic moving web of the length l is considered. The web moves at constant velocity c. The co-ordinates system and geometry are shown in Fig.1. The equation governing the transverse motion of the two dimensional axially moving plate were derived by Marynowski and Kołakowski (1999) and have the following form

$$\begin{aligned}&\rho h(-w_{,tt} - 2cw_{,xt} - \kappa c^2 w_{,xx}) + M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + q + \\ &+(N_x w_{,x})_{,x} + (N_y w_{,y})_{,y} + (N_{xy} w_{,x})_{,y} + (N_{xy} w_{,y})_{,x} = 0\end{aligned} \tag{2.1}$$

Neglecting the velocity dependet terms in Eq (2.1) and taking into account the linear damping in the transverse direction leads to the results obtained by Tylikowski (1988).

In this paper, a non-linear simplificated form of the governing equation has been taken into consideration. In the case of thin web, the results of earlier investigations show that an 1D beam model approximates accurately the dynamic behaviour of the web. Taking into account the non-linear geometric relation

$$\varepsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2 \tag{2.2}$$

the non-linear component appears in the governing equations (cf Fung et al., 1998 for the string)

$$(N_x w_{,x})_{,x} = \frac{3Eh}{2(1-\nu^2)} w_{,x}^2 w_{,xx} \tag{2.3}$$

Hereinafter, one assumes that the web is subject to tension only in its longitudinal direction, hence

$$N_x \neq 0 \qquad q = 0 \qquad N_y = N_{xy} = 0 \tag{2.4}$$

Then the non-linear equation governing transverse motion of the axially moving beam model is

$$\rho h(-w_{,tt} - 2cw_{,xt} - \kappa c^2 w_{,xx}) + Rw_{,xx} - Dw_{,xxxx} + \frac{3Eh}{2(1-\nu^2)} w_{,x}^2 w_{,xx} = 0 \tag{2.5}$$

where the flexural stiffness of the beam is equal to the flexural stiffness of the isotropic plate.

A general velocity proportional damping force of the form (Ulsoy and Mote, 1982; Marynowski, 1997) $b_1 w_{,t} + b_2 c w_{,x}$ where $b_1 = b_e + b_i$ and $b_2 = b_e$ has been introduced into the left-hand side of the governing equation (2.5). Finally, the mathematical model of the moving web system has the following form

$$\begin{aligned} &\rho h(w_{,tt} + 2cw_{,xt} + \kappa c^2 w_{,xx}) - Rw_{,xx} + Dw_{,xxxx} + b_1 w_{,t} + \\ &+ b_2 c w_{,x} + \frac{3Eh}{2(1-\nu^2)} w_{,x}^2 w_{,xx} = 0 \end{aligned} \tag{2.6}$$

The boundary conditions

$$w(0,t) = w(l,t) = 0 \qquad w_{,xx}(0,t) = w_{,xx}(l,t) = 0 \tag{2.7}$$

Let the dimensionless parameters be

$$\begin{array}{llll} z = \frac{w}{h} & \xi = \frac{x}{l} & c_1 = \sqrt{\frac{R}{\rho h}} & s = \frac{c}{c_1} \\ \tau = \frac{tc_1}{l} & \varepsilon = \frac{D}{Rl^2} & \beta_1 = \frac{b_1 l}{\sqrt{R\rho h}} & \beta_2 = \frac{b_2 l s}{\sqrt{R\rho h}} \end{array} \tag{2.8}$$

Substitution of Eq (2.8) into Eq (2.6) gives the dimensionless equation of motion

$$z_{,\tau\tau} + 2sz_{,\xi\tau} + (\kappa s^2 - 1)z_{,\xi\xi} + \varepsilon z_{,\xi\xi\xi\xi} + \beta_1 z_{,\tau} + \beta_2 z_{,\xi} + \frac{1}{8}\varepsilon z_{,\xi}^2 z_{,\xi\xi} = 0 \tag{2.9}$$

The eigenvalue problem represented by Eq (2.9) together with the appropriate boundary conditions Eq (2.7) has been solved using the Galerkin method. The following finite series representation of the dimensionless transverse displacement has been assumed

$$z_i(\xi,\tau) = \sum_{i=1}^{m} \sin(i\pi\xi) q_i(\tau) \tag{2.10}$$

For example, for $m = 3$, Eq (2.9) is reduced to the following second order ordinary differential equations

$$\begin{aligned}
\ddot{q}_1(\tau) &= (\kappa s^2 - 1)\pi^2 q_1(\tau) - \varepsilon\pi^4 q_1(\tau) + (16/3)s\dot{q}_2(\tau) - \beta_1\dot{q}_1(\tau) + \\
&+ (8/3)\beta_2 q_2(\tau) - 0.045\varepsilon q_2^2(\tau) q_1(\tau) \\
\ddot{q}_2(\tau) &= 4(\kappa s^2 - 1)\pi^2 q_2(\tau) - 16\varepsilon\pi^4 q_2(\tau) - (16/3)s\dot{q}_1(\tau) + (48/5)s\dot{q}_3(\tau) + \\
&- \beta_1\dot{q}_1(\tau) - (8/3)\beta_2 q_1(\tau) + (24/5)\beta_2 q_3(\tau) - 0.18\varepsilon q_1^2(\tau) q_2(\tau) + \\
&+ 0.648\varepsilon q_1(\tau) q_2(\tau) q_3(\tau) - 0.584\varepsilon q_2(\tau) q_3^2(\tau) \qquad (2.11) \\
\ddot{q}_3(\tau) &= 9(\kappa s^2 - 1)\pi^2 q_3(\tau) - 81\varepsilon\pi^4 q_3(\tau) + (48/5)s\dot{q}_2(\tau) - \beta_1\dot{q}_3(\tau) + \\
&+ (24/5)\beta_2 q_2(\tau) - 5.252\varepsilon q_2^2(\tau) q_3(\tau)
\end{aligned}$$

3. Numerical results and discussion

Numerical simulations have been carried out for a thin steel web. The following parameter values have been taken: $l = 1\,\mathrm{m}$, $b = 0.2\,\mathrm{m}$, $h = 1.5\,\mathrm{mm}$, $\rho = 7800\,\mathrm{kg/m^3}$, $E = 0.2\cdot 10^{12}\,\mathrm{N/m^2}$, $R = 2.5\cdot 10^3\,\mathrm{N/m}$, $\nu = 0.3$, $\varepsilon = 0.025$, $c_1 = 14.618\,\mathrm{m/s}$.

First, the stability of the linearized system was investigated. Only first, linear terms in Eq (2.9) have been taken into consideration. The complex eigenvalue problem of the set of ordinary differential equations has been solved using the iterative method (Press et al., 1989).

To test accuracy of the computational method the absolute values of lowest imaginary parts of eigenvalues of undamped moving steel web have been calculated ($m = 2$, $m = 3$) and compared with the exact solution for $J = 10$ rectangular plate segments of the web (Marynowski and Kołakowski, 1999). The calculation results are shown in Fig.2. For $m = 3$ the discrepancy in the

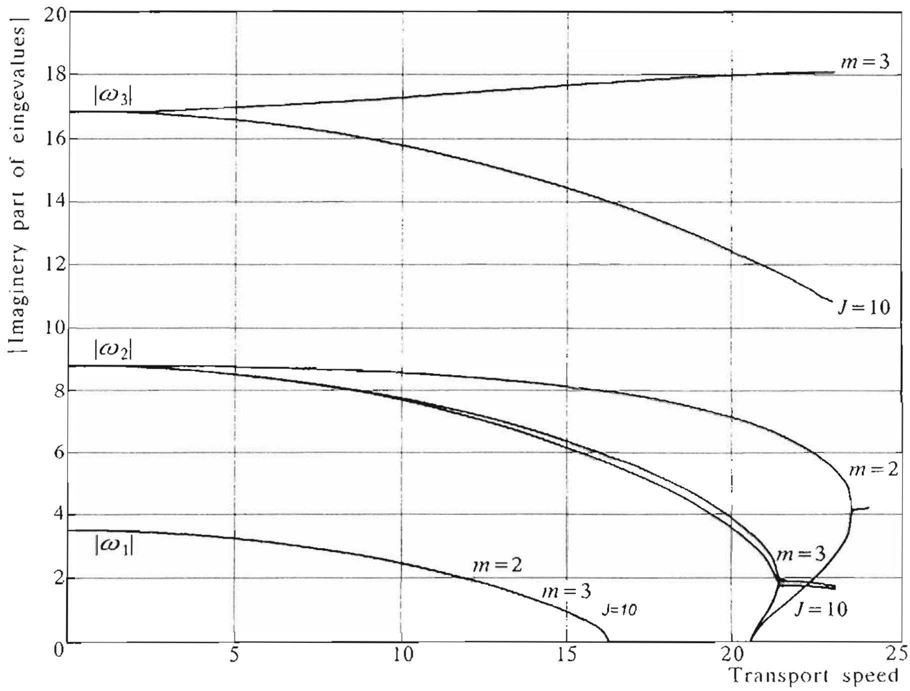

Fig. 2. Lowest eingevalues of the undamped web; $\kappa = 1$, $\beta_1 = 0$

first eigenvalue of the beam model is less than 0.5% within the second eigenvalue is less than 2%. Thus, three approximating functions in the Galerkin procedure have been introduced into simulations.

The plots of the first two eigenfrequencies of the damped web model versus the transport velocity are shown in Fig.3 and Fig.4. In all these cases the external damping was taken into account $(b_1 = b_2,\ \beta_2 = \beta_1 s)$ and the roll stiffness coefficient $\kappa = 1$.

For the system with small damping (Fig.3a) the axial velocity c decreases absolute values of the first two imaginary eigenvalues until the first eigenvalue vanishes at the critical value c_{crI}. Then, the positive real part of the first eigenvalue appears, i.e. the divergence type of instability. Dynamic behaviour of the linearized system was investigated above the critical transport speed. At supercritical transport speeds, the web experiences first the divergent instability (the fundamental mode with non-zero σ and zero ω) and next the flutter instability (non-zero σ and non-zero ω). Between them there is a second stable region where $\sigma = 0$. The flutter instability appears at the second critical transport speed c_{crII} (Fig.3a). For higher transport speeds above the

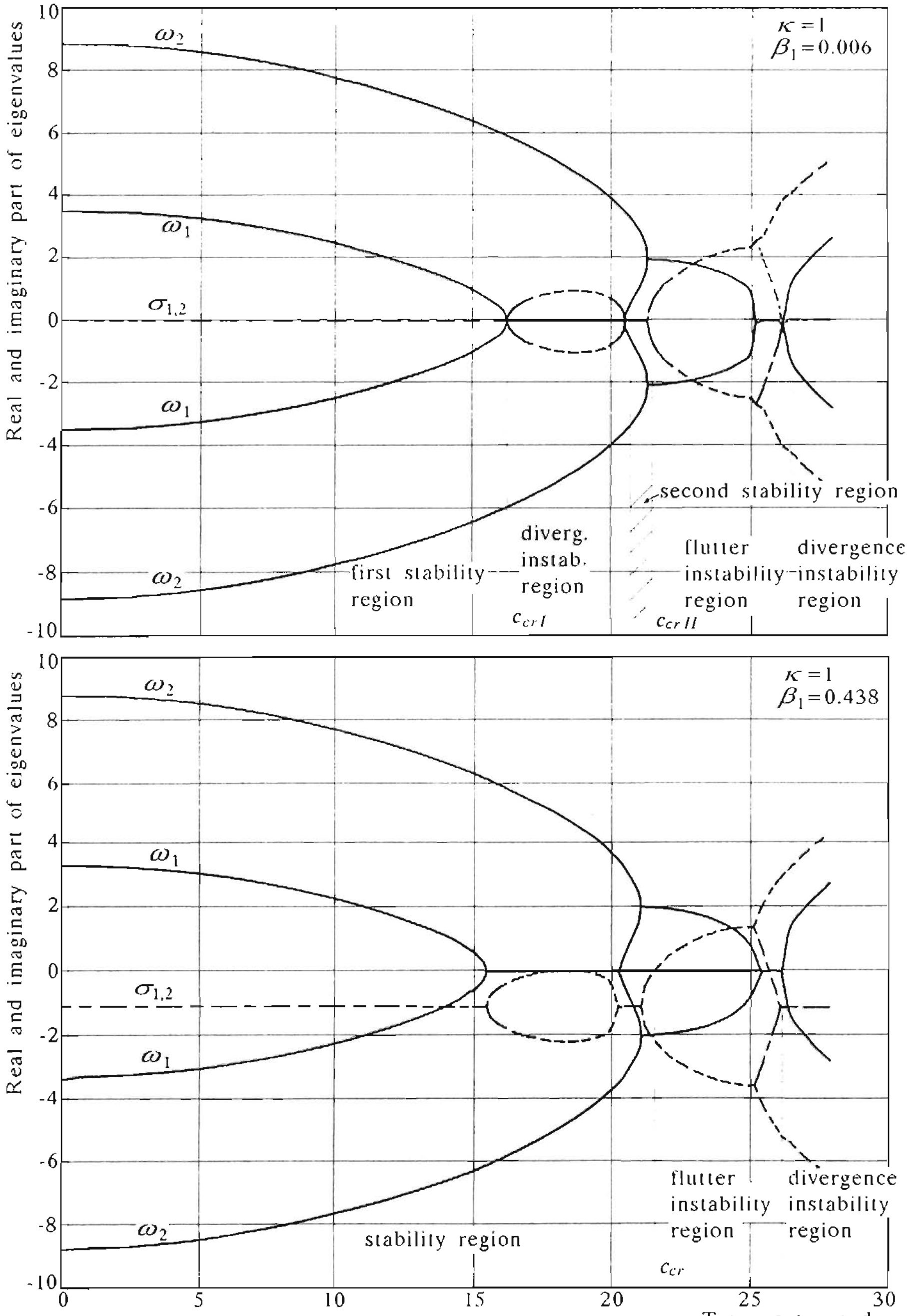

Fig. 3. Two lowest eingevalues of the web

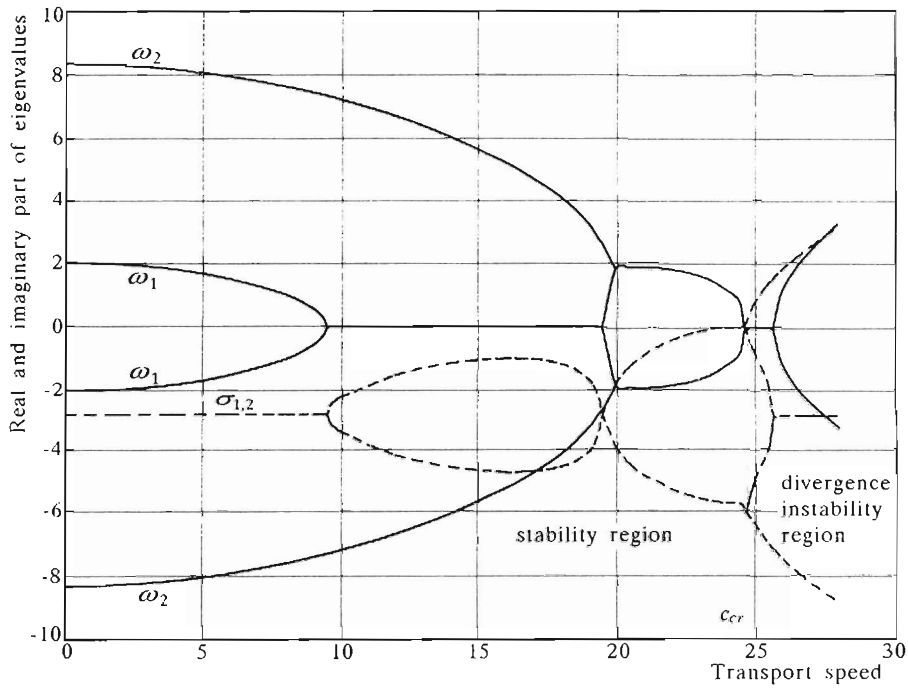

Fig. 4. Two lowest eingevalues of the web; $\kappa = 1$, $\beta_1 = 1.14$

flutter region only the divergence instability of the web motion is observed. It was shown earlier (Marynowski and Kołakowski, 1999) that appearance of the second stability region for the undamped system depends on the slenderness ratio and orthotropy factor of the web.

As the external damping of the web motion increases the width of the first divergence region diminishes. For $\beta_1 = 0.438$ the first divergence region vanishes (Fig.3b). At the critical transport speed c_{cr} one of the real parts of conjugate complex eigenvalues passes through zero i.e., the flutter instability of the web motion appears. Further increasing of damping causes vanishing of the flutter instability region. For $\beta_1 = 1.14$ above the stability region only the divergence instability of the web motion is observed (Fig.4).

Numerical calculations have been made for different values of the rolls support stiffness κ. For $\kappa = 1$, the two rolls are rigidly fixed with respect to each other. For $\kappa = 0$ the rolls can move relative to each other when tension varies. Decreasing of the rolls support stiffness changes positions of both the divergence and flutter instability regions. The plots of dimensionless external damping coefficient β_1 versus the critical transport speed c_{cr} for different values of the rolls support stiffness are shown in Fig.5.

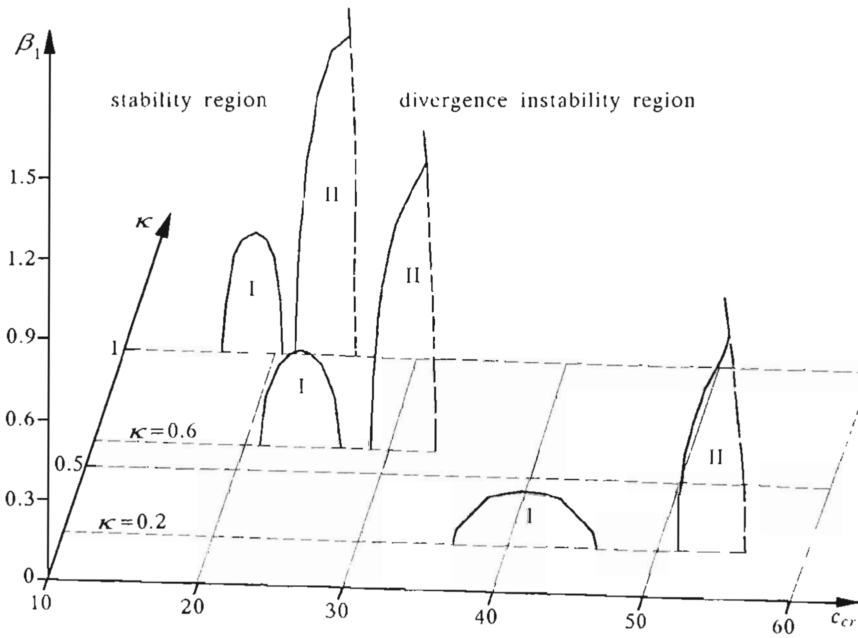

Fig. 5. Positions of the instability regions; I – divergence instability region, II – flutter instability region

Next, the dynamic behaviour of the non-linear web system was studied so all terms in Eq (2.9) have been taken into consideration. The investigation procedure follows that derived from the Hopf bifurcation theory by Iooss and Joseph (1980). The attention has focused on free vibrations in the neighbourhood of some points on the stability boundary in the flutter instability region of the linearized system. The kind of the Hopf bifurcation (sub- or supercritical) of the moving web has been investigated.

Sample investigation results have been presented for a constant value of the rolls support stiffness and for different values of external damping. The plots of the estimated radius of the bifurcation solution in the neighbourhood of the flutter instability threshold versus the transport speed are shown in Fig.6 for $\kappa = 1$ and $\kappa = 0.6$, respectively. In the case of rigid rolls support $\kappa = 1$, for the damping coefficient range $0 \leqslant \beta_1 \leqslant 0.61$ the subcritical Hopf bifurcation with the unstable limit cycle can be observed. Beyond this range the region of supercritical Hopf bifurcation with the stable limit cycle occurs. In the case of more flexible rolls support $\kappa = 0.6$, the subcritical Hopf bifurcation with the

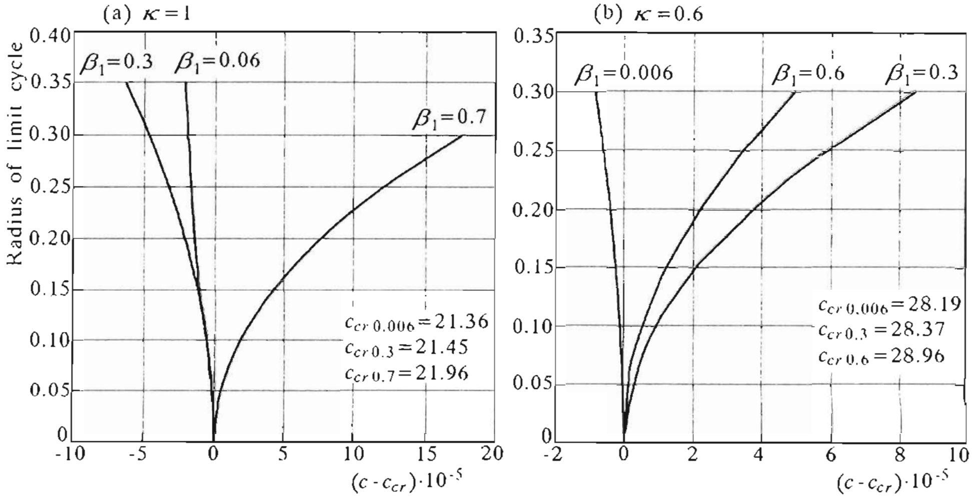

Fig. 6. Bifurcation solutions

unstable limit cycle can be observed within the damping range $0 \leqslant \beta_1 \leqslant 0.06$. Investigation results show that the type of bifurcation depends on the rolls support stiffness and damping of the web motion. Fig.7 shows the boundary line between subcritical and supercritical Hopf bifurcations for different values of stiffness and damping coefficients.

4. Conclusions

Dynamic stability and bifurcations in axially moving web have been investigated. To analyse the supercritical dynamic behaviour of thin web the beam model with a general velocity proportional damping force has been taken into consideration. Geometric nonlinearity has been introduced the non-linear governing equation.

The dynamic analysis of the linearized system shows that the moving web may encounter divergent or flutter instability at supercritical transport speeds. For small damping, when the transport speed increases the web experiences first the divergent instability and next the flutter instability. Between them, there is a second stable region. For higher transport speeds above the flutter region only the divergence instability of the web motion is observed.

The critical value of transport speed depends on the rolls stiffness and

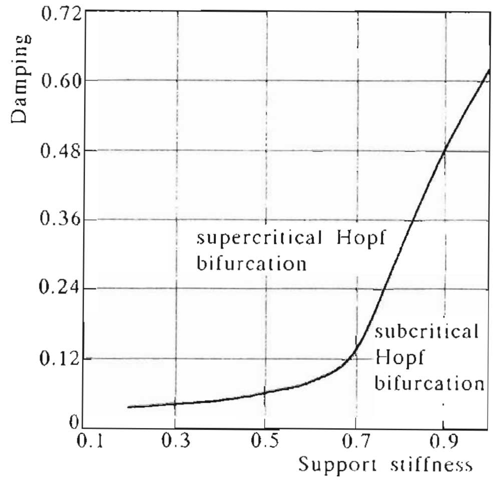

Fig. 7. Sub- and supercritical Hopf bifurcation regions

damping of the web motion. When the external damping of the web motion increases the width of the first divergence instability region more and more diminishes until vanishes. Further damping increasing causes vanishing of the flutter instability region and above the stability region of transport speeds only the divergence instability of the web motion is observed.

Dynamic analysis of non-linear system shows that both subcritical and supercritical Hopf bifurcations appear in the region of flutter instability of the web motion. The bifurcation kind depends mainly on the rolls support stiffness and external damping of the web motion. The rolls stiffness decreasing diminishes the damping coefficient at which the sub- and supercritical Hopf bifurcation threshold appears.

It is worth noting that because of the existence of unstable limit cycle neglecting of nonlinear components in the governing equation especially in the regions of subcritical Hopf bifurcation may yield incorrect results of dynamic investigations.

Acknowledgement

This paper was supported by the State Committee for Scientific Research (KBN) under grant No. 7 T08E 028 12.

References

1. FUNG R.F., HUANG J.S., CHEN Y.C., YAO C.M., 1998, Nonlinear Dynamic Analysis of the Viscoelastic String with a Harmonically Varying Transport Speed, *Computers and Structures*, **66**, 6, 777-784
2. IOOSS G., JOSEPH D.D., 1980, *Elementary Stability and Bifurcation Theory*, Springer-Verlang, Berlin
3. LIN C.C., 1997, Stability and Vibration Characteristics of Axially Moving Plates, *Int. Journal Solid Structures*, **34**, 24, 3179-3190
4. LIN C.C., MOTE C.D. JR., 1995, Equilibrium Displacement and Stress Distribution in a Two Dimensional, Axially Moving Web under Transverse Loading, *Journal of Applied Mechanics ASME*, **62**, 772-779
5. LIN C.C., MOTE C.D. JR., 1996, The Wrinkling of Thin, Flat, Rectangular Webs, *Journal of Applied Mechanics ASME*, **63**, 774-779
6. MARYNOWSKI K., 1997, Stability Analysis of Axially Moving System by Using a Rotor's Model, *Mechanics and Mechanical Engineering*, **1**, 2, 131-136
7. MARYNOWSKI K., KOŁAKOWSKI Z., 1999, Dynamic Behaviour of an Axially Moving Thin Orthotropic Plate, *Journal of Theoretical and Applied Mechanics*, **37**, 1, 109-128
8. PRESS W.H., FLANNERY B.P., TEUKOLSKY S.A., VATTERLING W.T., 1989, *Numerical Recipes in Pascal*, Cambrdge Univ. Press
9. TYLIKOWSKI A., 1988, Dynamic Stability of Antisymmetrically Laminated Cross-Ply Cylindrical Shells, *Journal of Theoretical and Applied Mechanics*, **26**, 4, 655-661
10. ULSOY A.G., MOTE C.D. JR., 1982, Vibration of Qide Band Saw Blades, *Journal of Engineering for Industry ASME*, **104**, 1, 71-78
11. WICKERT J.A., MOTE C.D. JR., 1988, Current Research on the Vibration and Stability of Axially Moving Materials, *Shock and Vibration Digest*, **20**, 3-13
12. WICKERT J.A., MOTE C.D. JR., 1990, Classical Vibration Analysis of Axially-Moving Continua, *Journal of Applied Mechanics ASME*, **57**, 738-744

Stateczność nadkrytyczna i bifurkacje w poruszającej się osiowo wstędze

Streszczenie

W pracy badano stateczność dynamiczną i bifurkacje w poruszającej się osiowo wstędze. Do analizy zachowania dynamicznego w obszarze nadkrytycznych prędkości przesuwu wykorzystano model belkowy, w którym uwzględniono nieliniowość geometryczną oraz siłę tłumienia proporcionalną do uogólnionej prędkości wstęgi. Przybliżone rozwiązanie równania ruchu o pochodnych cząstkowych otrzymano stosując metodę Galerkina. Zastosowano metodę badawczą wyprowadzoną z teorii bifurkacji Hopfa przez Iooss'a i Joseph'a. Polega ona na poszukiwaniu przybliżonych okresowych rozwiązań nieliniowych równań ruchu w postaci parametrycznej. W obszarze nadkrytycznym może występować niestateczność dywergencyjna oraz niestateczność typu flater. Skoncentrowano się na badaniach drgań własnych w pobliżu granicy stateczności typu flater. W tych punktach badano charakter bifurkacji (pod- i nadkrytyczny).

Manuscript received October 28, 1998; accepted for print January 15, 1999