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TORSIONAL VIBRATION OF A SANDWICH SHAFT WITH DAMPING INTERLAYER

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This paper presents an analytical method of solving torsional vibration problems concerning a sandwich circular shaft with a viscoelastic soft and light interlayer. The elasticity and damping coefficients of the interlayer are assumed to be dependent on its geometrical characteristic and viscoelastic properties of the interlayer material. Complex functions of a real variable are applied in the solution to free and forced vibration problems. Then, the property of orthogonality of complex modes of the free vibration, which is the basis for solving the free vibration problem for arbitrary initial conditions, has been demonstrated. The solution to the problem of real stationary forced vibration has been obtained on the grounds of the complex stationary modes of vibration.

Key words: sandwich shaft, damping, torsional vibration

Notations

ψ_1, ψ_2	–	angles of torsion of shafts I and II, $\psi_i = \psi_i(x,t)$, $i = 1,2$
m_2	–	distributed load torque of the shaft II, $m_2 = m_2(x,t)$
μ	–	moment transfered through the interlayer from one shaft to the other, $\mu = \mu(x,t)$
τ	–	tangential stress on the cylindrical surface of radius ρ, $\tau = \tau(x,\rho,t)$ and $r_1 \leqslant \rho \leqslant r_2$
r_1, r_2	–	internal and external radius of the interlayer
r	–	external radius of the sandwich shaft
γ	–	shear strain on a surface of the interlayer, $\gamma = \gamma(x,\rho,t)$
G_1, G_2	–	Kirchhoff's moduli of shafts I and II

G	–	Kirchhoff's moduli of the interlayer
b	–	viscosity coefficient of the interlayer
E_1, E_2	–	Young's moduli of shafts for I and II
E	–	Young's modulus of the interlayer
c	–	damping coefficient of the interlayer
k	–	elasticity coefficient of the interlayer
ρ_1, ρ_2	–	mass density of the material of shafts I and II per unit length
I_{01}, I_{02}	–	polar cross-section moments of inertia of shafts I and II
l	–	length of shafts I and II
x	–	longitudinal axis of shafts I and II
t	–	time.

1. Introduction

Complex torsional systems coupled together by viscoelastic constraints play an important role in various engineering and building structures. Vibration analysis of laminated layer elements such as plates, shells, beams and shafts have been presented by Kurnik and Tylikowski (1997). Application of piezoelectric vibration dampers in various elements have been discussed by Tylikowski (1999), Przybyłowicz (1995).

Vibration analysis of complex structural systems with damping is a difficult problem. In the above complex cases, especially where viscous and discrete elements occur, it is recommended to adopt a method of solving the dynamic problem of the given system in the real domain of a variable complex function.

The property of orthogonality of free vibration complex modes in discrete systems with damping was first presented by Tse et al. (1978), in discrete-continuous systems with damping by Nizioł and Snamina (1990) and in continuous systems with damping by Cabańska-Płaczkiewicz (1998, 1999a,b), Cabańska-Płaczkiewicz and Pankratova (1999).

Dynamic analysis of discrete-continuous complex torsional systems with damping were also presented in the papers by Bogacz and Szolc (1993), Nadolski (1994), Pielorz (1995), Kasprzyk (1996).

In the papers by Cabańska-Płaczkiewicz (1998, 1999a,b), an analytical method of solving the free vibration problem of continuous one- and two-dimensional sandwich systems with damping, with manifold boundary conditions and different initial conditions was presented.

The aim of this paper is to conduct a dynamic analysis of free and forced vibration problems of a continuous torsional sandwich circular shaft with damping in the interlayer, in which the outer layers are made of an elastic material, while the internal one possesses some viscoelastic properties and is a soft and light structure.

2. Formulation of the problem

2.1. Physical model of the system

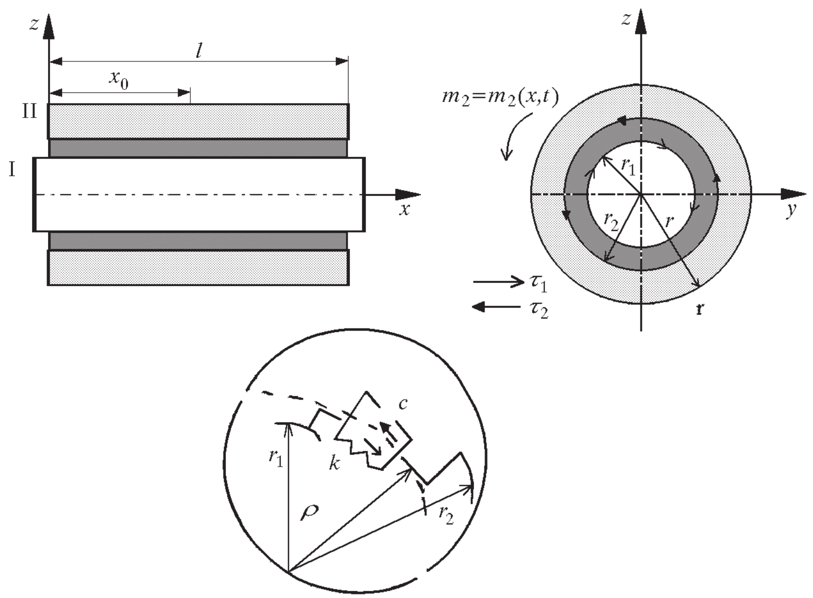

Fig. 1. Model of torsional vibration of the sandwich shaft with damping in the interlayer

The sandwich system consists of an internal solid shaft I, and outer ring roller II, coupled together by a viscoelastic ring-shaped interlayer (Fig. 1). Internal and outer layers I and II are made of a homogeneous, elastic material. The viscoelastic interlayer is made of a light soft material with circumferential characteristic. A shearing, which is described by the Voigt-Kelvin model (cf Nowacki, 1972; Osiński, 1979) is observed on the cylindrical surface of the viscoelastic interlayer. It has been assumed that the interlayer does not transfer torsional stresses in the transverse sections. Outer shaft II is subjected to a torque acting at the point $x_0 = 0.5l$, varying in time t, described by the function $m_2 = M_2\delta(x-x_0)\sin(\omega_0 t)$. The load from shaft II on shaft I is transferred through tangential stresses on the cylindrical surface of the interlayer. Deformation of separated segment of the interlayer is shown in Fig. 2.

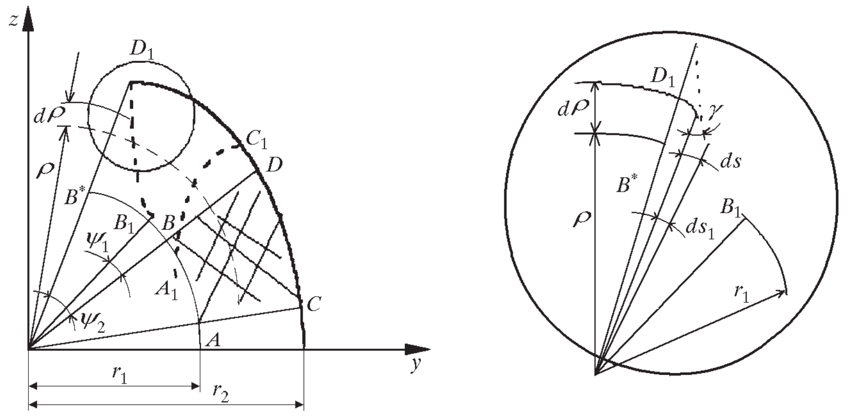

Fig. 2. Deformation of a separated segment of the interlayer

The transfered moment $\mu = 2\pi\rho^2\tau$ occurring in the interlayer takes for $\rho = r_1$, $\tau = \tau_1$ the form $\mu = 2\pi r_1^2\tau_1$, which implies the following relation

$$\tau = \frac{r_1^2}{\rho^2}\tau_1 \tag{2.1}$$

Making use of the constitutive equations of the Voigt-Kelvin model (cf Nowacki, 1972; Osiński, 1979) into Eq. (2.1) we obtain a relationship for the shear strain on the interlayer surface

$$\gamma = \frac{r_1^2}{\rho^2}\gamma_1 \tag{2.2}$$

In order to define the next geometrical relationships a segment of the interlayer has been separated, as presented in Fig. 2, and then a deformation of this interlayer shown.

Having transformed the absolute shear strain $ds = \gamma d\rho$ on the cylindrical surface of the internal shaft we obtain

$$ds_1 = \frac{r_1^3}{\rho^3}\gamma_1 d\rho \tag{2.3}$$

The arc length B^*B_1 (Fig. 2) has been denoted by Δs_1. Then, the geometrical dependence has been determined

$$\Delta s_1 = (\psi_2 - \psi_1)r_1 \tag{2.4}$$

Integrating Eq. (2.3) within the limits from r_1 to r_2, the following form has been obtained

$$\varDelta s_1 = \frac{r_1\gamma_1(r_1^2 - r_2^2)}{2r_2^2} \tag{2.5}$$

After comparing Eq. (2.4) and Eq. (2.5), the shear strain of the interlayer has been calculated

$$\gamma_1 = \frac{2(\psi_2 - \psi_1)r_2^2}{r_1^2 - r_2^2} \tag{2.6}$$

After substituting Eq. (2.6) in the constitutive equations of the Voigt-Kelvin, the transfered moment can be rewritten in the following form

$$\mu = \Big(k + c\frac{\partial}{\partial t}\Big)(\psi_1 - \psi_2) \tag{2.7}$$

where

$$k = 4\pi G\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \qquad\qquad c = 4\pi b\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \tag{2.8}$$

2.2. Mathematical description of the model

The phenomenon of torsional vibration of the sandwich shaft with damping in the interlayer is described by the following heterogeneous system of conjugate partial differential equations

$$\begin{aligned} &R_1\frac{\partial^2\psi_1}{\partial x^2} - \varGamma_1\frac{\partial^2\psi_1}{\partial t^2} - \Big(k + c\frac{\partial}{\partial t}\Big)(\psi_1 - \psi_2) = 0 \\ &R_2\frac{\partial^2\psi_2}{\partial x^2} - \varGamma_2\frac{\partial^2\psi_2}{\partial t^2} + \Big(k + c\frac{\partial}{\partial t}\Big)(\psi_1 - \psi_2) = m_2(x,t) \end{aligned} \tag{2.9}$$

where

$$R_i = G_i J_{0i} \qquad\qquad \varGamma_i = \rho_i I_{0i} \qquad\qquad i = 1,2$$

3. Solution to the boundary-value problem

By substituting (3.1) (cf Nowacki, 1972; Tse et al., 1978; Osiński, 1979; Nizioł and Snamina, 1990) to the system of differential equations (2.9), on the assumption that $m_2 = 0$

$$\psi_1 = \varPsi_1(x)\exp(\mathrm{i}\nu t) \qquad\qquad \psi_2 = \varPsi_2(x)\exp(\mathrm{i}\nu t) \tag{3.1}$$

the homogenous system of conjugate ordinary differential equations describing the complex modes of vibration of the shafts is obtained

$$
\begin{aligned}
&\frac{d^2\Psi_1}{dx^2} + R_1^{-1}\Big[(\Gamma_1\nu^2 - k - \mathrm{i}c\nu)\Psi_1 + (k + \mathrm{i}c\nu)\Psi_2\Big] = 0\\
&\frac{d^2\Psi_2}{dx^2} + R_2^{-2}\Big[(\Gamma_2\nu^2 - k - \mathrm{i}c\nu)\Psi_2 + (k + \mathrm{i}c\nu)\Psi_1\Big] = 0
\end{aligned}
\tag{3.2}
$$

where $\Psi_1(x)$, $\Psi_2(x)$ are the complex modes of the free vibration of shafts I and II, and ν is the complex eigenfrequency of the sandwich shaft.

The general solution to the system of differential equations (3.2) has been presented in the following form (cf Cabańska-Płaczkiewicz, 1998)

$$
\begin{aligned}
\Psi_1(x) &= \sum_{\upsilon=1}^{2} A_\upsilon^* \sin\lambda_\upsilon x + A_\upsilon^{**}\cos\lambda_\upsilon x\\
\Psi_2(x) &= \sum_{\upsilon=1}^{2} a_\upsilon(A_\upsilon^* \sin\lambda_\upsilon x + A_\upsilon^{**}\cos\lambda_\upsilon x)
\end{aligned}
\tag{3.3}
$$

where λ_υ are parameters describing the roots of the characteristic equation, a_υ are coefficients of amplitudes (cf Cabańska-Płaczkiewicz, 1998), and A_υ^*, A_υ^{**} are integration constants.

In order to solve the boundary value problem, the following boundary conditions are applied

$$
\Psi_1(0) = \Psi_1(l) = \Psi_2(0) = \Psi_2(l) = 0 \tag{3.4}
$$

The following frequency equation of the free vibration has been obtained

$$
\begin{aligned}
&\nu^4 - \Big[(R_1\lambda_s^2 + k + \mathrm{i}c\nu)\Gamma_1^{-1} + (R_2\lambda_s^2 + k + \mathrm{i}c\nu)\Gamma_2^{-1}\Big]\nu^2 +\\
&+\lambda_s^2\Big[R_1R_2\lambda_s^2 + (k + \mathrm{i}c\nu)(R_1 + R_2)\Big](\Gamma_1\Gamma_2)^{-1} = 0
\end{aligned}
\tag{3.5}
$$

from which a sequence of complex eigenfrequencies was determined

$$
\nu_n = \mathrm{i}\eta_n \pm \omega_n \tag{3.6}
$$

where

$$
\lambda_s = \frac{s\pi}{l} \qquad n = 2s - \delta_{n,(2s-1)} \qquad \eta = \frac{c}{2\Gamma_p} \qquad s = 1, 2, ...
$$

and $\delta_{n,(2s-1)}$ is the Kronecker number.

The coefficients of amplitudes have been found as

$$a_n = \frac{R_1\lambda_s^2 - \Gamma_1\nu_n^2 + k + \mathrm{i}c\nu_n}{k + \mathrm{i}c\nu_n} = \frac{k + \mathrm{i}c\nu_n}{R_2\lambda_s^2 - \Gamma_2\nu_n^2 + k + \mathrm{i}c\nu_n} \tag{3.7}$$

By incorporating the sequences of λ_s and a_n to Eqs (3.3), the two following sequences of modes of the free vibration for the two shafts have been obtained

$$\Psi_{1n}(x) = \sin\lambda_s x \qquad\qquad \Psi_{2n}(x) = a_n \sin\lambda_s x \tag{3.8}$$

4. Solution to the initial value problem

The complex equation of motion, when $\nu = \nu_n$ has the following form

$$T_n = \Phi_n \exp(\mathrm{i}\nu_n t) \tag{4.1}$$

where Φ_n denote Fourier's coefficients.

The free vibration of the shafts is presented in the form of Fourier's series, based on the complex eigenfunctions, i.e.

$$\psi_s(x,t) = \sum_{n=1}^{\infty} \Psi_{sn}\Phi_n \exp(\mathrm{i}\nu_n t) \qquad\qquad s = 1,2 \tag{4.2}$$

From the system of Eqs (3.2), after making some algebraic transformations, adding the equations together, and then integrating both sides within the limits from 0 to l, the property of orthogonality of the eigenfunctions is obtained (cf Cabańska-Płaczkiewicz, 1998, 1999)

$$\int\limits_0^l \mathrm{i}(\nu_n + \nu_m)(\Gamma_1\Psi_{1n}\Psi_{1m} + \Gamma_2\Psi_{2n}\Psi_{2m}) + c(\Psi_{1n} - \Psi_{2m})\;dx = N_n\delta_{nm} \tag{4.3}$$

where δ_{nm} is Kronecker's delta and

$$N_n = 2\int\limits_0^l \Big[2\mathrm{i}\nu_n(\Gamma_1\Psi_{1n}^2 + \Gamma_2\Psi_{2n}^2 + c(\Psi_{1n} - \Psi_{2m})^2\Big]\;dx \tag{4.4}$$

The problem of the free vibration of the shafts is solved by application of the following conditions

$$\begin{aligned} \psi_1(x,0) &= \psi_{01} \quad & \psi_2(x,0) &= \psi_{02} \\ \dot{\psi}_1(x,0) &= \dot{\psi}_{01} \quad & \dot{\psi}_2(x,0) &= \dot{\psi}_{02} \end{aligned} \tag{4.5}$$

Applying conditions (4.5) in series (4.2) and taking into consideration the property of orthogonality (4.3), the following formula for Fourier's coefficients is obtained

$$\begin{aligned}\varPhi_n &= \frac{1}{N_n}\int\limits_0^l \Big[\varGamma_1(\mathrm{i}\nu_n\varPsi_{1n}\psi_{01}+\varPsi_{1n}\dot{\psi}_{01})+ \\ &+ \varGamma_2(\mathrm{i}\nu_n\varPsi_{2n}\psi_{02}+\varPsi_{2n}\dot{\psi}_{02})+c(\varPsi_{1n}-\varPsi_{2n})(\psi_{01}-\psi_{02})\Big]\;dx\end{aligned} \tag{4.6}$$

Substituting Eqs (3.8), (4.1) and Eq. (4.6) to Eqs (4.2) and performing trigonometrical and algebraical transformations, the final form of free vibration of the sandwich shaft with damping in the interlayer is obtained

$$\psi_s=\sum_{n=1}^{\infty}\mathrm{e}^{-\eta_n t}|\varPhi_n||\varPsi_{sn}|\cos(\omega_n t+\varphi_n+\chi_{sn}) \qquad s=1,2 \tag{4.7}$$

where

$$\begin{array}{lll} |\varPsi_{sn}|=\sqrt{X_{sn}^2+Y_{sn}^2} & \chi_{sn}=\arg\varPsi_{sn} & s=1,2 \\ |\varPhi_n|=\sqrt{C_n^2+D_n^2} & \varphi_n=\arg\varPhi_n & \end{array}$$

and

$$\begin{array}{lll} X_{sn}=\mathrm{re}\,\varPsi_{sn} & Y_{sn}=\mathrm{im}\,\varPsi_{sn} & s=1,2 \\ C_n=\mathrm{re}\,\varPhi_n & D_n=\mathrm{im}\,\varPhi_n & \end{array}$$

5. Solution to the forced vibration problem

In the case when $\nu=\omega_0$ (Eq. 3.3)

$$\lambda_1\neq\lambda_2 \qquad\qquad a_1\neq a_2 \tag{5.1}$$

where ω_0 is the frequency of the real stationary forced vibration.

After incorporating of Eqs (3.4), (5.1) to Eqs (3.3), the general solution to the system of ordinary differential equations (3.2) in the following matrix form is obtained

$$\boldsymbol{\varPsi}^*(x)=\left[\begin{array}{c} A_1\sin\lambda_1 x+A_2\sin\lambda_2 x \\ a_1A_1\sin\lambda_1 x+a_2A_2\sin\lambda_2 x\end{array}\right] \tag{5.2}$$

The particular solutions, using the operator method (cf Osiowski, 1972), are derived.

The system of equations (2.9) (after elimination of time) in a matrix form is as follows

$$\mathbf{C}_2 \frac{d^2\boldsymbol{\psi}}{dx^2} + \mathbf{C}_0\boldsymbol{\psi} = \boldsymbol{f}(x) \tag{5.3}$$

where

$$\boldsymbol{\psi}(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} \qquad \boldsymbol{f}(x) = \begin{bmatrix} 0 \\ R_2^{-1}M_2\delta(x-x_0) \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.4}$$

$$\mathbf{C}_0 = \begin{bmatrix} R_1^{-1}(\varGamma_1\omega_0^2 - k - \mathrm{i}c\omega_0) & R_1^{-1}(k + \mathrm{i}c\omega_0) \\ R_2^{-1}(k + \mathrm{i}c\omega_0) & R_2^{-1}(\varGamma_2\omega_0^2 - k - \mathrm{i}c\omega_0) \end{bmatrix}$$

where $\delta(x-x_0)$ is the Dirac delta function.

The system of Eqs (5.3) is a normal system, because

$$\det \mathbf{C}_2 \neq 0 \tag{5.5}$$

Using Laplace's transformations to Eqs (5.3), the operational equation is obtained

$$\mathbf{G}(s) \times \mathbf{Y}(s) = \mathbf{F}(s) + \mathbf{L}(s) \tag{5.6}$$

where

$$\mathbf{G}(s) = \mathbf{C}_2 s^2 + \mathbf{C}_0 \tag{5.7}$$

is the characteristic matrix, $\mathbf{Y}(s)$ and $\mathbf{F}(s)$ – matrix transforms, and $\boldsymbol{\psi}^{**}(s) = \mathbf{Y}(s)$ on the assumption that the volume matrix $\mathbf{L}(s) \equiv \mathbf{0}$.

Equations (5.6) can be written in the following form

$$\mathbf{Y}(s) = \mathbf{G}^{-1}(s)[\mathbf{F}(s) + \mathbf{L}(s)] \tag{5.8}$$

where

$$\mathbf{G}(s) = \begin{bmatrix} s^2 + R_1^{-1}(\varGamma_1\omega_0^2 - k - \mathrm{i}c\omega_0) & R_1^{-1}(k + \mathrm{i}c\omega_0) \\ R_2^{-1}(k + \mathrm{i}c\omega_0) & s^2 + R_2^{-1}(\varGamma_2\omega_0^2 - k - \mathrm{i}c\omega_0) \end{bmatrix} \tag{5.9}$$

and

$$\mathbf{K}(s) = \mathbf{G}^{-1}(s) \tag{5.10}$$

Using inverse Laplace's transformation $\mathcal{L}^{-1}$ of Eq. (5.6), the particular solutions in the matrix form are obtained

$$\boldsymbol{y}(x) = \mathbf{k}(x)\boldsymbol{f}(x) \tag{5.11}$$

where

$$\mathbf{k}(x) = \mathcal{L}^{-1}\mathbf{K}(s) \tag{5.12}$$

The elements of matrix (5.12) are described in the form of

$$\mathbf{k}(x) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{5.13}$$

After substituting Eq. (5.13) to Eq. (5.11), the particular solution of the system of Eqs (5.3) in the matrix form is obtained

$$\boldsymbol{\Psi}^{**}(x) = \begin{bmatrix} \int\limits_0^l M_2 k_{12}(x-\tau)\delta(\tau - x_0)\, d\tau \\ \int\limits_0^l M_2 k_{22}(x-\tau)\delta(\tau - x_0)\, d\tau \end{bmatrix} \tag{5.14}$$

where

$$\boldsymbol{y}(x) = \boldsymbol{\Psi}^{**}(x) \tag{5.15}$$

The modes of the stationary forced vibrations of the two shafts can be written in the form

$$\boldsymbol{\Psi}^(x) = \boldsymbol{\Psi}^{*}(x) + \boldsymbol{\Psi}^{**}(x) \tag{5.16}$$

The steady-state forced vibration of the sandwich shaft is

$$\boldsymbol{\psi}(x,t) = \boldsymbol{\Psi}(x)\exp(\mathrm{i}\omega_0 t) \tag{5.17}$$

Substituting Eq. (5.16) to Eq. (5.17) and making trigonometric and algebraic transformations, the forced vibration of the sandwich shaft with damping in the interlayer is obtained

$$\psi_s = |\Psi_s|\sin(\omega_0 t + \chi_s) \qquad s = 1,2 \tag{5.18}$$

where

$$\begin{aligned} |\Psi_s| &= \sqrt{X_s^2 + Y_s^2} & \chi_s &= \arg\Psi_s \\ X_s &= \mathrm{re}\,\Psi_s & Y_s &= \mathrm{im}\,\Psi_s \quad s = 1,2 \end{aligned}$$

and $|\Psi_1|$, $|\Psi_2|$ – amplitudes of shafts I and II.

6. Calculations

Calculations are carried out for the following data

$$
\begin{array}{lll}
E_1 = E_2 = 2.1 \cdot 10^{11}\ \mathrm{Nm^{-2}} & E = 10^7 \mathrm{Nm^{-2}} & r_1 = 0.02\ \mathrm{m} \\
\rho_1 = \rho_2 = 7.8 \cdot 10^3\ \mathrm{Ns^2m^{-4}} & r_2 = 0.05\ \mathrm{m} & r = 0.06\ \mathrm{m} \\
M_2 = 4000\ \mathrm{Nm} & \nu_0 = 0.2 & l = 5\ \mathrm{m} \\
c = 2.5\ \mathrm{Ns} & k = 2.5 \cdot 10^4\ \mathrm{N} & x_0 = 0.5l
\end{array}
$$

Constants occurring in Eqs (5.2) are described in the following forms

$$
A_1 = \frac{M_2 \sin \lambda_1 (l - x_0)}{R_2 \lambda_1 \sin \lambda_1 l} \qquad\qquad A_2 = -\frac{M_2 \sin \lambda_2 (l - x_0)}{R_2 \lambda_2 \sin \lambda_2 l} \tag{6.1}
$$

Tables 1-3 present values of the complex eigenfrequencies $\nu_n = \mathrm{i}\eta_n \pm \omega_n$ for $s = 1, 2, 3$. The effects of the damping coefficients for $c = 2.5\,\mathrm{Ns}$ (Table 1), $c = 5\,\mathrm{Ns}$ (Table 2) and $c = 7\,\mathrm{Ns}$ (Table 3) on the system frequencies are shown. The investigation of the complex eigenfrequencies of the sandwich shaft has shown, that in the case, when $s = 1$ $(n = 1)$, $s = 2$ $(n = 3)$, $s = 3$ $(n = 5)$, the real parts ω_n of the complex eigenfrequencies for the damping coefficients: $c = 2.5\,\mathrm{Ns}$ (Table 1), $c = 5\,\mathrm{Ns}$ (Table 2) and $c = 25\,\mathrm{Ns}$ (Table 3) do not change. In the case, when $s = 1$ $(n = 2)$, $s = 2$ $(n = 4)$, $s = 3$ $(n = 6)$, the real parts ω_n of the complex eigenfrequencies for the damping coefficient $c = 2.5\,\mathrm{Ns}$ (Table 1) are correspondingly larger than for the damping coefficient $c = 5\,\mathrm{Ns}$ (Table 2) and $c = 25\,\mathrm{Ns}$ (Table 3). In the case, when $s = 1$ $(n = 2)$, $s = 2$ $(n = 4)$, $s = 3$ $(n = 6)$, the imaginary parts η_n of the complex eigenfrequencies for the damping coefficient $c = 2.5\,\mathrm{Ns}$ (Table 1) are correspondingly smaller than for the damping coefficient $c = 5\,\mathrm{Ns}$ (Table 2) and $c = 25\,\mathrm{Ns}$ (Table 3).

Table 1. Complex eigenfrequencies ν_n for $c = 2.5\,\mathrm{Ns}$ and $s = 1, 2, 3$

s	ν_n $(n = 1, ..., 6)$	
1	$\nu_1 = \pm 2104.44$	$\nu_2 = \pm 4126.35 + 651.102\mathrm{i}$
2	$\nu_3 = \pm 4208.88$	$\nu_4 = \pm 5505.71 + 651.102\mathrm{i}$
3	$\nu_5 = \pm 6313.32$	$\nu_6 = \pm 7242.66 + 651.102\mathrm{i}$

Table 2. Complex eigenfrequencies ν_n for $c = 5\,\mathrm{Ns}$ and $s = 1, 2, 3$

s	ν_n $(n = 1, ..., 6)$	
1	$\nu_1 = \pm 2104.44$	$\nu_2 = \pm 3969.25 + 1302.2\mathrm{i}$
2	$\nu_3 = \pm 4208.88$	$\nu_4 = \pm 5388.97 + 1302.2\mathrm{i}$
3	$\nu_5 = \pm 6313.32$	$\nu_6 = \pm 7154.32 + 1302.2\mathrm{i}$

Table 3. Complex eigenfrequencies ν_n for $c = 25\,\mathrm{Ns}$ and $s = 1, 2, 3$

s	ν_n $(n = 1, ..., 6)$	
1	$\nu_1 = \pm 2104.44$	$\nu_2 = \pm 0 + 11505.3\mathrm{i}$
2	$\nu_3 = \pm 4208.88$	$\nu_4 = \pm 0 + 9925.21\mathrm{i}$
3	$\nu_5 = \pm 6313.32$	$\nu_6 = \pm 3238.31 + 6511.02\mathrm{i}$

After analysing the results shown in Tables 1-3 we state that:

- a decrease in the real parts ω_n of the complex eigenfrequencies corresponds to a larger period of damped vibration of shafts I and II
- an increase in the imaginary parts η_n of the complex eigenfrequencies corresponds to smaller amplitudes (damping decrement) of shafts I and II, Eq. (4.7).

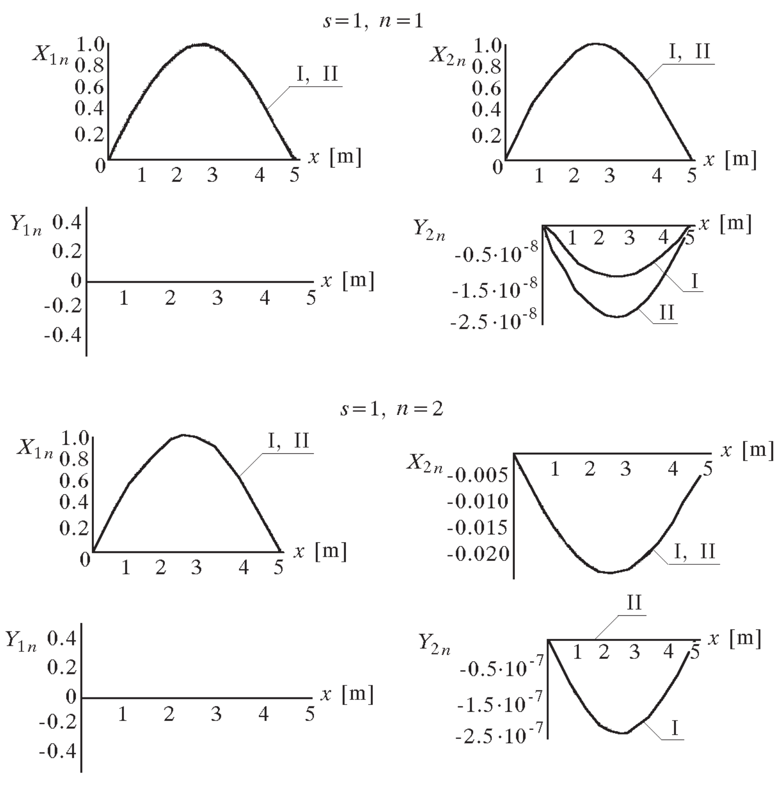

Fig. 3. Complex modes of free vibrations of shafts I and II for $s = 1$, $(n = 1, 2)$ – variants I, II

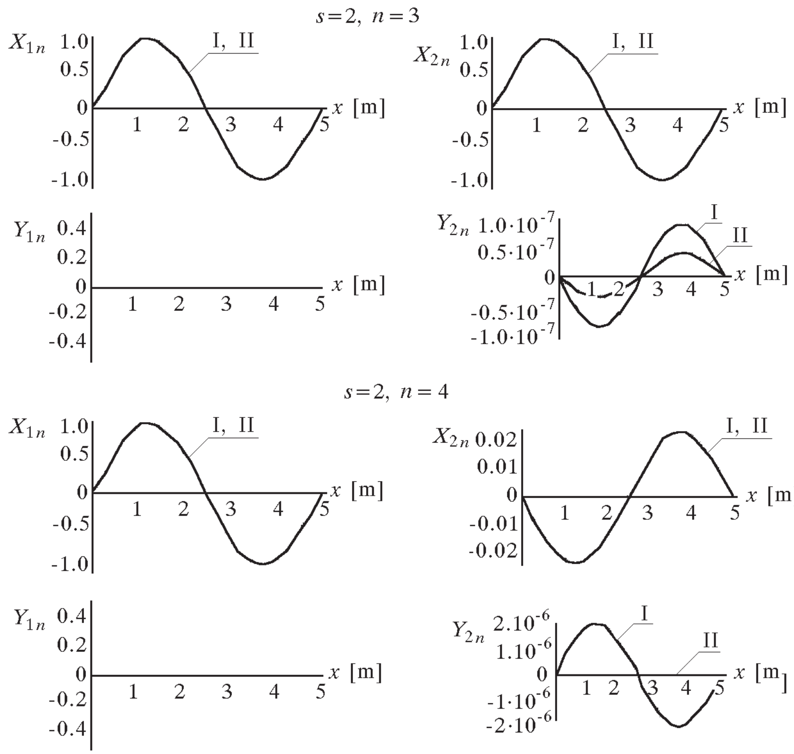

Fig. 4. Complex modes of free vibrations of shafts I and II for $s = 2$, $(n = 3, 4)$ – variants I, II

The influence of the complex eigenfrequencies for small damping coefficient $c = 2.5\,\mathrm{Ns}$ (Table 1), and for large damping coefficient $c = 25\,\mathrm{Ns}$ (Table 3) on the complex modes of free vibrations is illustrated in the Fig. 3 and Fig. 4 (variants I, II). The complex modes of the free vibrations for the eigenfrequencies presented in Table 1 are given in variant I, for the eigenfrequencies presented in Table 3 – given in variant II. The diagrams in Fig. 3 and Fig. 4 show the complex modes of the free vibrations for shafts I and II. The results are given for $s = 1$ $(n = 1, 2)$ – Fig. 3, $s = 2$ $(n = 3, 4)$ – Fig. 4. The complex modes for the real eigenfrequencies $\nu_n = \pm\omega_n$ $(n = 1, 3)$ – Tables 1-3 have synchronous character (Fig. 3-4). In the case of higher complex eigenfrequencies $\nu_n = \mathrm{i}\eta_n \pm \omega_n$ $(n = 2, 4)$ – Tables 1-3, the modes have asynchronous character (Fig. 3-4).

Figure 5 shows the complex modes of the stationary forced vibration of the torsional sandwich circular shaft with damping in the interlayer for the set of

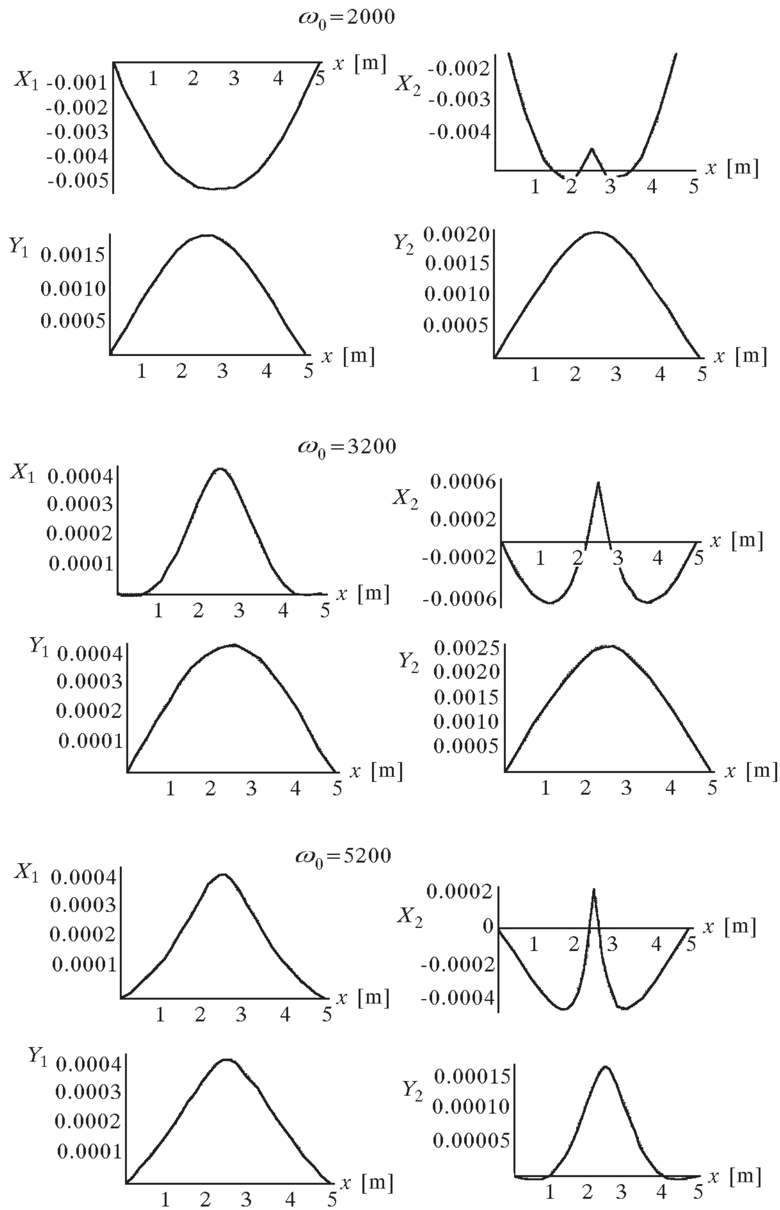

Fig. 5. Complex modes of stationary forced vibrations of shafts I and II

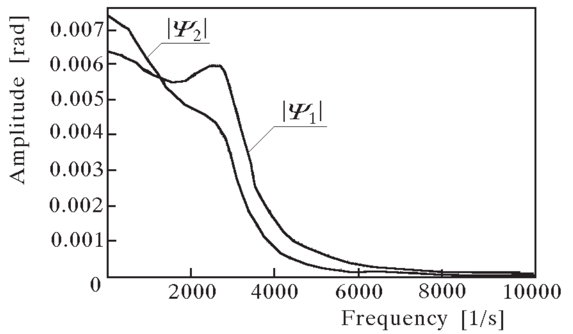

Fig. 6. Amplitude-frequency characteristic of torsional vibration of the sandwich shaft with damping in the interlayer at the point $x_0 = 0.5l$

real frequencies $\omega_0 = \{2000, 3200, 5200\}$. Figure 6 presents the amplitude-frequency diagrams of the torsional sandwich shaft with damping in the interlayer. The amplitude-frequency diagrams show changes of the amplitudes $|\Psi_1|$ and $|\Psi_2|$ of shafts I and II for real stationary frequencies in the range of $0 < \omega_0 < 10000$. After analysing the results presented in Fig. 6 we state, that internal shaft I can be a dynamic vibration damper for outer shaft II, which is subjected to a torque acting at the point $x_0 = 0.5l$, varying in time, described by the function $m_2 = M_2\delta(x - x_0)\sin(\omega_0 t)$.

In this paper the system of two conjugate differential equations (2.9) of torsional vibration of the system of two shafts coupled by a viscoelastic soft light interlayer is solved. The viscoelastic interlayer is made of a light soft material with a circumferential characteristic. The obtained solution can be applicable to systems with interlayers of a small thickness, as well as small damping for variant I and large damping for variant II.

7. Conclusions

- Complex modes of vibration and the property of orthogonality of these modes have been presented in this paper. They a basis for solving the free and forced vibration problems of the torsional sandwich circular shaft with damping in the interlayer.

- The method presented in this paper is correct for small and large damping. In the case of small damping, the free vibrations have periodic character. For large (critical) damping, the lower components of the free vibrations have non-periodic character.

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Drgania skrętne sandwiczowego wału z tłumieniem w przekładce

Streszczenie

W pracy przedstawiono analityczną metodę rozwiązywania problemu drgań skrętnych sandwiczowego okrągłego wału z lepko-sprężystą, miękką i lekką przekładką. Współczynnik sztywności i tłumienia przekładki uzależniono od jej cech geometrycznych oraz od lepko-sprężystych własności materiału przekładki. W rozwiązaniu zagadnienia drgań swobodnych i wymuszonych zastosowano funkcje zespolone zmiennej rzeczywistej. Następnie wykazano własność ortogonalności zespolonych postaci drgań własnych, która jest podstawą rozwiązania zagadnienia drgań własnych przy dowolnych warunkach początkowych. Rozwiązanie zagadnienia rzeczywistych ustalonych wymuszonych drgań otrzymano za pośrednictwem zespolonych ustalonych postaci drgań.

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