JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **37**, 1999

VIBRATIONS OF A PLATE IN FLUID AND ASSOCIATED DAMPING DUE TO ENERGY TRANSMISSION BY DILATATIONAL WAVES[1]

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The paper presents a solution to the problem of harmonic vibrations of a plate submerged in an unbounded medium of inviscid compressible fluid. The solution is obtained, as a limiting case, by means of a solution to the problem of an infinite elliptic cylinder vibrating in the fluid. The latter problem is solved with the help of the Fourier method of separation of variables in the elliptic coordinate system. For comparison purposes, a similar problem of circular cylinder vibrating in the fluid is also investigated. From the discussion presented it follows, that the fluid compressibility is essential in estimating hydrodynamical forces, especially in calculating damping of plate vibrations for higher frequencies.

Key words: vibrations, fluid, damping, silatational waves

1. Introduction

In offshore engineering we usually deal with the problem of dynamical interaction of structures and fluids. In many cases, in theoretical descriptions of such problems, it is justified to neglect the small fluid compressibility. On the other hand, there are also cases when the compressibility of the fluid must be taken into account. Examples of the latter cases are sea piers, breakwaters and elements of offshore structures loaded with impact pressure forces resulting from water waves. For practical and theoretical reasons, vibrations

[1]A part of this work was presented at the XIII-th Conf. on Fluid Mechanics held at the Technical University of Częstochowa, 1998.

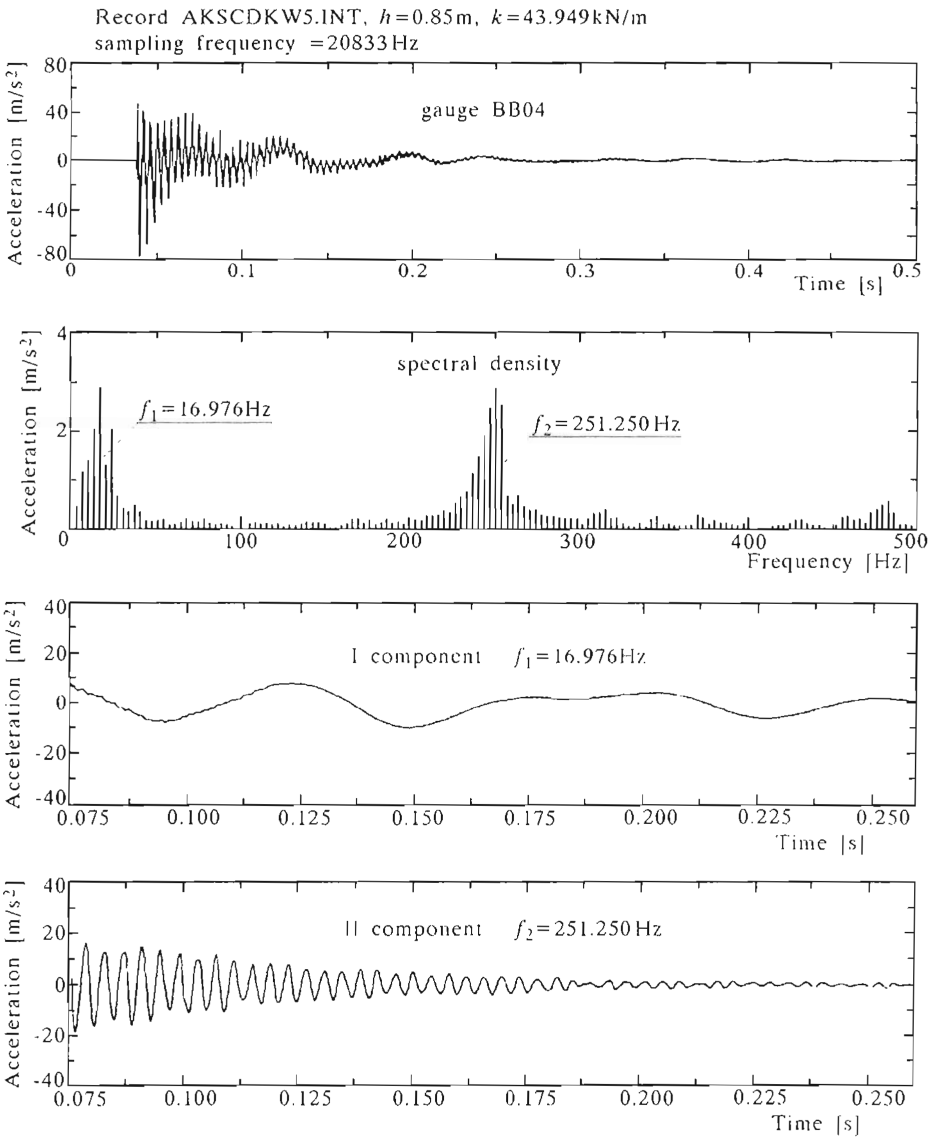

Fig. 1. Constant force induced vibrations of the floating plate

of an elastic plate submerged in the fluid seems to be one of the fundamental problems in this field. In order to learn more about the phenomenon, experimental investigations of such a plate under impact the hydrodynamical forces were carried out at the Institute of Hydro-engineering of PAS in Gdańsk. The experiments consisted in investigations into dynamic behaviour of a horizontal plate suspended elastically in a hydraulic flume an loaded with water wave forces. Among the experiments, there were also cases of the plate having a contact with the surface of calm water (floating plate) forced to move by external forces suddenly applied to its upper surface. In particular, the plate was loaded with a constant force (the Heaviside function of time) or, with an impulsive force of short duration. The plate model was equipped with pressure, acceleration and displacements gauges together with a recording unit. The data obtained in experiments has the form of a sequence of numbers corresponding to the sampling frequency of electronic devices used in the experiments. The experimental records were then processed with the help of the Kalman filter method which allowed for decomposition of vibrations into the components corresponding to the dominant frequencies of the system mentioned. Typical results obtained in this way are shown in Fig.1, where the acceleration record of the plate together with the spectral density plot are given. The case shown in the figure corresponds to a constant force applied to the plate at a certain moment of time. The experiments performed reveal the importance of the fluid compressibility as well as the flexibility of the structure in proper estimation of the dynamical interaction of the plate – fluid system. From theoretical point of view, a very important task is to find a solution to the problem of the floating plate vibrating with an assumed frequency. The problem considered is a classical one involving coupling between the plate and fluid. The solution to the problem in question is obtained indirectly. In the first step, a solution to the problem of an elliptic cylinder vibrating in the fluid is constructed. Then, in the second step, by making a limit in the results derived, the desired solution for the plate is obtained. For comparison purposes, an analytical solution to the problem of infinite circular cylinder vibrating in the fluid is also constructed. We confine our considerations to the plane problem of steady state vibrations of the cylinder cross-section in the fluid. The equations derived allow for calculating the added mass of fluid and damping of vibrations associated with energy radiation by means of the outgoing dilatational waves.

2. Harmonic vibrations of a circular cylinder in a fluid

We consider the plane problem of harmonic vibrations of a rigid circle immersed in a compressible fluid. The geometry and the assumed coordinate systems of the considered problem are shown schematically in Fig.2.

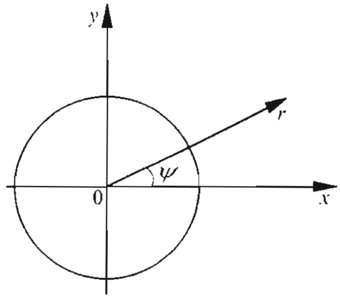

Fig. 2. Rigid circle with the coordinate systems

The mutual relation between the Cartesian and polar coordinate systems is

$$x = r\cos\psi \qquad y = r\sin\psi \qquad 0 \leqslant r \leqslant \infty \qquad o \leqslant \psi \leqslant 2\pi \tag{2.1}$$

The relevant unit base vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\psi$ of the polar coordinate system are expressed in terms of the Cartesian base $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as follows

$$\begin{aligned} \boldsymbol{e}_r &= \boldsymbol{e}_1\cos\psi + \boldsymbol{e}_2\sin\psi \\ \boldsymbol{e}_\psi &= -\boldsymbol{e}_1\sin\psi + \boldsymbol{e}_2\cos\psi \end{aligned} \tag{2.2}$$

It is well known (cf Lamb, 1975), that, for small disturbances, the fluid motion starting from rest may be described by means of the velocity potential $\Phi(x, y, t)$ satisfying the wave equation

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \tag{2.3}$$

and the appropriate initial and boundary conditions. The boundary condition imposed on the cylinder surface is that the normal components of the cylinder and fluid velocities must be equal. For the discussed case of the steady-state harmonic vibrations, the solution should satisfy the Sommerfeld condition at infinity. In Eq (2.3) c is the velocity of sound in the fluid and ∇^2 is the Laplace operator. In the discussed case, the wave equation in polar coordinates assumes the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \tag{2.4}$$

The classical method of separation of variables in the velocity potential

$$\Phi(r,\theta,t) = R(r)\Psi(\theta)T(t) \tag{2.5}$$

yields the ordinary differential equations

$$\begin{aligned} &\frac{d^2T}{dt^2} + \omega^2 T = 0 \qquad \Rightarrow \qquad T(t) = A\mathrm{e}^{\pm \mathrm{i}\omega t} \\ &\frac{d^2\Psi}{dt^2} + m^2\Psi = 0 \qquad \Rightarrow \qquad \Psi(\theta) = B\mathrm{e}^{\pm \mathrm{i}\theta t} \\ &\frac{d^2R}{dr^2} + \frac{1}{r}\Big(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\Big)R = 0 \end{aligned} \tag{2.6}$$

The third equation in (2.6) is the classical Bessel equation. Its solution is expressed in the form of a linear combination of the Bessel or Hankel functions (cf Morse and Feshbach, 1958, 1960). Let the circular cylinder vibrates in the vertical direction according to the formula

$$\boldsymbol{w}(t) = w_0 \mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{e}_2 \tag{2.7}$$

The relevant velocity and acceleration are described by the equations

$$\begin{aligned} \boldsymbol{v}(t) &= \frac{dw}{dt} = -\mathrm{i}\omega w_0 \mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{e}_2 \\ \boldsymbol{a}(t) &= \frac{dv}{dt} = -\omega^2 w_0 \mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{e}_2 \end{aligned} \tag{2.8}$$

With the help of the latter relations we can define the normal component of the cylinder velocity

$$v_n = -\mathrm{i}w_0\omega \mathrm{e}^{-\mathrm{i}\omega t} \tag{2.9}$$

For the discussed case of outgoing dilatational waves, one has to substitute $m = 1$ in Eqs (2.6). The procedure leads to the following formula for the velocity field potential

$$\Phi(r,\theta,t) = AH_1^{(1)}\Big(\frac{\omega r}{c}\Big)\sin\theta \mathrm{e}^{-\mathrm{i}\omega t} \qquad\qquad H_1^{(1)} = J_1 + \mathrm{i}Y_1 \tag{2.10}$$

where: J_1 and Y_1 are the first order Bessel functions of the first and second kinds, respectively and $H_1^{(1)}$ is the Hankel function of the first kind and first order.

The component of velocity field normal to the cylinder surface $r = a$ is

$$v_n = \frac{\partial \Phi}{\partial r}\Big|_{r=a} = \frac{A\omega}{c}\Big[H_0^{(1)}\Big(\frac{\omega a}{c}\Big) - \frac{c}{\omega a}\Big)H_1^{(1)}\Big(\frac{\omega a}{c}\Big)\Big] \sin\theta e^{-i\omega t} \tag{2.11}$$

On the cylinder surface, the normal components of the cylinder and fluid are the same, and thus

$$A = \frac{-icw_0}{H_0^{(1)}(\alpha) - \frac{1}{\alpha}H_1^{(1)}(\alpha)} \tag{2.12}$$

where $\alpha = \omega a/c$ is a dimensionless variable.

Having the velocity potential

$$\Phi = \frac{-icw_0 H_1^{(1)}\left(\frac{\omega r}{c}\right)\sin\theta}{H_0^{(1)}(\alpha) - \frac{1}{\alpha}H_1^{(1)}(\alpha)} e^{-i\omega t} \tag{2.13}$$

it is a simple task to calculate of the fluid pressure on the cylinder surface

$$p = -\rho\frac{\partial \Phi}{\partial t} = \frac{-\rho\omega c w_0 H_1^{(1)}(\alpha)\sin\theta}{H_0^{(1)}(\alpha) - \frac{1}{\alpha}H_1^{(1)}(\alpha)} e^{-i\omega t} \tag{2.14}$$

and finally, the differential of the normal force acting on the cylinder surface

$$d\boldsymbol{F} = \frac{\rho\omega c w_0 H_1^{(1)}(\alpha)\sin\theta}{H_0^{(1)}(\alpha) - \frac{1}{\alpha}H_1^{(1)}(\alpha)} e^{-i\omega t}\boldsymbol{e}_r a\, d\theta \tag{2.15}$$

Integration of the equation with respect to θ yields the vertical resultant of the fluid pressure

$$P_\nu = \rho\pi a^2 w_0 \omega^2 \aleph(\alpha) \qquad\qquad \aleph(\alpha) = \frac{H_1^{(1)}(\alpha)}{-\alpha H_0^{(1)}(\alpha) + H_1^{(1)}(\alpha)} \tag{2.16}$$

where $\aleph$ is the complex number

$$\aleph = F_M + iF_T \tag{2.17}$$

Knowing the acceleration and velocity of the cylinder, Eqs (2.8), we may calculate the corresponding hydrodynamical force applied to the cylinder

$$R_c = -P_\nu = \rho\pi a^2 F_M \ddot{w}(t) + \rho\pi a^2 \omega F_T \dot{w}(t) \tag{2.18}$$

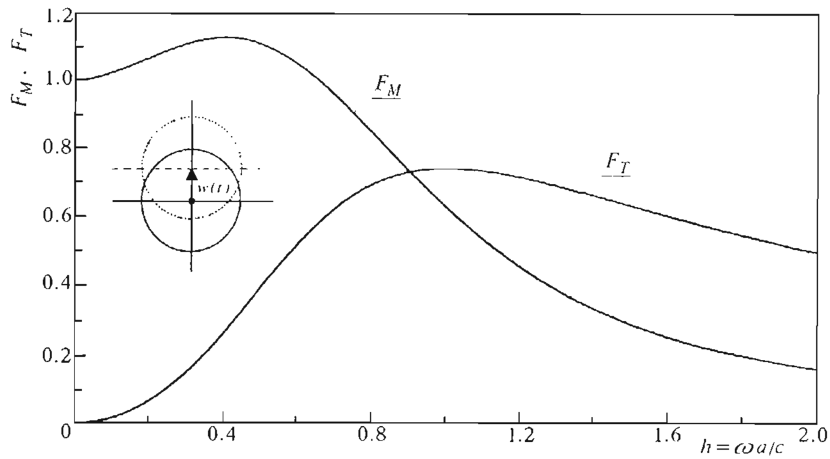

Fig. 3. Components $F_M(h)$ and $F_T(h)$ for vertical vibrations of the circular cylinder

The first term in Eq (2.18) defines the added mass of fluid and, the second one describes the damping of vibrations due to transmission of energy by outgoing dilatational waves. For small values of α, the following relation holds

$$\aleph \cong \frac{1+\mathrm{i}\frac{\pi\alpha^2}{4}}{1+\mathrm{i}\frac{\pi\alpha^2}{4}-\frac{\pi\alpha^2}{2}\Big(-\ln\frac{\alpha}{2}+0.11593+\mathrm{i}\Big)} \tag{2.19}$$

The components of the resultant force (2.17) are depicted in Fig.3. From the plots it is seen, that for $h \to 0$ ($h = \omega a/c$), no damping is obtained, i.e., in order to calculate the damping it is necessary to take the fluid compressibility into account.

3. Harmonic vibrations of an elliptical cylinder in a compressible fluid

Now, like in the previous case, let us consider, the plane problem of harmonic vibrations of a rigid ellipse in the direction of its smaller axis. In order to construct a solution to the problem mentioned, it is convenient to introduce the elliptical coordinate system (see Fig.4).

The Cartesian and the curvilinear coordinates are related through the transformation

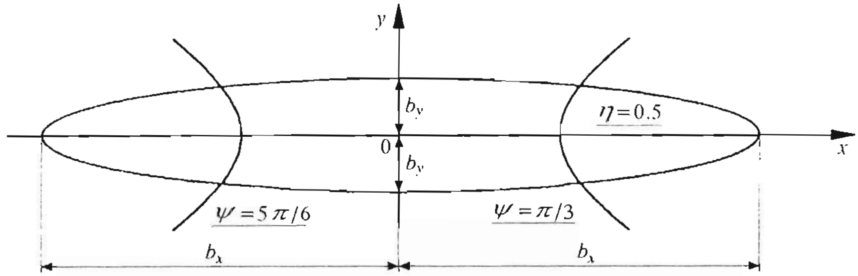

Fig. 4. Eliptic system of coordinates

$$x = a\cosh\eta\cos\psi$$
$$y = a\sinh\eta\sin\psi \tag{3.1}$$

where $2a$ is the focal length of the ellipse and $0 \leqslant \eta \leqslant \infty$, $0 \leqslant \psi \leqslant 2\pi$.

The base vectors of the elliptic coordinate axes are expressed as follows

$$\boldsymbol{a}_1 = \frac{\partial \boldsymbol{r}}{\partial \eta} = a(\boldsymbol{e}_1 \sinh\eta\cos\psi + \boldsymbol{e}_2\cosh\eta\sin\psi)$$
$$\boldsymbol{a}_2 = \frac{\partial \boldsymbol{r}}{\partial \theta} = a(-\boldsymbol{e}_1 \cosh\eta\sin\psi + \boldsymbol{e}_2\sinh\eta\cos\psi) \tag{3.2}$$

From the dot products of the vectors, the components of the metric tensor result

$$g_{11} = g_{22} = a^2(\cosh^2\eta - \cos^2\psi) \qquad\qquad g_{12} = 0 \tag{3.3}$$

Accordingly, the determinant of the tensor is

$$g = a^4(\cosh^2\eta - \cos^2\psi)^2 \tag{3.4}$$

The outward unit vector normal to the surface $\eta = \eta_0$ has the form

$$\boldsymbol{a}_\eta = \frac{a}{\sqrt[4]{g}}(\boldsymbol{e}_1\sinh\eta_0\cos\psi + \boldsymbol{e}_2\cosh\eta_0\sin\psi) \tag{3.5}$$

For harmonic vibrations it is convenient to introduce the spatial potential $\phi(x,y)$

$$\varPhi(x,y,t) = \phi(x,y)\mathrm{e}^{-\mathrm{i}\omega t} \tag{3.6}$$

where ω is the circular frequency of vibrations.

Upon substituting Eq (3.6) into the wave equation (2.3) we obtain the Helmholtz equation for the spatial potential

$$\nabla^2\phi + \left(\frac{\omega}{c}\right)^2 \phi = 0 \tag{3.7}$$

Keeping in mind the time factor, we will confine ourselves to the space variables. In the elliptic coordinate system introduced, the velocity field is defined as

$$\boldsymbol{v} = \operatorname{grad}\phi(\eta,\psi) = \frac{1}{\sqrt{g}}\Big(\frac{\partial\phi}{\partial\eta}\boldsymbol{a}_1 + \frac{\partial\phi}{\partial\psi}\boldsymbol{a}_2\Big) \tag{3.8}$$

At the same time, the Laplace operator assumes the form

$$\nabla^2\phi(\eta,\psi) = \frac{1}{\sqrt{g}}\Big(\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2}\Big) \tag{3.9}$$

From substitution of Eq (3.9) into Eq (3.7) it follows

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2} + h^2(\cosh^2\eta - \cos^2\psi)\phi = 0 \qquad h = \frac{\omega a}{c} \geqslant 0 \tag{3.10}$$

In order to find a solution to the equation, the method of separation of variables is introduced, namely

$$\phi(\eta,\psi) = G(\eta)H(\psi) \tag{3.11}$$

Upon inserting Eq (3.11) into Eq (3.10) the following equations are obtained

$$\begin{aligned} &H''(\psi) + \Big(b - \frac{1}{2}h^2 - \frac{1}{2}h^2\cos 2\psi\Big)H(\psi) = 0 \\ &G''(\eta) - \Big(b - \frac{1}{2}h^2 - \frac{1}{2}h^2\cosh 2\eta\Big)G(\eta) = 0 \end{aligned} \tag{3.12}$$

where b is a separation constant and primes denote the differentiation with respect to independent variables.

The Mathieu equations (cf Morse and Feshbach, 1958, 1960) are obtained. Substituting for $\eta = \mathrm{i}\psi$ into the second equation, the form of the first equation is obtained. In the equations, h is a relatively small number (let us say $h \leqslant 1.5$), while $H(\psi)$ must be a periodic function with the period equal to 2π. In what follows, we confine ourselves to the symmetrical, with respect to the small axis, vibrations of the ellipse. Thus, let us consider now the vertical rigid body motion of the ellipse according to the formulae

$$\boldsymbol{w}(t) = w_0 \mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{e}_2 \qquad \boldsymbol{v}(t) = -\mathrm{i}\omega w_0 \mathrm{e}^{-\mathrm{i}\omega t}\boldsymbol{e}_2 \tag{3.13}$$

Multiplication of the second relation of Eq (3.13) by the vector (3.5) yields the normal component of the cylinder velocity

$$v_n = -\frac{\mathrm{i}\omega w_0 \cosh\eta_0 \sin\psi}{a\sqrt{\cosh^2\eta_0 - \cos^2\psi}} \tag{3.14}$$

For the assumed motion, the solution to Eq $(3.12)_1$ is expressed in the form of the following series (for details see Morse and Feshbach, 1958, vol.1, pp.532)

$$H(\psi) = \sum_{n=1}^{\infty} B_n \sin[(2n-1)\psi] \tag{3.15}$$

Making use of the relation

$$\cos 2\psi H(\psi) = \frac{1}{2}(-B_1 + B_2)\sin\psi + \frac{1}{2}\sum_{n=1}^{\infty}(B_{n-1} + B_{n+1})\sin[(2n-1)\psi] \tag{3.16}$$

and substituting Eq (3.15) into Eq $(3.12)_1$, an infinite homogeneous system of algebraic equations in the constants B_n, $n = 1, 2, \ldots$ is obtained. The system of equations is approximated by the following finite system

$$\begin{aligned}
&\Big(1 + \frac{h^2}{4} - b\Big)B_1 + \frac{h^2}{4}B_2 = 0\\
&\frac{h^2}{4}B_1 + \Big(9 + \frac{h^2}{2} - b\Big)B_2 + \frac{h^2}{4}B_3 = 0\\
&\frac{h^2}{4}B_2 + \Big(25 + \frac{h^2}{2} - b\Big)B_3 + \frac{h^2}{4}B_4 = 0\\
&\ldots\\
&\frac{h^2}{4}B_{n-1} + \Big[(2n-1)^2 + \frac{h^2}{2} - b\Big]B_n + \frac{h^2}{4}B_{n+1} = 0
\end{aligned} \tag{3.17}$$

Numerical calculations show, that for the cases we are interested in, the convergence is very fast and thus, such approximation is justified. Nontrivial solution to the equations exists only in the case when the parameter b is an eigenvalue of the system of equations. In such a case, the constants B_i, $i = 1, 2, \ldots, n$ are not independent, but depend on the form of the relevant eigenvector. In general, in the case of n equations (3.17) and n distinct eigenvalues, we have the $n \times n$ eigenvector matrix. As $h \to 0$ we deal with the incompressible fluid for which the matrix of the system of equations (3.17) is diagonal. In this case the eigenvalues of the system and the corresponding solutions are as follows

$$b_r = (2r-1)^2 \qquad H_r = \sin[(2r-1)\psi] \qquad r = 1, 2, \ldots \tag{3.18}$$

An interesting case is for small influence of the compressibility i.e. when the parameter h is a small number. In this case it is convenient to expand the relevant eigenvalues into a power series with respect to the parameter. For instance (cf Morse and Feshbach, 1958, 1960)

$$\begin{aligned} &b_1 = 1 + \alpha_1 h^2 + \beta_1 h^4 \qquad b_2 = 9 + \alpha_2 h^2 + \beta_2 h^4 \\ &b_3 = 25 + \alpha_3 h^2 + \beta_3 h^4 \\ &H_r = B_{r1} \sin\psi + B_{r2} \sin 3\psi + B_{r3} \sin 5\psi \qquad r = 1,2,3 \end{aligned} \tag{3.19}$$

Substitution of the relations into Eqs (3.17) yields

$$\begin{aligned} &m = 1 \\ &b_1 = 1 + \frac{1}{4}h^2 - \frac{1}{128}h^4 \qquad B_{11} = 1 + \frac{3}{32}h^2 - \frac{13}{3072}h^4 \\ &B_{21} = -\frac{1}{32}h^2 - \frac{1}{512}h^4 \qquad B_{31} = \frac{1}{3072}h^4 \\ &m = 2 \\ &b_2 = 9 + \frac{1}{2}h^2 + \frac{1}{2568}h^4 \qquad B_{12} = \frac{1}{96}h^2 - \frac{1}{6144}h^4 \\ &B_{22} = \frac{1}{3} + \frac{1}{192}h^2 - \frac{7}{36864}h^4 \qquad B_{32} = -\frac{1}{192}h^2 - \frac{1}{12288}h^4 \\ &m = 3 \\ &b_3 = 25 + \frac{1}{2}h^2 + \frac{1}{256}h^4 \qquad B_{13} = \frac{1}{30720}h^4 \\ &B_{23} = \frac{1}{320}h^2 - \frac{3}{102400}h^4 \qquad B_{33} = \frac{1}{5} - \frac{3}{1600}h^2 - \frac{1}{1536000}h^4 \end{aligned} \tag{3.20}$$

In a general case the eigenvectors of the system should be normalised according to the formula (cf Morse and Feshbach, 1958, 1960))

$$\sum_{n=1}^{N} (2n-1)B_n = 1 \tag{3.21}$$

where N stands for the number of equations.

Knowing r, $r = 1,2,...,N$ solutions to Eqs (3.17) we have to construct solutions to the following r differential equations

$$G_r''(\eta) - (b_r - h^2 \cosh^2 \eta) G_r(\eta) = 0 \tag{3.22}$$

where $r = 1,2,...,N$.

By virtue of the above relations the solution to the wave equation may be expressed in the form (cf Morse and Feshbach, 1958, 1960)

$$\Phi(\eta,\psi,t) = \sum_m G_m(\eta) H_m(\psi) e^{-i\omega t} \tag{3.23}$$

where H_m are defined above and

$$G_m(\eta) = \sqrt{\frac{\pi}{2}} \tanh\eta \sum_{n=1}^{\infty} (-1)^{n-m}(2n-1)B_{nm}[J_{2n-1}(z) + \mathrm{i}Y_{2n-1}(z)] \tag{3.24}$$

where $z = h\cosh\eta$, B_{nm} are the elements of the eigenvector matrix and J_{2n-1} and Y_{2n-1} are the Bessel functions (cf Morse and Feshbach, 1958, 1960).

The Bessel function entering the last equation has the following asymptotic expansion

$$J_{2n-1}(z) + \mathrm{i}Y_{2n-1}(z) \cong \sqrt{\frac{2}{\pi z}} \exp\Big[\mathrm{i}z - \frac{\pi}{2}\Big(2n - \frac{1}{2}\Big)\Big] \tag{3.25}$$

For small values of h it is convenient to use the expressions

$$\begin{aligned} J_{0,2m-1} &= \mathrm{Re}(G_m) = (-1)^{m-1}\sqrt{\frac{2}{\pi}} \frac{\sum_{n=1}^{\infty} B_{nm}J_{2n-1}(h\sinh\eta)}{\sum_{n=1}^{\infty}(-1)^{n-1}B_{nm}} \\ Y_{0,2m-1} &= \mathrm{Im}(G_m) = \sqrt{\frac{2}{\pi}} \frac{1}{B_{1m}} \sum_{n=1}^{\infty} (-1)^{n-m} B_{nm} \cdot \\ &\cdot \Big[J_{n-1}\Big(\frac{h}{2}\mathrm{e}^{-\eta}\Big)Y_n\Big(\frac{h}{2}\mathrm{e}^{\eta}\Big) - J_n\Big(\frac{h}{2}\mathrm{e}^{-\eta}\Big)Y_{n-1}\Big(\frac{h}{2}\mathrm{e}^{\eta}\Big)\Big] \end{aligned} \tag{3.26}$$

In Morse and Feshbach (1960) (vol.2, pp.529-530), there are given formulae for derivatives of the Bessel functions in the limiting case $b_y \to \infty$

$$\begin{aligned} \Big[\frac{dJ_{0,2m-1}}{d\eta}\Big]_{\eta=0} &= \frac{1}{2}h(-1)^{m-1}\sqrt{\frac{2}{\pi}} \frac{B_{1m}}{\sum_{n=1}^{\infty}(-1)^{n-1}B_{nm}} \\ \Big[\frac{dY_{0,2m-1}}{d\eta}\Big]_{\eta=0} &= \sqrt{\frac{2}{\pi}\frac{1}{B_1}} \sum_{n=1}^{\infty} (-1)^{n-m} B_{nm} \cdot \\ &\cdot \Big[-(2n-1)(J_{n-1}Y_n + J_nY_{n-1}) + h(J_{n-1}Y_{n-1} + J_nY_n)\Big] \end{aligned} \tag{3.27}$$

where the arguments of all functions are equal to $h/2$.

For the assumed number of equations, Eq (3.23) may be rewritten in the following form

$$\Phi(\eta,\psi,t) = \boldsymbol{C}\mathsf{G}\mathsf{B}^\top\boldsymbol{S}\exp(-\mathrm{i}\omega t) \tag{3.28}$$

where

- $\boldsymbol{C}$ – vector of constants
- G – diagonal matrix with elements $G_1, G_2, \ldots$, being the solutions to Eq (3.22) and finally
- $\boldsymbol{S}$ – vector with elements $\sin\psi, \sin 3\psi, \ldots$.

Knowing the normal component of the fluid velocity at the ellipse boundary

$$v_n = \frac{1}{\sqrt[4]{g}} \frac{\partial \Phi}{\partial \eta}\Big|_{\eta=\eta_0} \tag{3.29}$$

we can solve the system of equations

$$-\mathrm{i}w_0\omega a \cosh \eta_0 [1, 0, 0, \ldots] = \boldsymbol{C} \frac{\partial \mathbf{G}}{\partial \eta}\Big|_{\eta=\eta_0} \mathbf{B}^\top \tag{3.30}$$

and obtain the vector $\boldsymbol{C}$ of constants of the solution. Having the solution for the velocity potential we can straightforward calculate the fluid pressure and the elementary force applied to the cylinder boundary

$$\begin{aligned} d\boldsymbol{F} &= \rho \frac{\partial \Phi}{\partial t} \boldsymbol{a}_1 \, d\psi = \\ &= -\rho \mathrm{i}\omega a \boldsymbol{C} \mathbf{G} \mathbf{B}^\top \mathbf{S} (\sinh \eta_0 \cos \psi \boldsymbol{e}_1 + \cosh \eta_0 \sin \psi \boldsymbol{e}_2) d\psi \end{aligned} \tag{3.31}$$

where $\mathbf{S}$ is a matrix of trigonometric functions.

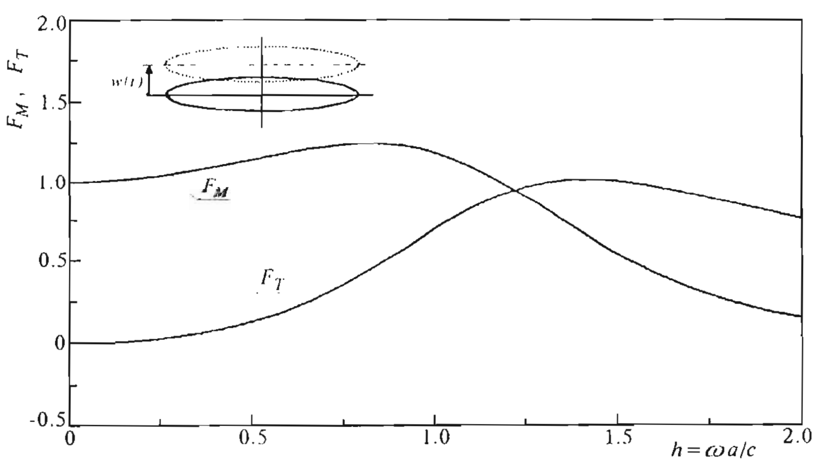

Fig. 5. Components $F_M(h)$ and $F_T(h)$ for vertical vibrations of the plate

Upon integrating the last equation with respect to the angle in the range $0 - 2\pi$, the resultant of the fluid pressure applied to the unit length of the elliptic cylinder is obtained. Taking the limit passage $b_y \to 0$ in the result mentioned, the solution to the plate vibrations is obtained. With respect to the displacement $w(t)$ of the plate, the force may be expressed in the following form

$$R = \ddot{w}(t) \rho \pi b_x^2 F_M(h) + \dot{w}(t) \omega \rho \pi b_x^2 F_T(h) \tag{3.32}$$

where the plots of the functions $F_M(h)$ and $F_T(h)$ are shown in Fig.5.

On the basis of the above results, one can calculate the hydrodynamic pressure in the case of flexural displacements of a simply supported plate. For instance, let the displacements are

$$w(x,t) = w_0\Big[1 - \Big(\frac{x}{b_x}\Big)^2\Big] \exp(-\mathrm{i}\omega t) \tag{3.33}$$

The corresponding velocity component, normal to the surface of the elliptic cylinder, is

$$v_n = \frac{3}{4}\frac{-aw_0\mathrm{i}\omega\cosh\eta_0}{a\sqrt{\cosh^2\eta_0 - \cos^2\psi}}\Big(\sin\psi - \frac{1}{3}\sin 3\psi\Big)\exp(-\mathrm{i}\omega t) \tag{3.34}$$

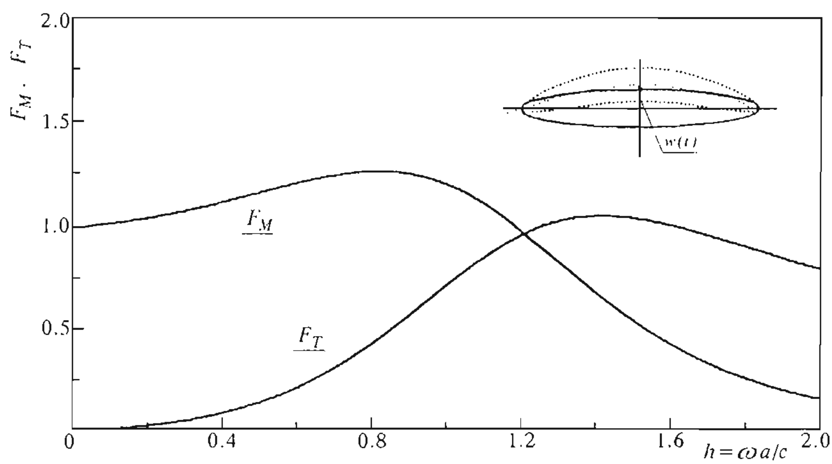

Fig. 6. Components $F_M(h)$ and $F_T(h)$ for flexural vibrations of the plate

Further calculations are similar to those presented above and lead to similar results. It may be interesting to note, that, within the accuracy of numerical computations, the resultant of the pressure for this case is equal to $3/4$ of the relevant value for the rigid body motion of the plate. The components $F_M(h)$ and $F_T(h)$ in this case are shown in Fig.6. In addition to the cases discussed so far, let us consider now the case of rigid rotational vibrations of the elliptic cylinder with respect to its longitudinal axis according to the formula

$$\boldsymbol{w}(x,t) = \varphi_0 \exp(-\mathrm{i}\omega t)a(-\sinh\eta_0\sin\psi\boldsymbol{e}_1 + \cosh\eta_0\cos\psi\boldsymbol{e}_2) \tag{3.35}$$

where the angle φ_0 represents the amplitude of rotation.

The associated normal component of the velocity is

$$v_n = -\mathrm{i}\omega\varphi_0 \exp(-\mathrm{i}\omega t)\frac{a^2 \sin 2\psi}{2a\sqrt{\cosh^2 \eta_0 - \cos^2 \psi}} \tag{3.36}$$

The resultant moment of the pressure reads

$$M_\varphi = \ddot{\varphi}(t)\frac{1}{8}\rho\pi a^4 F_M(h) + \dot{\varphi}(t)\frac{1}{8}\rho\pi a^4 \omega F_T(h) \tag{3.37}$$

The plots of $F_M(h)$ and $F_T(h)$ are depicted in Fig.7.

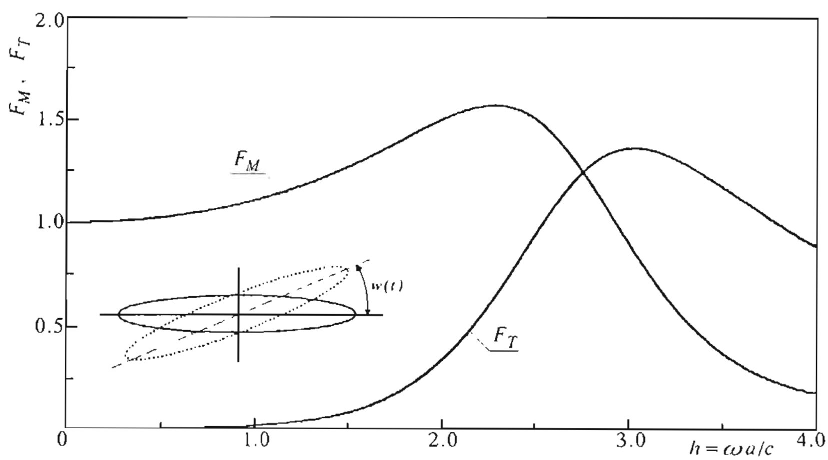

Fig. 7. Components $F_M(h)$ and $F_T(h)$ for rotational vibrations of the plate

With the help of the solution derived, other cases of transverse vibrations of an elastic plate can be investigated.

4. Concluding remarks

We have presented a solution to the plane problem of harmonic vibrations of an elastic plate submerged in an unbounded medium of compressible fluid. The solution was obtained by means of taking a limit passage in the solution derived in the case of harmonic vibration of an elliptic cylinder immersed in the fluid. In this way a kind of singularity at the end points of the plate was defined. The solution for the cylinder was obtained with the help of separation of variables in the elliptic coordinate system. With the procedure mentioned, the problem has been reduced to solutions of the Mathieu equations satisfying the Sommerfeld condition at infinity. For comparison, a problem of circular cylinder was also presented. From the analysis performed it is seen that the fluid compressibility plays a fundamental role in calculating the damping of vibrations resulting from transmission of energy by outgoing dilatational waves.

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Drgania płyty w cieczy i tłumienie wywołane transmisją energii za pośrednictwem fal dylatacyjnych

Streszczenie

W pracy przedstawiono rozwiązanie zagadnienia drgań harmonicznych płyty zanurzonej w nieograniczonym obszarze ściśliwej cieczy nielepkiej. Rozwiązanie to otrzymano za pomocą przejścia granicznego w zbudowanym rozwiązaniu dla nieskończonego walca eliptycznego drgającego w cieczy. Ten ostatni problem rozwiązano za pomocą metody Fouriera rozdzielenia zmiennych w eliptycznym układzie współrzędnych. Dla porównania, wyznaczono również rozwiązanie dla drgań walca kołowego zanurzonego w cieczy. Z przedstawionej dyskusji wynika, że ściśliwość cieczy jest podstawowym parametrem w opisie sił hydrodynamicznych, a szczególnie – tłumienia drgań płyty dla wyższych częstości.

Manuscript received November 12, 1998; accepted for print January 11, 1999