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VIBRATIONS AND STABILITY OF COLUMNS SUBJECTED TO A CERTAIN TYPE OF GENERALISED LOAD

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A classification of loading of cantilever columns regarding the way of load application has been established. Natural vibration frequency against the applied load and stability of two columns under the generalised load of the first kind have been investigated numerically. The influence of some geometrical parameters of the loading head as well as the concentrated mass fixed at the end of column upon both the eigenvalue curves and the divergence critical load were analysed. Some numerical results of eigenfrequency have been confirmed by an experiment performed on two constructed stands.

Key words: elastic column, divergence instability, natural frequency, generalised load

1. Introduction

The problem of stability of a cantilever column under the load due to which both the transverse force and the bending moment at the loaded end of column depend upon the displacement and deflection angle of this end (Fig.1a) was described by Kordas (1963). Those relation are

$$
\begin{aligned}
&EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} + Pe = 0\\
&EJ\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} + P\delta - m\frac{\partial^2 W(x,t)}{\partial t^2}\Big|_{x=l} = 0
\end{aligned}
\tag{1.1}
$$

where

$$e = \rho \frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \nu W(l,t) \qquad\qquad \delta = \mu \frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \gamma W(l,t)$$

and

ρ, ν, μ, γ	–	known coefficients
P	–	compressive force
$W(x,t)$	–	transverse displacement
m	–	concentrated mass
EJ	–	flexural rigidity
l	–	length of a column.

Under conditions (1.1) rotary inertia of the mass m has been neglected, because in the investigated range of the mass, its adequate inertial terms only slightly affected the calculated values of natural frequency.

The boundary conditions at the clamped end of a cantilever column are

$$W(0,t) = \frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{1.2}$$

The load is conservative when rotation of its vector field is equal to zero, what gives the relation (cf Gajewski and Życzkowski, 1970; Tomski et al., 1996)

$$\nu + \mu - 1 = 0 \tag{1.3}$$

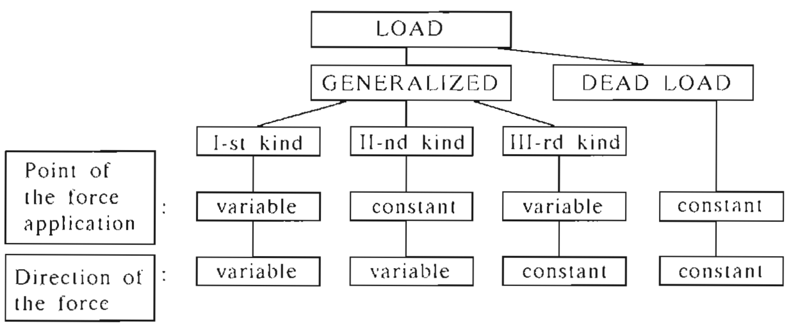

Fig. 1. Classification of generalised load with respect to the way of application of the loading

Taking into account the two features characterising the force P from Eqs (1.1), i.e. its direction which is determined in a plane and its point of application with regard to the column end, the classification of loading of columns can be established as follows (see Fig.1).

1.1. Generalised load of the first kind

In this case a column is under a compressive force, direction of which during column natural vibrations depends both on the displacement and deflection angle of the column end. A point of force application changes also with respect to the end of column – Fig.2a. Possible constructions of columns under this kind of load are depicted in Fig.2b,c,d.

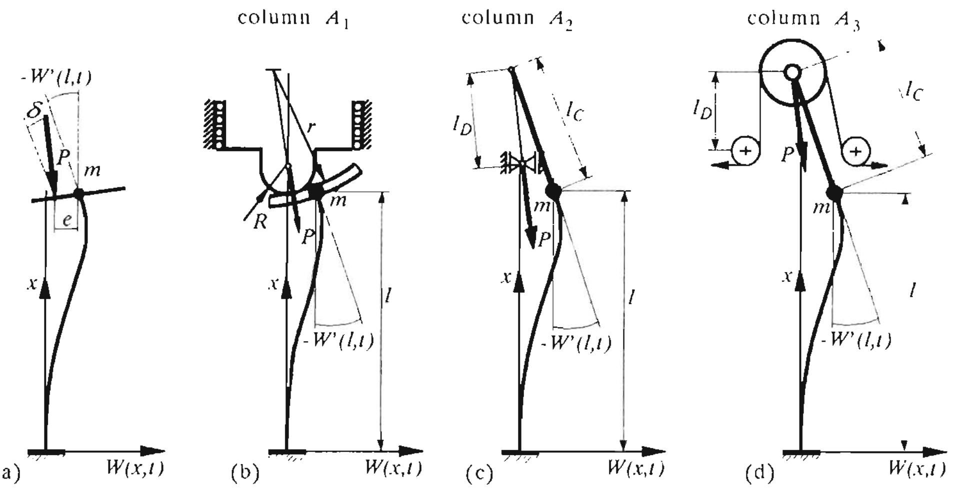

Fig. 2. Cantilever columns under the generalised load of the first kind, (a) scheme of the column and its load, (b), (c), (d) possible constructions

A particular case of this type of load is the follower load directed towards a pole (cf Timoshenko and Gere, 1961; Levinson, 1966). In this case the line of action of a compressive force passes through a pole, and depends both upon the displacement and deflection angle of the end of column. Between this displacement and the deflection angle the following relation holds $W(l,t) = \pm cW'(l,t)$, where c is the coefficient of proportionality. The distance of a pole from the end of column is called "positive" when the pole lies below the column end, and "negative" when it is placed over this end (cf Gajewski and Życzkowski, 1969). The vibrations and stability of a column loaded towards a "positive" pole were studied by Tomski et al. (1998), outwards of a "negative" pole is the subject of this work. A scheme of the column under considerations and its possible constructional variants are depicted in Fig.3.

According to the classification implemented here, the generalised load described by Gajewski and Życzkowski (1970), (1988), Kordas (1963) and Tomski et al. (1996) are proposed to be called the generalised load of the first kind. A

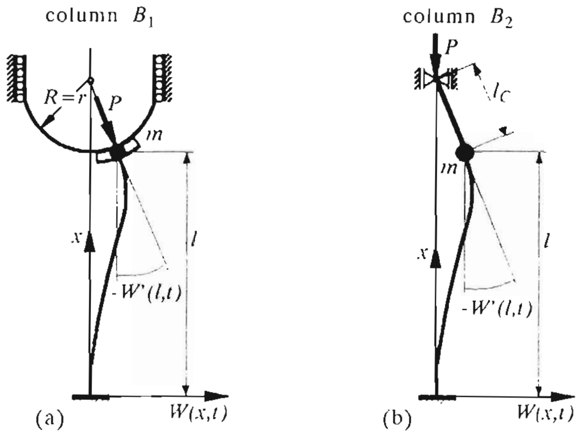

Fig. 3. Constructions of a cantilever column under the follower load directed outwards a "negative" pole

real construction of column under this kind of load was studied by Tomski et al. (1996) and a plane frame by Tomski et al. (1995).

1.2. Generalised load of the second kind

A compressive force has got a constant point of application, but the angle of its line of action changes (Fig.4a,b). Here the following subgroups can be specified:

1.2.1. Generalised Beck's column (cf Beck, 1953; Kordas and Życzkowski, 1963; Ziegler, 1968)

In this case a force has got a constant point of action, but its direction follows the deflection angle of the column end – Fig.4a. As it was specified by Kordas and Życzkowski (1963), the load can be anti-tangential for $\eta < 0$, sub-tangential $0 < \eta < 1$, tangential $\eta = 1$, and super-tangential for $\eta > 1$, where η is the coefficient of proportionality.

1.2.2. Force of direction passing through a fixed point

A force changes direction passing towards a fixed point (a pole) which is placed on the axis of an undeformed column (Fig.4b).

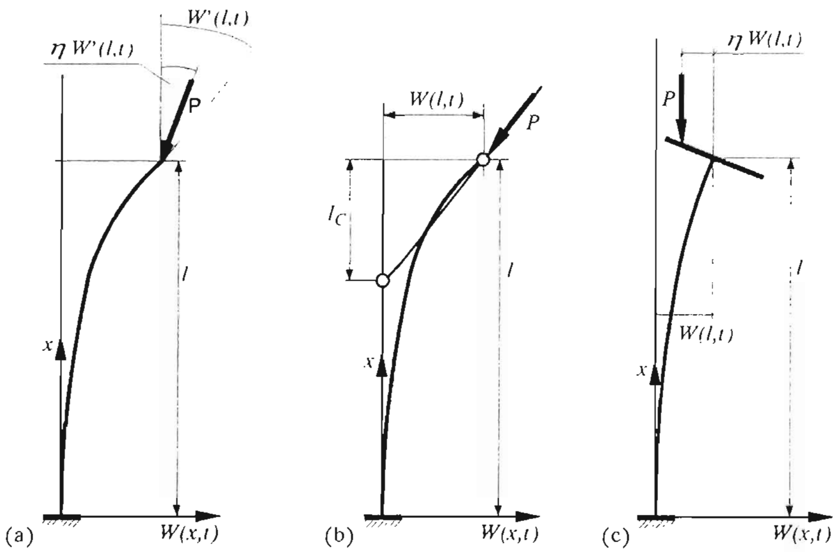

Fig. 4. Cantilever columns under (a), (b) the generalised load of the second kind, (a) Beck's column, (b) column loaded by a force passing through a fixed point, (c) the generalised load of the third kind – Reut's column

1.3. Generalised load of the third kind

A compressive force has got a constant direction, but altered point of its application. For Reut's column (cf Nemat-Nasser and Herrmann, 1966; Plaut, 1972) a load of constant direction changes its position proportionally to the displacement of the end of column – Fig.4c.

A few constructions of systems under the load of the second and third kind were presented by Gajewski and Życzkowski (1970), (1988). It should be noticed that the generalised load can be conservative or nonconservative (see Eq (1.3)), but this problem is not studied in this work. A very thorough classification of systems with respect to the form of instability and existence of a potential function was presented by Argyris and Symeonidis (1981).

In the case of "dead load" a force acting on a column remains constant both in direction and point of application so the problem reduces to the Euler's one, which has been solved by using the static approach.

2. Formulation of the problem

In this work the stability and vibrations of elastic and prismatic columns presented in Fig.2d and Fig.3b are examined. The columns are composed of two identical rods of flexural rigidity $E_1J_1 = E_2J_2 = EJ$ and mass per unit length $\kappa_1 = \kappa_2 = \kappa$. Due to constructional demands the two rods have been used.

The Bernoulli-Euler equation for the transverse displacement is as follows

$$EJ\frac{\partial^4 W_i(x,t)}{\partial x^4} + P_i\frac{\partial^2 W_i(x,t)}{\partial x^2} + \kappa\frac{\partial^2 W_i(x,t)}{\partial t^2} = 0 \tag{2.1}$$

where $P_i = P/2$, $i = 1,2$.

For small vibrations

$$W_i(x,t) = y_i(x)e^{j\omega t} \tag{2.2}$$

The values of parameters ρ, ν, μ and γ from Eqs (1.1) for columns A_1 and A_2, A_3 are given in Table 1.

Table 1. Values of parameters ρ, ν, μ and γ for columns A_1 and A_2, A_3

	ρ	ν	μ	γ
A_1	$\frac{rR}{r-R}$	$\frac{r}{r-R}$	$-\frac{R}{r-R}$	$-\frac{1}{r-R}$
A_2, A_3	$\frac{l_C(l_C-l_D)}{l_D}$	$\frac{l_C}{l_D}$	$-\frac{l_C-l_D}{l_D}$	$-\frac{1}{l_D}$

Those columns are adequate to each other regarding the relations $l_d = r - R$, $l_C = r$, and are under the generalised load of the first kind. The values of radii r and R as well as the lengths l_C and l_D, describing the way of columns loading have got the reverse signs of values than those of columns investigated by Tomski et al. (1996).

The rigidity of element l_C from Fig.2c,d is much greater than that of both rods $E_CJ_C \ll 2EJ$, so it is assumed that this element is infinitely rigid. Using such an approximation the boundary conditions for columns A_1 and A_2, A_3 are as follows

$$\begin{aligned} &EJ[y_1''(l) + y_2''(l)] + P[\rho y_1'(l) + \nu y_1(l)] = 0 \\ &EJ[y_1'''(l) + y_2'''(l)] + P[\mu y_1'(l) + \gamma y_1(l)] + m\omega^2 y_1(l) = 0 \end{aligned} \tag{2.3}$$

where

$$y_1(l) = y_2(l) \qquad y_1'(l) = y_2'(l) \tag{2.4}$$

The load applied to columns B_1 and B_2 from Fig.3 is a particular case of the load of columns A_1 and A_2, A_3 from Fig.2, respectively.

The boundary conditions (2.3) after substituting the values of parameters ρ, ν, μ and γ for column A_1 can be rewritten as follows

$$\begin{aligned} &(r-R)EJ[y_1''(l)+y_2''(l)]+Pr[Ry'(l)+y(l)]=0 \\ &r(r-R)EJ[y_1'''(l)+y_2'''(l)]-Pr[Ry'(l)+y(l)]+r(r-R)m\omega^2 y_1(l)=0 \end{aligned} \tag{2.5}$$

Subtracting Eq $(2.5)_1$ from Eq $(2.5)_2$, and adding Eqs (2.5) one obtains

$$\begin{aligned} &y(l)=-ry'(l) \\ &EJ[y_1'''(l)+y_2'''(l)]+\frac{l}{r}[y''(l)+y_2''(l)]+m\omega^2 y_1(l)=0 \end{aligned} \tag{2.6}$$

The load of columns A_1, A_2, A_3, B_1, B_2 is conservative, because in each case the condition (2.1) is fulfilled.

The boundary condition $(2.6)_2$ for a column loaded by a follower force passing through a fixed point was derived by Tomski et al. (1998) on the basis of the Hamilton principle.

3. Experimental set-up. Construction of columns

A stand for vibration tests of column A_3 from Table 1 is presented in Fig.5. The column is composed of the two bars (1) and (2) clamped by the holder (3) set to the plate (4). In this way the conditions $y_i(0)=y_i'(0)=0$, $i=1,2$ are fulfilled. Free ends of the rods (1) and (2) are connected by means of the block (5) which ensures boundary the conditions (2.4) to be fulfilled. To the block (5) the rigid element (6) is mounted. One end of the element (6) has got the shape of fork (7) in which the pulley (8) is seated with the use of rolling bearing. A pair of the loading systems (9) is symmetrically placed with respect to the holder (3) on the plate (4). Each of the loading systems generates the loading force equal to $P/2$ by means of the dynamometers (10). On the plate (4) the two towers (11) are placed symmetrically also. These towers can be shifted horizontally. In the towers (11) the two pulleys (12_1) and $(12-2)$ are mounted. The pulleys (13) mounted on the beam (14) can be moved vertically to change the length l_C. The column is loaded by using the rope (15). The value of the loading force is measured by the dynamometers (10).

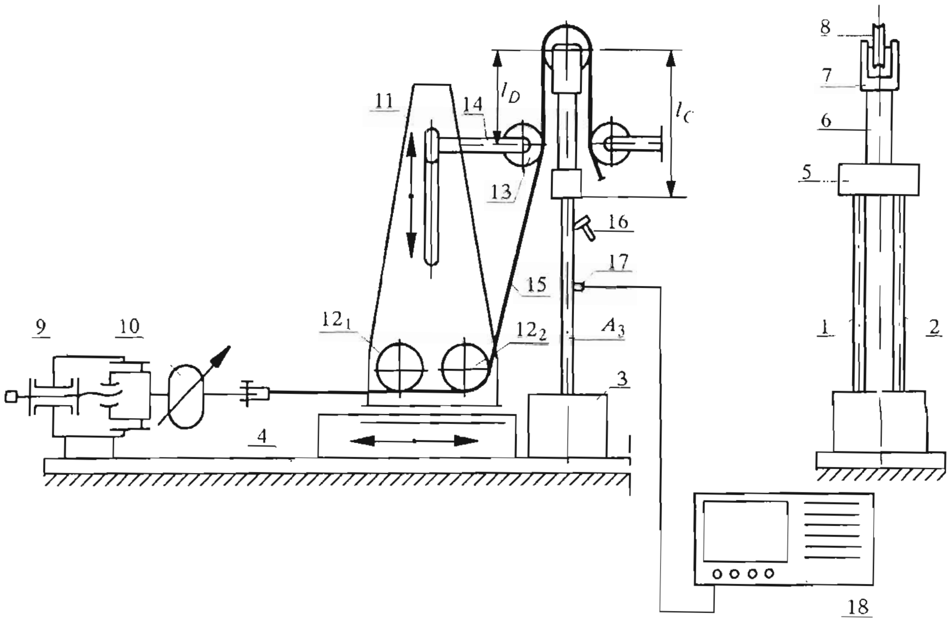

Fig. 5. Experimental set-up for frequency measuring of the A-type column

Another stand was used for experiments conducted for column B_2 which was loaded by a force directed outwards a "negative" pole – Fig.6. The main difference between both stands results from the way of column loading. Here the load to the column was transferred by a rolling bearing instead of a system of pulleys and ropes. That stand was thoroughly described by Tomski et al. (1998).

The experimental investigations into natural frequencies were conducted with the aid of the 2115 Bruell and Kjaer vibration analyser.

4. Experimental results for columns A_3 and B_2

Geometrical and physical data of columns A_3 and B_2 are included in Table 2. Two variants of each column were taken for the experiment and numerical investigation. The solutions of boundary value problems described by Eqs (2.1) and the boundary conditions (2.3) and (2.6) were obtained by using Mathematica 2.0 and Mathcad 5.0 Plus programs.

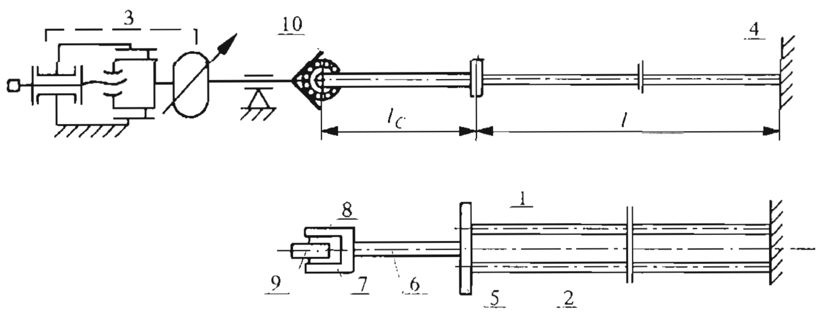

Fig. 6. Experimental set-up for frequency measuring of the B-type column

Table 2. Geometrical and physical data of columns A_3 and B_2

	$\kappa_1+\kappa_2$ [kg/m]	$E_1J_1+E_2J_2$ [Nm^2]	m [kg]	l [m]	l_C [m]	l_D [m]
$A_3^{(1)}$	0.438	73.63	1.438	0.435	0.285	0.145
$A_3^{(2)}$	0.438	73.63	1.438	0.435	0.285	0.265
$B_2^{(1)}$	0.858	282.86	0.350	0.640	0.130	0
$B_2^{(2)}$	0.858	282.86	0.266	0.725	0.045	0

In Fig.7 experimental and numerical results for columns $A_3^{(1)}$ and $A_3^{(2)}$ are shown. The results concern the first three natural frequencies f for the external load P. Natural frequency curves marked f_i^S concern vibrations with symmetrical modes which are characteristic for the columns composed of two identical rods. Those modes are independent of the values of lengths l_C and l_D so the adequate frequency curves overlap each other for columns $A_3^{(1)}$ and $A_3^{(2)}$ of different lengths l_D.

In Fig.8 the curves $P-f$ for columns $B_2^{(1)}$ and $B_2^{(2)}$ are presented.

Theoretical and experimental results obtained for the first three frequencies are of good accuracy; the maximal difference between the theoretical and experimental values was equal to 10%.

5. Numerical results of critical force and natural frequencies for the column of type A

For the considered systems depicted in Fig.2 (column A_1) the transcen-

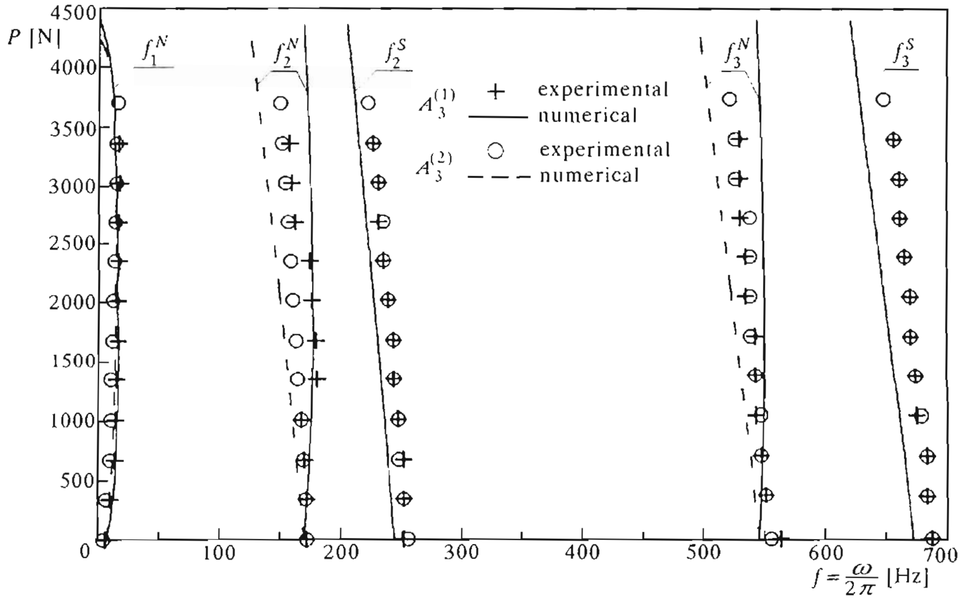

Fig. 7. Eigenvalue curves for A-type column (numerical and experimental results)

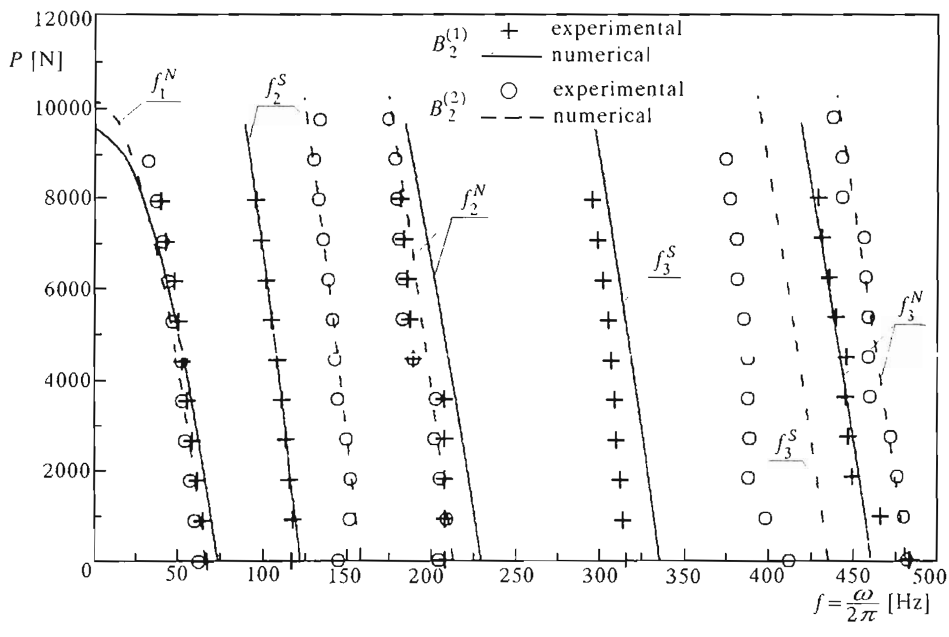

Fig. 8. Eigenvalue curves for B-type column (numerical and experimental results)

dental equation for the critical force is as follows

$$\frac{\cos(l\sqrt{\lambda})[-l\sqrt{\lambda}+r\sqrt{\lambda}-R\sqrt{\lambda}]-\sin(l\sqrt{\lambda})[1+lr\lambda+rR\lambda]}{r-R}=0 \tag{5.1}$$

where

$$\lambda=\frac{P}{E_1J_1+E_2J_2}$$

In figures that follow the dimensionless quantities are introduced

$$R^*=\frac{R}{l} \qquad r^*=\frac{r}{l} \qquad \lambda^*=\lambda l^2 \qquad m^*=\frac{m}{2\kappa l} \tag{5.2}$$

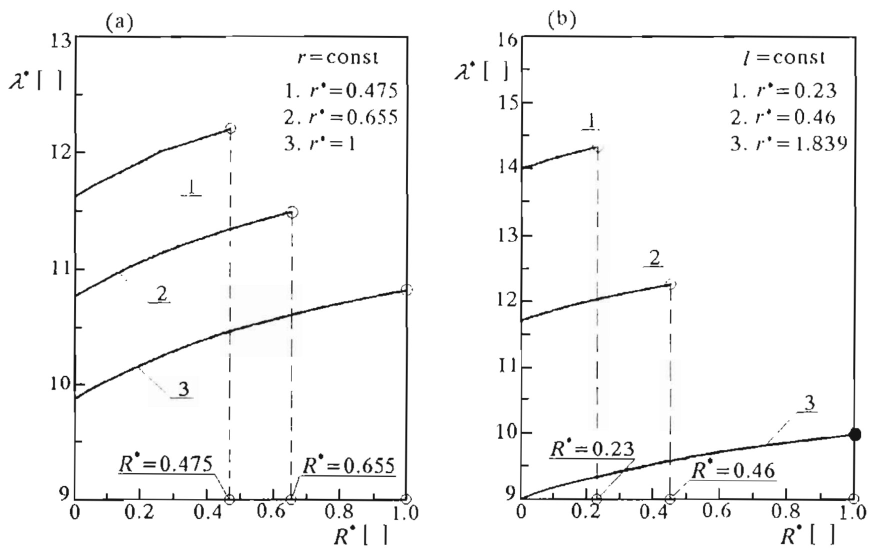

Fig. 9. Change of the first critical load λ^* for A-type column as a function of parameter R^* for: (a) $r^*=$ const, (b) $l=$ const

In Fig.9 the changes in the critical force λ^* as a function of parameter R^* for $r^*=$ const (Fig.9a), and for $l=$ const (Fig.9b) are depicted, respectively. Due to the fact that $r^*>R^*$ the curves are not presented in the whole range of parameter R^*.

The natural vibration frequency curves obtained from numerical calculations are plotted for the dimensionless loading parameter λ^* and dimensionless frequency Ω^*, where

$$\Omega^*=\frac{\rho_1A_1\omega^2l^4}{E_1J_1} \tag{5.3}$$

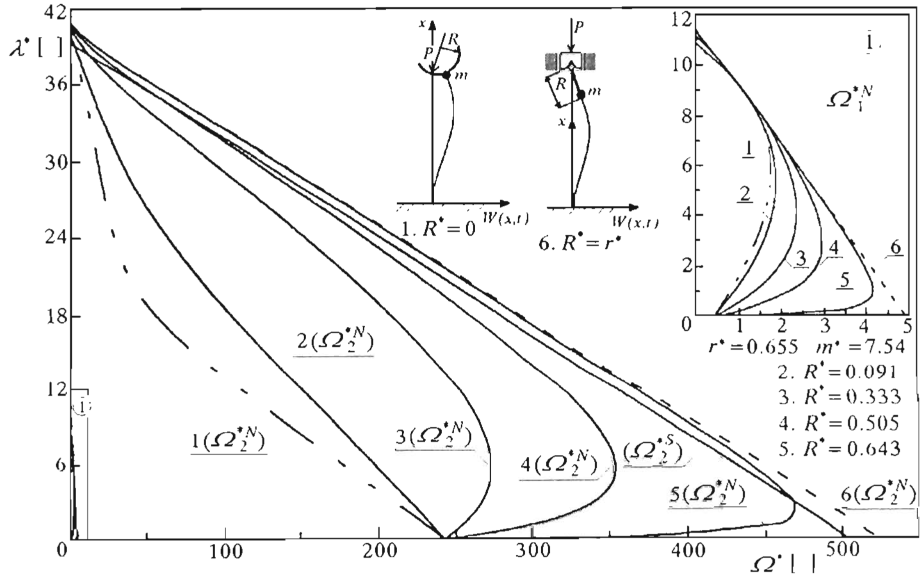

Fig. 10. Eigenvalue curves for different values of parameter R^* for the A-type column

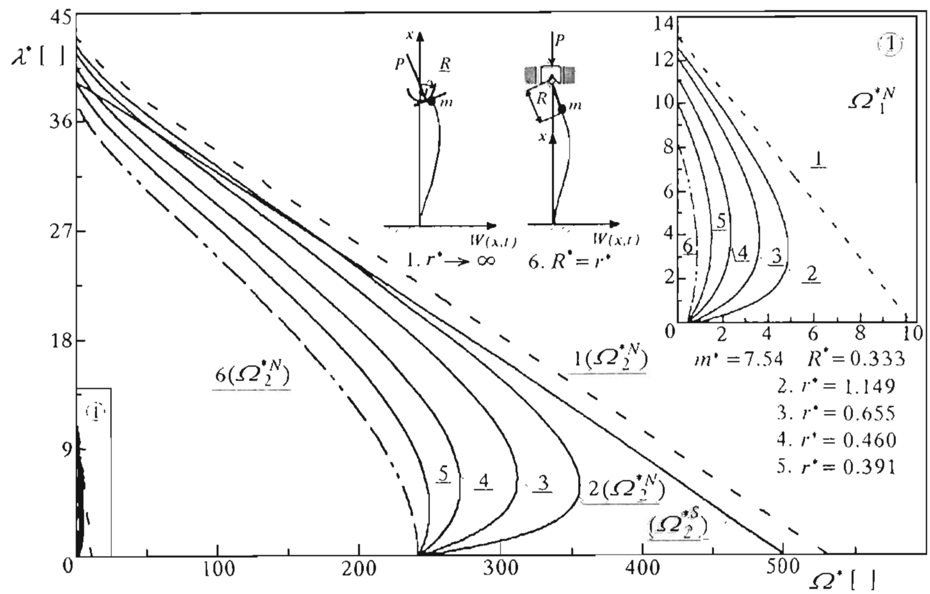

Fig. 11. Eigenvalue curves for different values of parameter r^* for the A-type column

Fig.10 provides an opportunity to study the first two eigencurves for a few values of parameter R^*. The range of natural frequencies is limited by curves (1) and (6) drawn for the boundary values of R^*, i.e. for $R^* = 0$ and $R^* = r^*$, respectively. Implemented notation of the natural frequencies describes:

- Ω_1^{*N}, Ω_2^{*N} the first and second natural frequencies, respectively, adequate to the first mode without a node, and the second mode with one node, respectively. These modes are characteristic for single rod columns,
- Ω_2^{*S} an additional natural frequency related to the symmetrical mode which is characteristic for systems the composed of two identical rods. The values of frequency in this case are independent of r^* and R^*, as well as of the value of concentrated mass m placed at the free end of the column.

The influence of parameter r^* on the course of eigencurves Ω^* for constant values of both the concentrated mass m^* and parameter R^*, is presented in Fig.11. The range of eigenvalues is limited by curves (1) and (6) which are drawn with broken lines, for $r^* \to \infty$ and $R^* = r^*$, respectively. The value of the critical load parameter for the symmetrical modes of a column (curve Ω_2^{*S}) is independent of the considered geometrical and physical parameters r^*, R^*, m^* and is constant.

The influence of the concentrated mass m^* for given R^* and r^* is shown in Fig.12. Computation has approved that for the same geometry of the system, the concentrated mass cannot influence the value of the critical load, changing only the courses of both the first and second eigencurves related to asymmetrical modes. The course of the natural frequency curve Ω_2^{*S} is not affected by the value of the concentrated mass for the reasons presented above.

6. Numerical results of critical force and natural frequencies for the column of type B

For the system presented in Fig.2b (column A_1), the influence of geometrical and physical parameters on the natural frequency and the critical load has been discussed when the relation $r^* > R^*$ holds. For $R^* = r^*$ one obtains the boundary course of eigenvalues depicted in Fig.10 and Fig.11. Such a condition leads to the constructional variant of column B_2 from Fig.3b for which

$$R = r = l_C \tag{6.1}$$

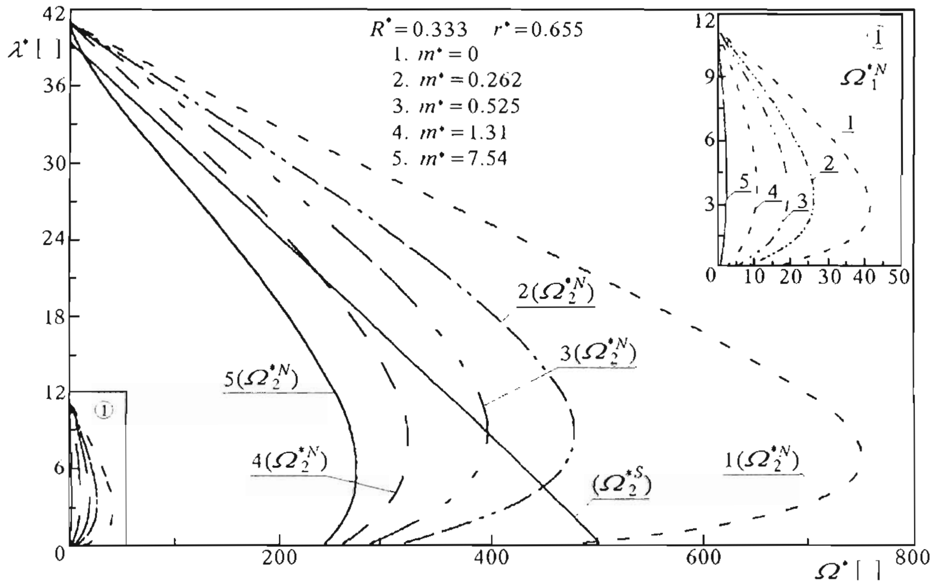

Fig. 12. Influence of the concentrated mass m^* on the eigencurves for the A-type column

For that column the boundary value problem represented by Eqs (2.1) with boundary conditions (1.2), (2.4) and (2.6), leads to the transcendental equation for the critical load in the form

$$l\sqrt{\lambda}\cos(l\sqrt{\lambda}) - \sin(l\sqrt{\lambda})[1 - l_C^2\lambda - l_C l\lambda] = 0 \tag{6.2}$$

Fig.13 illustrates the change of the first critical load λ^* as a function of dimensionless parameter for c^* ($c^* = l_C/l$).

Fig.14 gives the variation of two first natural vibrations Ω^* against the load parameter λ^* for a chosen length c^* and constant value of m^*. Curves (1) and (6) concern the courses for $c^* = 0$ and $c^* \to \infty$, respectively.

7. Conclusions

Taking into account the two features characterising load of a cantilever column, i.e. its direction and its point of application with regard to the end of column, three kinds of the generalised load have been determined.

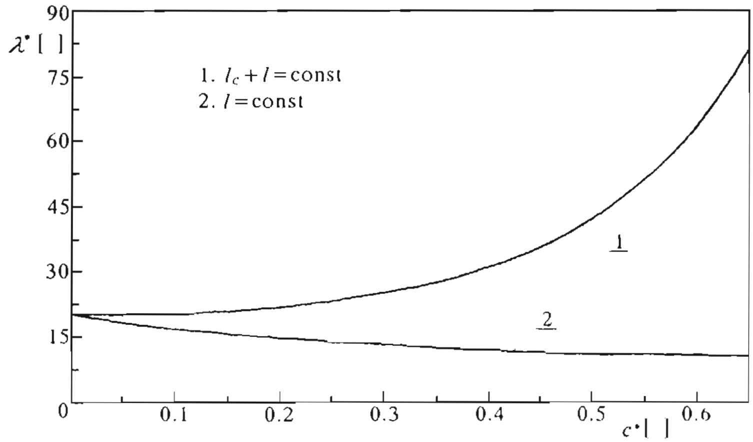

Fig. 13. Change of the first critical load λ^* for the B-type column as a function of parameter c^* for $l_C + l$ = const, and for l = const

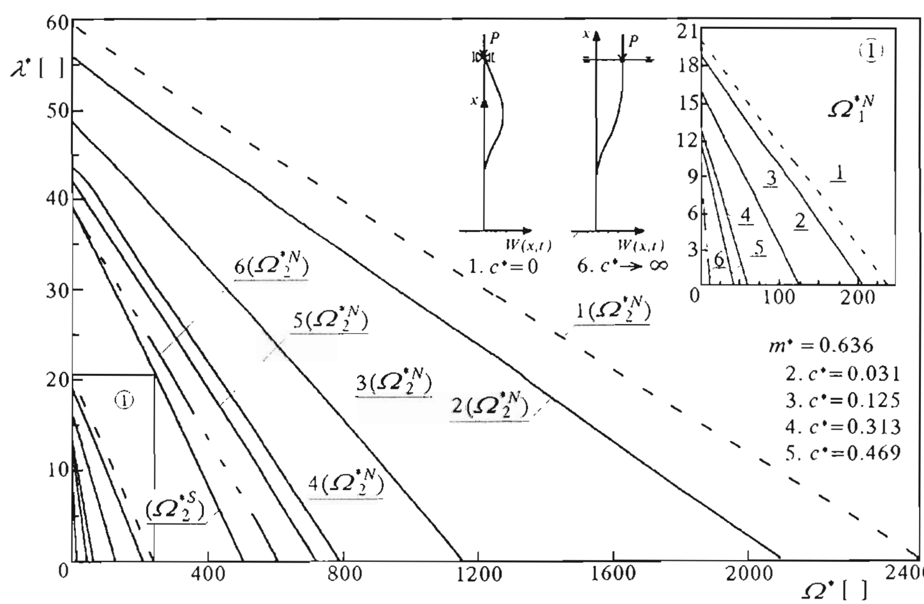

Fig. 14. Eigenvalue curves for different values of parameter c^* for the B-type column

Investigated columns being under the generalised load of the first kind belong to conservative systems. For the A-type column both the shearing force and the bending moment at the end of a column depend on the deflexion and the deflection angle at this point. For the B-type column the shearing force and the bending moment depend additionally on each other.

The obtained numerical results of eigenfrequency curves for two types of columns were in a good agreement with with those of an experiment.

It has been proved that the divergence critical load as well as the course of eigencurves depend on the geometry of loading head in relation to the length of a column.

Acknowledgement

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Stateczność i drgania kolumny poddanej pewnemu typowi obciążenia uogólnionego

Streszczenie

W pracy podano klasyfikację obciążeń kolumn wspornikowych z uwagi na sposób przyłożenia obciążenia. Zbadano numerycznie przebiegi częstości drgań własnych kolumn i ich niestateczność dywergencyjną przy obciążeniach uogólnionych pierwszego rodzaju. Analiza dotyczyła wpływu parametrów wynikających z geometrii głowic obciążających, jak również zmiany masy skupionej na krzywe częstości drgań i siłę krytyczną. Część wyników numerycznych dotyczących częstości drgań własnych potwierdzono eksperymentalnie na dwóch skonstruowanych stanowiskach badawczych.

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