JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **38**, 2000

VIBRATIONS AND STABILITY OF A NON-CONSERVATIVELY COMPRESSED PRISMATIC COLUMN UNDER NONLINEAR CREEP CONDITIONS

Antoni Gajewski

Institute of Physics, Cracow University of Technology
e-mail: dantek@fizyk.ifpk.pk.edu.pl

The aim of this paper is to investigate the influence of the physical nonlinearity of the material under creep conditions on vibrations and stability of a non-conservatively compressed column. Especially, behaviour of the characteristic curves is examined for a column compressed by a tangential force. The additional effects such as: compressibility of the column axis, rotational inertia and external damping are taken into account.

Key words: column, nonlinear creep, vibrations, stability

1. Introduction

There exists comprehensive literature devoted to the stability of linearly elastic structural systems, mainly columns, loaded by non-conservative forces (Bolotin, 1961; Gajewski and Życzkowski, 1988). Considerably fewer papers take into account the rheological properties of material. In particular, some of them consider a very interesting destabilization phenomenon due to the internal damping of material. The early papers dealing with the above problem are; e.g., Zoriy and Leonov (1961), Herrmann and Jong (1965), Dżygadło and Solarz (1970), Gajewski and Życzkowski (1972), Gajewski (1972). The internal damping of material has been characterized by a linear rheological model of visco-elastic material of the Kelvin-Voigt type.

The influence of internal damping of material on optimum shapes of structural elements was taken into account by several authors; e.g., Plaut (1972, 1975), Claudon (1975, 1978), Claudon and Sunakawa (1981), Langthjem (1993). Material was also described by means of the Voigt-Kelvin linear model. Many other problems of analysis and synthesis of columns compressed by follower forces in view of their stability have been discussed by Bogacz and Janiszewski (1987).

Stability of columns compressed by a non-conservative concentrated force under nonlinear creep conditions was investigated by Kowalski (1975) in his Ph.D.Thesis (under supervision of M.Życzkowski). Some elements of the work have been presented by Życzkowski and Kowalski (1984). The authors took into account the Norton and Zhukov-Rabotnov-Churikov (1953) nonlinear models of material. The dependence of the critical force causing loss of static and kinetic stability on the so-called tangency coefficient has been considered.

Some problems of the optimal design of conservative systems with respect to nonlinear creep stability were investigated:
- for bars by Życzkowski and Wojdanowska (1972), Błachut and Życzkowski (1984), Wróblewski (1989)
- for trusses by Wojdanowska and Życzkowski (1972), Wojdanowska (1974)
- for arches by Wróblewski and Życzkowski (1989)
- for plates by Wróblewski (1992).

A comprehensive review of papers (187 references) devoted to the optimal structural design under creep conditions was presented by Życzkowski (1996). Recently, an attempt at optimization of a column compressed by a non-conservative force in nonlinear creep conditions was presented by Gajewski (1997).

The principal aim of the present work is to examine the influence of nonlinear creep of material on vibrations and stability of column compressed by the tangential force. Especially, behaviour of the characteristic curves will be considered. We will also take into account the additional effects; namely, compressibility of the axis, rotational inertia and external damping.

2. Nonlinear creep law

Following the state equation hypothesis of Davenport (1938) it is assumed that the stress and strain components of the basic uniaxial stress state are interrelated by the following creep law, accounting for the strain hardening

$$\Phi(\overline{\sigma}, p, \dot{p}) = 0 \tag{2.1}$$

where
- p – creep strain, $p = \varepsilon - \overline{\sigma}/\overline{E}$
- ε – full strain

$\overline{\sigma}$ – stress
$\overline{E}$ – elastic modulus
Φ – given material function

and the bar over a symbol denotes dimensional quantities. Among various creep stability theories the Rabotnov-Shesterikov (1957) theory of perfect, straight bars seems to be most suitable here. Generally, it can be assumed that during vibrations of a system (or as a result of buckling), the stress and strain components in a basic state are subject to small variations and the creep law (2.1) can be linearized with respect to them. The behaviour of the variations determines the stability of the basic state (precritical) at a critical time t^*. In the basic state the relation between stress $\overline{\sigma}$ and strain ε is determined by Eq (2.1), while the *tangent creep modulus* $\overline{E}_{tc}$ should be evaluated on the basis of Rabotnov-Shesterikov theory from the following equation

$$\frac{\partial \Phi}{\partial \dot{p}}\Big|_0 \delta\dot{p} + \frac{\partial \Phi}{\partial p}\Big|_0 \delta p + \frac{\partial \Phi}{\partial \overline{\sigma}}\Big|_0 \delta\overline{\sigma} = 0 \tag{2.2}$$

Assuming that the variations of stress and strain components are subject to small linear vibrations of complex frequency $\overline{\Omega}$

$$\delta\varepsilon = \delta\varepsilon^a \mathrm{e}^{\overline{\Omega}t} \qquad\qquad \delta\overline{\sigma} = \delta\overline{\sigma}^a \mathrm{e}^{\overline{\Omega}t} \tag{2.3}$$

and substituting them into Eq (2.2) we obtain the tangent creep modulus

$$\overline{E}_{tc} = \frac{\delta\overline{\sigma}^a}{\delta\varepsilon^a} = \frac{\overline{\Omega}\frac{\partial \Phi}{\partial \dot{p}}\Big|_0 + \frac{\partial \Phi}{\partial p}\Big|_0}{\frac{\overline{\Omega}}{\overline{E}}\frac{\partial \Phi}{\partial \dot{p}}\Big|_0 + \frac{1}{\overline{E}}\frac{\partial \Phi}{\partial p}\Big|_0 - \frac{\partial \Phi}{\partial \overline{\sigma}}\Big|_0} \tag{2.4}$$

In this paper, the commonly used strain hardening creep law suggested by Rabotnov (1966) has been adopted

$$\Phi = \dot{p} - \Gamma\frac{\overline{\sigma}^n}{p^\mu} = \dot{p} - \Gamma\overline{\sigma}^n p^{-\mu} \tag{2.5}$$

where μ, n, Γ denote the material constants (temperature dependent). Moreover, all the results are obtained using material constants for copper at a temperature 200°C, $n = 32.8$, $\mu = 9.52$, $\overline{E} = 1.22 \cdot 10^5$ MPa, $\Gamma = 2.18 \cdot 10^{-113+n}\,(\mathrm{MPa})^{-n}\mathrm{h}^{-1}$ (see: Zhukov et al., 1953).

In the basic precritical state under the assumptions of constant stress ($\overline{\sigma}(t) = \mathrm{const}$) and with the initial condition $p(0) = 0$, we obtain from Eqs (2.1) and (2.5)

$$\varepsilon_0 = \frac{\overline{\sigma}_0}{\overline{E}}\Big\{1 + \overline{E}[(1+\mu)\Gamma\overline{t}_*]^{\frac{1}{1+\mu}}|\overline{\sigma}_0|^{\frac{n-1-\mu}{1+\mu}}\Big\} \tag{2.6}$$

and the "secant modulus"

$$E_{sc} = \frac{\overline{E}_{sc}}{\overline{E}_0} = \frac{\overline{E}}{\overline{E}_0\Big\{1 + \overline{E}[(1+\mu)\Gamma\overline{t}_*]^{\frac{1}{1+\mu}}|\overline{\sigma}_0|^{\frac{n-1-\mu}{1+\mu}}\Big\}} \tag{2.7}$$

According to the Rabotnov-Shesterikov (1957) theory, the "tangent modulus" for the nonlinear creep law (2.5) can be written in the form

$$E_{tc} = \frac{\overline{E}\Big(1 + \frac{1+\mu}{\mu}\overline{t}_*\overline{\Omega}\Big)}{\overline{E}_0\Big\{1 + \frac{1+\mu}{\mu}\overline{t}_*\overline{\Omega} + \frac{n(1+\mu)}{\mu}\overline{t}\Gamma\overline{E}[(1+\mu)\Gamma\overline{t}_*]^{-\frac{\mu}{1+\mu}}|\overline{\sigma}_0|^{\frac{n-1-\mu}{1+\mu}}\Big\}} \tag{2.8}$$

It is a function of the critical time $\overline{t}_*$ and complex frequency of vibration $\overline{\Omega} = \overline{\delta} + \mathrm{i}\overline{\omega}$. $\overline{E}_0$ is a certain constant of the stress dimension.

3. State equation

The vibrations and stability analysis of a non-prismatic column compressed by a non-conservative force requires first the calculation of displacement under uniaxial compression (precritical state) and then the application of the kinetic stability criterion, making use of small superposed vibrations. The general equations of precritical and vibration states were derived and presented in the monograph by Gajewski and Życzkowski (1988), where the effects of: extensibility of the axis, shear deformations, rotational inertia and nonlinear properties of the material were introduced. The equations of small vibrations of complex frequency $\overline{\Omega}$ superposed on the momentless precritical state can be transformed to four complex or to eight real linear ordinary differential equations with the appropriate boundary conditions. In the case of cantilever compressed by a follower force P it is a non-self-adjoint boundary value problem. For a prismatic column the equations of vibration state can be reduced one fourth order differential equation of constant complex coefficients

$$\begin{aligned}
&v^{IV} + fv'' + gv = 0\\
&f = \widetilde{E}_{tc}[\varepsilon_{01}P(1+hP) - r\alpha\widetilde{\rho}\Omega^2] - \varepsilon_{01}h\widetilde{\rho}\Omega^2 \qquad\qquad \widetilde{E}_{tc} = E_{tc}^{-1}\\
&g = \varepsilon_{01}\widetilde{E}_{tc}\Big\{\gamma\varepsilon_{01}P(1+hP)^2\Omega + [\varepsilon_{01}(1+hP) + r\alpha h\widetilde{\rho}\Omega^2]\widetilde{\rho}\Omega^2\Big\}
\end{aligned} \tag{3.1}$$

where the following dimensionless quantities have been introduced:
— independent variables

$$x = \frac{s}{l} \qquad\qquad t = \frac{\overline{t}}{t_0}$$

— external force

$$P = \frac{\overline{P}l^2}{E_0 I_0}$$

— constant parameters

$$\alpha = \frac{I_0}{A_0 l^2} \qquad t_0 = \sqrt{\frac{\rho_0 A_0 l^4}{E_0 I_0}} \qquad t_* = \frac{\overline{t}_*}{t_0} \qquad \widetilde{\rho} = \frac{\overline{\rho}}{\rho_0}$$

— elongation of the axis

$$\varepsilon_{00} = -\frac{\alpha P}{E_{sc}} \qquad \varepsilon_{00} = -\alpha P\Big(1 + TP^{\frac{n-1-\mu}{1+\mu}}\Big) \qquad \varepsilon_{01} = 1 + \varepsilon_{00}$$

— quantities connected with the physical law

$$T = T_{00}\Big(\frac{\tau}{\tau_0}\Big)^{\frac{1}{1+\mu}}\Big(\frac{\alpha}{\alpha_0}\Big)^{\frac{n-1-\mu}{1+\mu}} \qquad T_{00} = e[(1+\mu)\Gamma E_0^n \tau_0]^{\frac{1}{1+\mu}}\alpha_0^{\frac{n-1-\mu}{1+\mu}}$$

$$E_{tc} = e\frac{1 + \frac{1+\mu}{\mu}t_*\Omega}{1 + \frac{1+\mu}{\mu}t_*\Omega + \frac{n}{\mu}TP^{\frac{n-1-\mu}{1+\mu}}} \qquad e = \frac{\overline{E}}{\overline{E}_0}$$

$$\tau = t_0 t_* = \overline{t}_* \qquad \tau_0 = 3600\,\mathrm{s} \qquad \alpha_0 = 10^{-4}$$

A_0 denotes the cross-sectional area and I_0 denotes the moment of inertia of the prismatic column cross-sections, ρ_0 is a constant of density dimension, t_0 denotes a certain constant which may be treated as a unit of time and η, γ denote the tangency coefficient and external viscous damping coefficient, respectively. The parameter r characterises the magnitude of the cross-sectional rotational inertia while the function $h(\Omega)$ characterizing the shear effects is defined by the equation

$$h(\Omega) = \frac{\alpha\varepsilon_{01}}{k_1 E_{tc} - \alpha\varepsilon_{01}P} \tag{3.2}$$

where k_1 denotes the shear coefficient. Taking into account the physical constants for copper, given above, and $e \equiv 1$ we have $T_{00} = 0.781408$.

In further parts of this work we neglect the shear effect $(h = 0)$, which considerably simplifies numerical calculations. In such a case the boundary conditions for the cantilever column can be written in the form

$$\begin{aligned} &v(0) = 0 \qquad && v'(0) = 0 \\ &v''(1) = 0 \qquad && v'''(1) + c_3 v'(1) = 0 \end{aligned} \tag{3.3}$$

where

$$c_3 = \widetilde{E}_{tc}[\varepsilon_{01}(1-\eta)P - r\alpha\widetilde{\rho}\Omega^2] \tag{3.4}$$

The solution of Eq (3.1) can be looked for in the following form

$$v(x) = A\sin s_1 x + B\cos s_1 x + C\sin s_2 x + D\cos s_2 x \tag{3.5}$$

where s_1 and s_2 fulfil the algebraic complex characteristic equation

$$s^4 + fs^2 + g = 0 \tag{3.6}$$

Substituting Eq (3.5) into the boundary conditions (3.3) we obtain a system of four linear and homogeneous equations on constants A, B, C and D. The principal determinant of these equations equalled to zero allows one to calculate the relation between complex frequencies of vibrations and compressive force P and other parameters of the problem (characteristic curves). It can be written in the form analogous to the equation presented by Gajewski (1972)

$$F_1 - c_3 F_2 = 0 \tag{3.7}$$

and

$$\begin{aligned} F_1 &= (s_1^4 + s_2^4) - s_1 s_2 (s_1^2 + s_2^2)\sin s_1 \sin s_2 - 2s_1^2 s_2^2 \cos s_1 \cos s_2 \\ F_2 &= (s_1^2 + s_2^2) - 2s_1 s_2 \sin s_1 \sin s_2 - (s_1^2 + s_2^2)\cos s_1 \cos s_2 \end{aligned}$$

After separating the real and imaginary parts in Eq (3.7) we obtain two real algebraic equations, which determine the real δ and imaginary ω parts of complex frequency Ω.

4. Numerical calculations and results

Using the Newton-Raphson method to solve Eq (3.7) numerical calculations have been made for various values of parameters characterising extensibility of axis, critical time, external damping and cross-sectional rotational

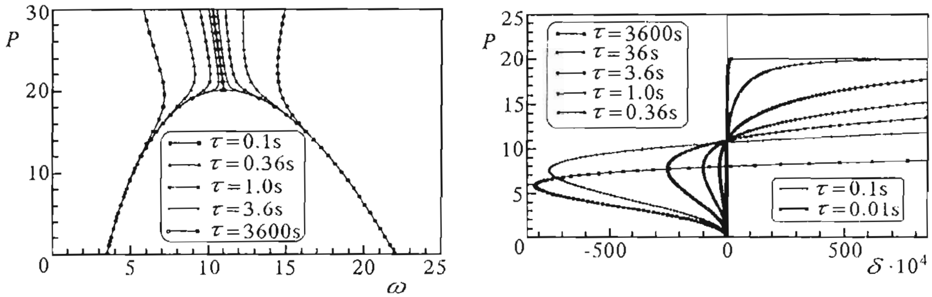

Fig. 1. Characteristic curves versus the critical time parameter τ for: $\eta = 1$, $\alpha = 10^{-5}$, $\gamma_0 = 0$, $r = 0$

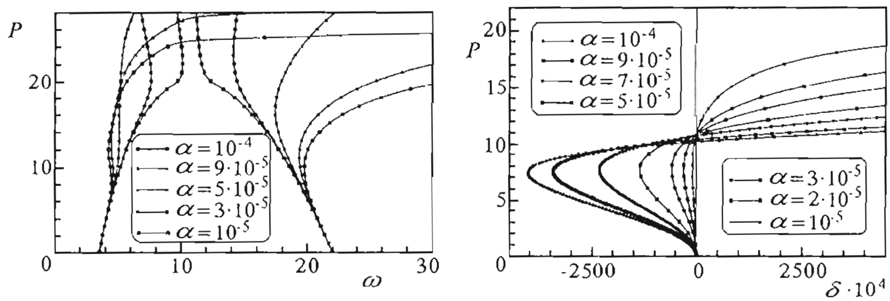

Fig. 2. Characteristic curves versus the extensibility parameter α for: $\eta = 1$, $\tau = 1\,\mathrm{s}$, $\gamma_0 = 0$, $r = 0$

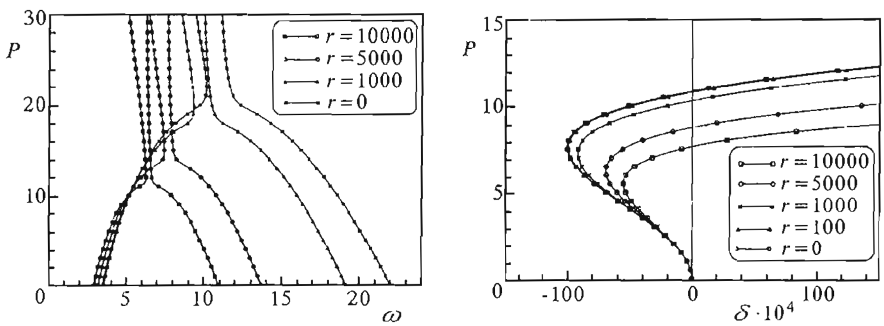

Fig. 3. Characteristic curves versus the rotational inertia parameter r for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 1\,\mathrm{s}$, $\gamma_0 = 0$

inertia. All the results presented in this paper are obtained for the tangential force, i.e. for $\eta = 1$ with the shear effects neglected. Fig.1 shows that the critical time parameter τ mostly influences the $(P - \omega)$ curves. In a minor degree it changes the critical magnitude of the force P (for which $\delta = 0$). A similar situation occurs when the characteristic curves depend on the extensibility parameter α. As it is seen in Fig.2 the critical value of the force P is nearly independent of the parameter $\alpha \in (10^{-5}, 10^{-4})$.

The cross-sectional rotational inertia parameter r exerts essential influence on the characteristic curves $(P - \omega)$ and $(P - \delta)$. A notable decrease in the critical force with increasing parameter r may be observed in Fig.3. The results obtained here are similar to those presented by Kounadis and Katsikadelis (1976) for a linearly elastic column.

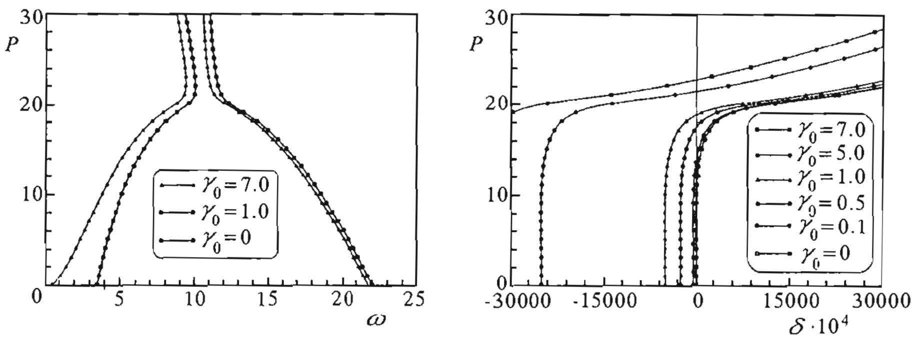

Fig. 4. Characteristic curves versus the external damping parameter γ_0 for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 1\,\mathrm{s}$, $r = 0$

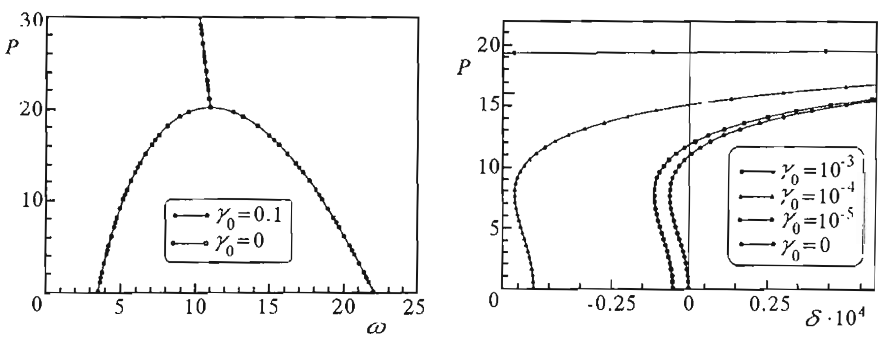

Fig. 5. Characteristic curves versus the external damping parameter γ_0 for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 3600\,\mathrm{s}$, $r = 0$

The external damping parameter γ_0 has a very small effect on the $(P-\omega)$ curves, however it strongly modifies the $(P-\delta)$ curves. As it is seen in Fig.4 and Fig.5 the external damping eliminates the destabilisation phenomenon. Such an effect has been previously observed for linearly visco-elastic columns by Herrmann and Jong (1966), Gajewski (1972), Gajewski and Życzkowski (1972), Seyranian (1987) and others.

5. Conclusions

The vibrations and stability of a tangentially compressed column in nonlinear creep conditions have been comprehensively examined. It has been assumed that the mechanical properties of the material of the column are characterised by the Zhukov-Rabotnov-Churikov (1953) law. Although the nonlinear creep law considered in this paper is essentially different from the linear visco-elastic models of material, the results presented here are close in character to those obtained for the Kelvin-Voigt model. Especially, the destabilizing phenomenon is quite similar to that very well known from the literature.

References

1. BŁACHUT J., ŻYCZKOWSKI M., 1984, Bimodal Optimal Design of Clamped-Clamped Columns Under Creep Conditions, *Int. J. Solids Struct.*, **20**, 6, 571-577

2. BOGACZ R., JANISZEWSKI R., 1987, Analysis and Synthesis of Column Under Follower Forces from the Point of View of Stability, *Adv. in Mechanics (Uspekhi mekhaniki),* **8**, 3, 3-52 (in Russian)

3. BOLOTIN W.W., 1961, Nonconservative Problems of Elastic Stability Theory, *Gos. Izd. Fiz.-Mat. Lit.*, Moscow (in Russian), English translation: Pergamon Press, New York 1963

4. CLAUDON J.-L., 1975, Characterisctic Curves and Optimum Design of Two Stuctures Subjected to Circulatory Loads, *Journal de Mécanique*, **14**, 3, 531-543

5. CLAUDON J.-L., 1978, Determination et maximization de la charge critique d'une colonne de Hauger en presence d'amortissement, *ZAMP*, **29**, 2, 226-236

6. CLAUDON J.-L., SUNAKAWA M., 1981, Optimizing Distributed Structures for Maximum Flutter Load, *AIAA Journal*, **19**, 7, 957-959

7. DAVENPORT C.C., 1938, Correlation of Creep and Relaxation Properties of Copper, *J. Appl. Mech.*, **5**, 2, A56

8. DŻYGADŁO Z., SOLARZ L., 1970, On Nonautonomous Vibrations of a Self-Excited System with Tangential Force, *Proc. Vibr. Probl.*, **11**, 2, 157-178

9. GAJEWSKI A., 1972, On the Destabilizing Effect in a Non-Conservative System with Slight Internal and External Damping, *Proc. Vibr. Probl.*, **13**, 2, 187-198

10. GAJEWSKI A., 1997, Optimization of a Column Compressed by Non-Conservative Force in Non-Linear Creep Conditions, *Proc. Second World Congress of "Structural and Multidisciplinary Optimization"*, (Edit. W.Gutkowski and Z.Mróz), May 26-30, Zakopane, Poland, 737-742

11. GAJEWSKI A., ŻYCZKOWSKI M., 1972, Influence of Simultaneous Non-Homogeneous External and Internal Damping on the Stability of Non-Conservative Systems, *J. Theor. Appl. Mech.*, **10**, 1, 127-142 (in Polish)

12. GAJEWSKI A., ŻYCZKOWSKI M., 1988, *Optimal Structural Design Under Stability Constraints*, Kluwer Academic Publishers, Dordrecht, Boston, London

13. HERRMANN G., JONG I.C., 1965, On the Destabilizing Effect of Damping in Non-Conservative Elastic Systems, *J. Appl. Mech.*, **32**, 3, 592-597

14. HERRMANN G., JONG I.C., 1966, On Non-Conservative Stability Problems of Elastic Systems with Slight Damping, *J. Appl. Mech.*, **33**, 1, 125-133

15. KOUNADIS A.N., KATSIKADELIS J.T., 1976, Shear and Rotatory Inertia Effect on Beck's Column, *J. Sound and Vibr.*, **49**, 2, 171-178

16. KOWALSKI A., 1975, Stability of Visco-Elastic Rods Compressed by Non-Conservative Loads, Ph.D. Thesis, Cracow University of Technology, Kraków

17. LANGTHJEM M., 1993, On the Influence of Damping in a Problem of Dynamic Stability Optimization, The Danish Center for Applied Math. and Mech., The Technical University of Denmark., Report No.467, 1-24

18. PLAUT R.H., 1972, The Effects of Various Parameters on an Aeroelastic Optimization Problem, *JOTA*, **10**, 5, 321-330

19. PLAUT R.H., 1975, Optimal Design for Stability under Dissipative, Gyroscopic or Circulatory loads, *Proc. IUTAM Symp. "Optimization in Structural Design"*, Warsaw 1973, Springer, Berlin-New York, (Edit.: A.Sawczuk and Z.Mróz), 168-180

20. RABOTNOV YU.N., 1966, *Creep of Structural Elements*, Nauka, Moskva (in Russian)

21. RABOTNOV YU.N., SHESTERIKOV S.A., 1957, Creep Stability of Columns and Plates, *Prikl. Mat. Mekch.*, **21**, 3, 406-412 (Russian version), *J. Mech. Phys. Solids*, **6**, 27-34 (English version)

22. SEYRANIAN A.P., 1987, Destabilization Paradox and Criteria of Vibrational Stability, Institute of Problems in Mechanics, USSR Academy of Sciences, Moscow, Preprint No 301, 1-59

23. WOJDANOWSKA R., 1974, Optimum Design of Lattice Structures in Creep with Consideration of the Rabotnov-Shesterikov Theory of Buckling, *Mech. Teor. Stos.*, **12**, 3, 245-263 (in Polish)

24. WOJDANOWSKA R., ŻYCZKOWSKI M., 1972, Optimum Design of Lattice Structures in Creep Conditions with Consideration of the Kempner-Hoff Theory of Buckling, *Arch. Inż. Ląd.*, **18**, 4, 511-530 (in Polish)

25. WRÓBLEWSKI A., 1989, Optimization of Rod and Plate Structural Elements with Respect to Stability in Creep Conditions, Ph.D. Thesis, Cracow University of Technology, Kraków (in Polish)

26. WRÓBLEWSKI A., 1992, Optimal Design of Circular Plates Against Creep Buckling, *Eng. Optim.*, **20**, 111-128

27. WRÓBLEWSKI A., ŻYCZKOWSKI M., 1989, On Multimodal Optimization of Circular Arches Against Plane and Spatial Creep Buckling, *Struct. Optim.*, **1**, 4, 227-234

28. ZHUKOV A.M., RABOTNOV YU.N., CHURIKOV F.S., 1953, Experimental Verification of Some Theories of Creep, *Inzh. Sbornik*, 17, 163-170 (in Russian)

29. ZORIY L.M., LEONOV M.YA., 1961, Influence of Damping on the Stability of Non-Conservative Systems, *Problems of Design and Strength in Machine Building,* **7**, 7, 127-136 (in Russian)

30. ŻYCZKOWSKI M., 1996, Optimal Structural Design Under Creep Conditions, *Appl. Mech. Rev.*, **49**, 9, 433-446

31. ŻYCZKOWSKI M., KOWALSKI A., 1984, Nonconservative Stability Problems for Columns Subject to Nonlinear Creep, *Proc. EUROMECH Colloquium 190: "Dynamical Stability of Inelastic Structures"*, Technische Universität Hamburg-Harburg Oct.1-4, 109-111

32. ŻYCZKOWSKI M., WOJDANOWSKA-ZAJĄC R., 1972, Optimal Structural Design with Respect to Creep Buckling, *Proc. IUTAM Symp. "Creep in Struct. II"*, Göteborg 1970, Springer, New York, 371-387

Drgania i stateczność kolumny pryzmatycznej ściskanej siłą niekonserwatywną w warunkach nieliniowego pełzania

Streszczenie

W pracy zbadano wpływ nieliniowości fizycznej materiału na drgania i stateczność niekonserwatywnie ściskanych kolumn pryzmatycznych w warunkach nieliniowego pełzania. W szczególności przedstawiono kształty tak zwanych krzywych charakterystycznych (zależności części rzeczywistej i urojonej zespolonej częstości drgań od wartości siły ściskającej) dla kolumny ściskanej siłą śledzącą. W pracy uwzględniono również dodatkowe czynniki, wpływające na kształty krzywych, a mianowicie: ściśliwość osi kolumny, bezwładność obrotów przekrojów oraz wiskotyczne tłumienie zewnętrzne drgań.

Manuscript received October 27, 1999; accepted for print November 19, 1999