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VARIATIONAL DESCRIPTIONS OF WEARING OUT SOLIDS AND WEAR PARTICLES IN CONTACT MECHANICS

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Experimental studies demonstrated the central role played by wear particles in sliding contacts. In this contribution, the classical variational formulation of contact problems is extended to wearing out solids by including the interfacial layer of wear debris between contacting bodies. Variational formulations are developed starting from strong forms of governing relations for the wearing solids and the layer. An implementation of a finite element representation of the contact problem is presented. The Lagrange multiplier method is applied to determine contact forces and to satisfy kinematic contact constraints. Discretized forms of the variational functionals lead to equations which can be useful in numerical analysis.

Key words: contact mechanics, friction, wear, variational methods

1. Introduction

Analysis of the interaction problem of solid bodies generally requires determination of stresses and strains within the individual contacting bodies, together with the information about distribution of displacements and stresses in the contact region, for the given boundary and initial conditions. An effective way to solve the contact problems is given by variational formulations and approximations with the aid of the finite element method, see Telega (1988), Szefer et al. (1994), Wriggers (1995), Stupkiewicz and Mróz (1999).

Easily observable results of a wear process can be identified as an increase of the gap between contacting bodies (worn contours) and as formation and circulation of the wear debris in contact interfaces. Wear particles trapped between two surfaces can be a source of significant mechanical phenomena. Development of a thin, uniform and almost continuous transfer film (intermediate layer) of the wear debris between contacting bodies belongs to the most important phenomena resulting from the wear process. The thin layer between the bodies (called sometimes the third body) is treated as a continuum consisting of solid particles with mechanical properties different from the mating bodies.

In the published literature, various continuum mechanics-based models for the third body have been proposed. Godet (1988) suggested modelling the third body (i.e. wear debris or a solid powder lubricant film) as a viscous fluid film. In Godet's opinion the load carrying mechanism can exist within the third body, as it occurs in the hydrodynamic fluid lubrication. Elrod (1992), Berker and Van Arsdale (1992) proposed phenomenological models for the third body materials taking into account granular media. In order to specialize the mechanics of granular media to the third body, it can be assumed that the interfacial layer consists of isolated discrete particles (spherical in shape, similar in size, non-cohesive and incompressible). Hou et al. (1997) proposed an elasto-plastic rheological model for the third body consisting of fine oxidized wear debris and contaminants. It was assumed that the wear and other debris, under traction and extreme pressure, form a uniform solid incompressible layer. The shear stresses in the layer are elastic below a critical shear stress and plastic above it. In the studies by Zmitrowicz (1987), the effect of wear particles was considered in the form of a two-dimensional interfacial layer. Various constitutive models have been analysed for the interface layer: micropolar thermoelastic, micropolar fluid and thermoviscous fluid. Stupkiewicz and Mróz (1999) modified the classical wear law taking into account the third body (oxides, wear particles) between tool and workpiece surfaces in metal forming processes.

In the paper by Szefer et al. (1994), the interface layer was assumed to be bonded to contacting bodies. Problems of bonding of two solids by a thin adhesive layer have been extensively studied in the literature, see e.g. Ganghoffer et al. (1997). A typical model contains two bodies joined by a thin layer, i.e. the thin layer adheres to the bodies.

The aim of this paper is to present a complete set of governing relations describing solids subject to wear process. We start with a local strong formulation of the contact problem being based on differential equations, then we pass to variational forms of the problem and to finite element approximations.

2. Experimental observations of wear particles

Friction process of elements operating in contact conditions always involves frictional heat and wear of their surfaces. Wear is often defined as gradual removal of the material from contacting and rubbing surfaces of solids during their relative sliding. Taking into account the amount of the removed material from the body, the measures of wear have been formulated with respect to changes of the following quantities: mass, volume and dimensions of the body (e.g. depth of the removed material). Wear of machine component parts and tools is often identified with irreversible changes of the shapes and sizes of rubbing elements (so called wear profiles). A significant increase of stresses have been observed in worn profiles of the elements.

Many mechanical, physical and chemical phenomena are responsible for wear of materials. Several types of wear have been recognized: adhesive, abrasive, contact fatigue, fretting, oxidation, corrosion, erosion. In machinery, wear occurs most frequently as the abrasive or contact fatigue process. Most of the wear processes are the consequences of the interfacial rubbing process. Wear in sliding systems is usually a very slow process, but it is steady and continuous.

The removal of the material takes the form of small particles which may come off in loose form. The wear particles are detached from solids during sliding by the following microscopic mechanisms: microcutting of adhesive junctions between surfaces, mechanical failure of contacting asperities, surface spalling, plastic deformations of the surfaces in a form of grooves and scratches, nucleation and propagation of the surface and subsurface cracks and voids, etc.

There are numerous studies in the field of wear revealing details about the size and nature of wear particles, and a large body of empirical data collected, see Dowson et al. (1992), Zanoria et al. (1995). The shape, size and number are the main parameters characterizing *morphological properties* of the wear particles. For example, when two metal surfaces slide against each other under a load, the wear debris is produced in the form of both macroscopic (size from a few to several micrometers) and microscopic (size from submicrometer to a few micrometers) particles. It was found that the number of macroscopic wear particles was small and the number of microscopic debris was great. The generated wear debris can assume different shapes (flakes, chips, thin platelets, filings, powder-like particles).

The entrapped wear particles remain within the wear track, and they become a part of the interfacial medium. Some of them are processed further e.g. crushed into finer particles, compacted and agglomerated into large debris. It is often observed that the wear debris in the form of flakes and chips etc. can be "rolled over" into balls, cylinders or needles. It takes place especially, when the contacting bodies are in oscillating sliding and when the number of sliding cycles increases. For instance, large cylindrical rolls are formed when using a rubber eraser.

Roll-like wear debris are created by rubbing forces. Each roll is subjected to opposing tangential forces at its top and bottom surfaces. The torque resulting from these forces makes the debris roll. The axes of the roll particles are aligned perpendicularly to the direction of sliding, indicating that they are actually rolling on the wear track (or that they are generated by rolling in the contact), and suggesting that the particle circulation is common in the contact. The rolls can grow as snow balls by collecting more and more finer wear particles.

A particularly intriguing type of a wear particle, often observed and subject to much interest, is the spherically shaped wear particle (Dowson et al., 1992). The spherical particles ranging in the size from 1 up to 80 μm have been observed. Small (less then 1 μm) spherical particles have also been seen by some investigators. Rolls and spheres were obtained in a very wide range of running conditions, and for very different materials (steels, ceramics, polymers, etc.).

The material which was removed from surfaces during the course of a wear process is not in fact the same either structurally or chemically, as the parent material. Instead, it is very fine-grained material which may be derived from both parts of the contacting system and may include contaminants from the environment as well. *Physical properties* of the wear particles include composition, microstructure, density, thermal expansion, thermal conductivity, melting temperature, etc. Depending on the type of the bulk material different kinds of wear particles are created: metals, metal oxides, alloys, plastics, ceramic materials, etc.

Formation of loose particles is the key of a wear process. Wear debris generally accompanies wear of any sliding system. Practical importance of wear particles in mechanical devices depends on the sliding system – it is small for an automobile tire rolling on a road; it is important for bearings and for knee and hip joints prostheses. Numerous experimental investigations and everyday practice indicate the formation of a wear debris layer on the sliding interface (i.e. an intermediate layer between the sliding surfaces) practically immediately after the rubbing process starts, transforming the contact of two bodies into a contact of two bodies with the interfacial layer. The surfaces slide on each other being separated by particles of the wear debris.

The particles can slide and roll in contacting surfaces and in relation to each other. In the case of the wear debris layer, the friction occurs partially between the surfaces of the contacting bodies, partially between the surfaces of each body and the wear particles, and inside the layer between the wear debris. The resistance to rolling (i.e. rolling friction) follows from rotations of single particles with respect to the neighboring particles.

Friction phenomenon is very sensitive to changes of sliding conditions. The wear particles entrapped between the sliding surfaces can significantly affect the frictional and wear behavior. The presence of debris implies modification of the friction coefficient and wear rate. The friction coefficient increases when the particles are accumulated and decreases when the particles are removed from the sliding interface. The steady-state wear rate may be larger in some sliding conditions when the wear particles cannot be removed from the contact area and act as abrasive particles. It has also been reported that as a result of formation of spherical and cylindrical particles, the coefficient of friction undergoes transition to lower values up to a factor of three and the wear decreases by several orders of magnitude (Zanoria et al., 1995). It has been suggested that the cylindrical debris can act as miniature roller bearings so that the sliding friction can be reduced when this debris forms.

Wear particles will be active in a rubbing system until they are removed from the contact. The ejection of particles from both the contact and the wear track (sliding path) is predominant in most situations. It can be helpful to eliminate the wear debris from a sliding system. Sometimes, wear particles are removed from the sliding interface by brushing off (the case of lubricated or open mechanical devices). Sometimes this can be done by filtering a circulating lubricant.

Complete separation with third bodies is not a rare occurrence. An interfacial layer can also be formed by hard particles and contaminants entrapped into a sliding system from the operating environment (e.g. airborne debris of dust and sand, combustion products such as fly ash, corrosion scale) or solid body particles specially introduced into the sliding interface with respect to their beneficial role during the friction process (e.g. typical lamellar solid lubricants such as carbon graphite powder, molybdenum disulphide, PTFE) or polishing slurry composed of hard abrasive particles (diamond, silicone carbide, alumina) in material-finishing operations.

A considerable effort has been made to characterize the *mechanical properties* of the third bodies and how they become deformed in the rubbing contact. The transfer material and debris trapped in the interface deform when subjected to high compressive and shear stresses. The stiffness of the third body can influence the deformation of the rubbing system. The wear debris give rise to a unique stress pattern and deformation behavior in the regions near the contact surface.

The layer has certain characteristic properties which depend upon the physical and mechanical properties of the wear particles and the layer as a whole. Identification and understanding of the third body mechanical behaviour is difficult since it requires direct observations of the interface during the wear process, i.e. in real working conditions. It might be some promise in treating the third body as a single continuum. For this purpose, one needs some equivalent but inscrutable properties to characterize the solid third body particles. Then, typical strength tests should be done, i.e. investigating the strength with respect to: tension, compression, shear, bending, torsion, creep, fatigue, hardness. Such fundamental tests have not been undertaken yet for the wear debris layer.

3. Governing equations

3.1. Wearing out of solids

The presence of a thin layer of the wear debris separating contacting solids is an extremely important topic of the present analysis. Therefore, the model of rubbing and wearing out solids consists of two contacting solids A, B and the interfacial layer S of wear debris. Elastic materials and linearized theories are taken into account.

Motion of the wearing body A is governed by the following equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma}_A - \widehat{\boldsymbol{f}}_A = \boldsymbol{0} \qquad \text{in } V_A \tag{3.1}$$

where

$\boldsymbol{\sigma}_A$ – Cauchy's stress tensor
$\widehat{\boldsymbol{f}}_A$ – vector of body forces.

The constitutive relations of the elastic body are as follows

$$\boldsymbol{\sigma}_A = \boldsymbol{E}_A \boldsymbol{\varepsilon}_A \qquad\qquad \boldsymbol{\varepsilon}_A(\boldsymbol{u}_A) = \frac{1}{2}(\operatorname{grad} \boldsymbol{u}_A + \operatorname{grad}^\top \boldsymbol{u}_A) \tag{3.2}$$

where

$\boldsymbol{E}_A$ – tensor of elasticity
$\boldsymbol{\varepsilon}_A$ – (right) Cauchy-Green's strain tensor
$\boldsymbol{u}_A(\boldsymbol{x}_A)$ – displacement function, $\boldsymbol{x}_A \in V_A$.

The boundary conditions for the body A are given by

$$\begin{aligned}
&\boldsymbol{u}_A(\boldsymbol{x}_A) = \widehat{\boldsymbol{u}}_A(\boldsymbol{x}_A) && \text{on} \quad \Omega_{uA} \\
&\boldsymbol{\sigma}_A \boldsymbol{n}_A = \widehat{\boldsymbol{q}}_A && \text{on} \quad \Omega_{qA} \\
&\boldsymbol{\sigma}_A \boldsymbol{n}^+ = \boldsymbol{p}_A && \text{on} \quad \Omega_{cA}
\end{aligned} \tag{3.3}$$

where

$\boldsymbol{n}_A$	–	unit vector normal to the body boundary out of the contact region
$\boldsymbol{n}^+$	–	unit vector normal to the boundary of the body A in the region of contact
$\widehat{\boldsymbol{u}}_A, \widehat{\boldsymbol{q}}_A$	–	displacements and loads
$\boldsymbol{p}_A$	–	contact stresses
Ω_{cA}	–	contact region of the body A.

The contact conditions with respect to the contact stresses (contact tractions) are following

$$
\begin{aligned}
& p_n^A \leqslant 0 && \text{on} \quad \Omega_{cA} \\
& |\boldsymbol{p}_t^A| \leqslant \mu_A |p_n^A| && \text{on} \quad \Omega_{cA} \quad \text{(stick)} \\
& \boldsymbol{p}_t^A = -\mu_A |p_n^A| \frac{\boldsymbol{V}_{AS}}{|\boldsymbol{V}_{AS}|} && \text{on} \quad \Omega_{cA} \quad \text{(slip)}
\end{aligned}
\tag{3.4}
$$

where

μ_A	–	friction coefficient
$\boldsymbol{p}_t^A, \boldsymbol{p}_n^A$	–	tangential and normal components of the contact stresses, i.e.

$$
\begin{aligned}
& \boldsymbol{p}_t^A = (\mathbf{1} - \boldsymbol{n}^+ \otimes \boldsymbol{n}^+)(\boldsymbol{\sigma}_A \boldsymbol{n}^+) && \text{(stick)} \\
& \boldsymbol{p}_n^A = (\boldsymbol{n}^+ \otimes \boldsymbol{n}^+)(\boldsymbol{\sigma}_A \boldsymbol{n}^+) \equiv p_n^A \boldsymbol{n}^+
\end{aligned}
\tag{3.5}
$$

$\boldsymbol{V}_{AS}$ is the relative sliding velocity between the body A and the layer S

$$
\boldsymbol{V}_{AS} = (\mathbf{1} - \boldsymbol{n}^+ \otimes \boldsymbol{n}^+)(\dot{\boldsymbol{u}}_A - \dot{\boldsymbol{u}}_S) \equiv \dot{\boldsymbol{u}}_t^A - \dot{\boldsymbol{u}}_t^S \tag{3.6}
$$

The power of the friction force is non-positive in any sliding direction, i.e. $\boldsymbol{p}_t^A \boldsymbol{V}_{AS} \leqslant 0, \ \forall \, \boldsymbol{V}_{AS}$. Various friction laws can be included in this formulation (Zmitrowicz, 1998).

The contact conditions with respect to displacements in the contact regions are as follows

$$
d_n = g_n - u_n^A + u_n^+ + u_n^B - u_n^- \geqslant 0 \qquad \text{on} \quad \Omega_{cA} \cup \Omega_{cB} \tag{3.7}
$$

The total clearance gap (d_n) between the two contacting bodies A and B consists of: initial gap (g_n), increase of the gap as a result of the wear process of the bodies A and B (u_n^+, u_n^-) and elastic deformations of the bodies in the normal direction to the contact (u_n^A, u_n^B).

The depth worn out – wear profiles (u_n^+, u_n^-) – are evaluated by the integrals of wear velocities v^+ and v^- for the interval of time $< t_0, t_1 >$ taken as the range of the integration, i.e.

$$u_n^+(\boldsymbol{x}_A, t_1) = \int_{t_0}^{t_1} v^+ \, dt \qquad\qquad u_n^-(\boldsymbol{x}_B, t_1) = \int_{t_0}^{t_1} v^- \, dt \tag{3.8}$$

The velocities normal to solid boundaries at the contact region (v^+, v^-) are dependent variables of the constitutive relations of wear, and they are given by the following equations

$$v^+ = -i_A |p_n^A| |\boldsymbol{V}_{AS}| \qquad\qquad v^- = -i_B |p_n^B| |\boldsymbol{V}_{BS}| \tag{3.9}$$

where

i_A, i_B – wear intensities
$\boldsymbol{V}_{BS}$ – relative velocity between B and S.

Relations (3.9) define the velocities at which the surfaces of the sliding bodies "travel" into other surfaces because of their wearing out. Wear equations (3.9) differ from the classical Archard law of wear in the omission of the term directly representing hardness of materials. Nevertheless, the effect can be incorporated in the term defining the wear intensity coefficients i_A, i_B. Furthermore, we do not establish the principal wear relationship for the volume of the removed material but for the wear velocity. With the aid of equations (3.8), the volume and the mass of the removed material can be easily calculated.

3.2. The wear debris layer

Perhaps the most difficult step in the analysis of a wear process is the development of an acceptable model of the wear debris between a pair of solid bodies as the wear debris exhibits properties quite distinct from those associated with interiors of the bodies. Let us consider the following assumptions:

a) It is natural to assign to wear debris the continuum models with microstructure which capture fundamental kinematical features of the wear particles. In particular, microrotational effects can be depicted in such models in a natural straightforward way. The micropolar layer can support stress and body force couples. The separate structure of the wear debris is characterized by constitutive equations independent of those characterizing the parent materials.

b) The evolution of displacements and stresses transmitted by wear debris is a complicated phenomenon in general case, since it implies that the wear debris can undergo large displacements or/and large strains. Therefore, a simplified model of the wear debris is needed, by specifying the kinematical and mechanical features of the wear debris, in order to make the problem tractable from both analytical and numerical points of view. We assume that the layer particles undergo only small displacements and microrotations and the deformations of the layer are elastic.

c) The fact that the wear debris layer can be regarded as thin makes it possible to introduce simplifying assumptions in the continuum description. The governing equations of the wear layer are simplified in order to eliminate the dependence on the through-the-thickness coordinate, thus considering the layer as a two-dimensional material continuum.

The mass continuity equation and equilibrium equations with respect to translational and rotational degrees of freedom of a micropolar elastic layer S (Zmitrowicz, 1987) are given by

$$\begin{aligned}
&\frac{\partial \rho_S}{\partial t} + \operatorname{div}_s\Big(\rho_S \frac{\partial \boldsymbol{u}_S}{\partial t}\Big) - m_A - m_B = 0 \\
&\operatorname{div}_s \boldsymbol{\sigma}_S - \widehat{\boldsymbol{f}}_S - m_A \boldsymbol{V}_{AS} - m_B \boldsymbol{V}_{BS} = \boldsymbol{p}_t^A + \boldsymbol{p}_t^B \\
&\operatorname{div}_s \boldsymbol{\mu} + (m_A + m_B)\boldsymbol{j}_S \frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{c}
\end{aligned} \tag{3.10}$$

where

ρ_S	–	mass density
$\boldsymbol{u}_S, \boldsymbol{\psi}$	–	vectors of the displacements and independent microrotations of the wear debris
$\boldsymbol{\sigma}_S$	–	Cauchy's stress tensor
$\widehat{\boldsymbol{f}}_S$	–	body force
$\boldsymbol{p}_t^A, \boldsymbol{p}_t^B$	–	friction forces between the bodies A, B and the layer S,
$\boldsymbol{\mu}$	–	stress couple tensor in the layer
$\boldsymbol{j}_S$	–	inertia tensor of the wear debris
$\boldsymbol{c}$	–	couple of friction forces.

The terms m_A and m_B determine the mass fluxes of the wear debris supplied to the layer, $m_A \boldsymbol{V}_{AS}$ and $m_B \boldsymbol{V}_{BS}$ define the momentum supplied to the layer by the wear debris, and $(m_A + m_B)\boldsymbol{j}_S \partial\boldsymbol{\psi}/\partial t$ is the moment of momentum of the supplied wear debris.

The boundary conditions for equilibrium equations $(3.10)_{2,3}$ are as follows

$$\begin{aligned} &\boldsymbol{u}_S(\boldsymbol{l}) = \widehat{\boldsymbol{u}}_S(\boldsymbol{l}) && \text{on} \quad L_u \\ &\boldsymbol{\sigma}_S\boldsymbol{\nu} = \widehat{\boldsymbol{q}}_S && \text{on} \quad L_q \\ &\boldsymbol{\psi}(\boldsymbol{l}) = \widehat{\boldsymbol{\psi}}(\boldsymbol{l}) && \text{on} \quad L_\psi \\ &\boldsymbol{\mu}\boldsymbol{\nu} = \widehat{\boldsymbol{m}} && \text{on} \quad L_m \end{aligned} \tag{3.11}$$

and the boundary and initial conditions for mass continuity equation $(3.10)_1$ are

$$\begin{aligned} &\rho_S(\boldsymbol{l},t) = \widehat{\rho}(t) && \text{on} \quad L_r \\ &\rho_S(\boldsymbol{l},t_0) = \rho_0(\boldsymbol{l}) && \text{in} \quad S \end{aligned} \tag{3.12}$$

where

$\widehat{\boldsymbol{u}}_S, \widehat{\boldsymbol{q}}_S, \widehat{\boldsymbol{\psi}}, \widehat{\boldsymbol{m}}$	–	boundary displacements, forces, microrotations and couples
$\boldsymbol{\nu}$	–	unit vector normal to the layer boundary
$\widehat{\rho}$	–	boundary mass efflux of the debris removed from the contact region
ρ_0	–	initial mass intensity.

The elastic micropolar layer S is characterized by the constitutive equations independent of those characterizing the bulk response properties of the parent bodies A and B

$$\begin{aligned} &\boldsymbol{\sigma}_S = \mathbf{E}_S\boldsymbol{\varepsilon}_S && \boldsymbol{\varepsilon}_S(\boldsymbol{u}_S) = \frac{1}{2}(\operatorname{grad}_s\boldsymbol{u}_S + \operatorname{grad}_s^\top\boldsymbol{u}_S) \\ &\boldsymbol{\mu} = \mathbf{D}_S\boldsymbol{\kappa} && \boldsymbol{\kappa}(\boldsymbol{\psi}) = \operatorname{grad}_s\boldsymbol{\psi} \end{aligned} \tag{3.13}$$

where

$\boldsymbol{\varepsilon}_S$	–	infinitesimal strain tensor
$\boldsymbol{\kappa}$	–	couple strain measure in S
$\mathbf{E}_S, \mathbf{D}_S$	–	tensors of elasticity with respect to the displacements and microrotations, respectively
$\operatorname{grad}_s(\cdot)$	–	surface (two-dimensional) gradient operator.

Here, $\rho_S(\boldsymbol{l},t)$, $\boldsymbol{u}_S(\boldsymbol{l})$, $\boldsymbol{\psi}(\boldsymbol{l})$, $\boldsymbol{l} \in S$, $t \in I$, $I \subset \mathcal{R}$ are functions of mass intensity, displacement and microrotation.

The wear debris mass fluxes (from the bodies A and B to the layer S) are

$$m_A = \rho_A v^+ \qquad\qquad m_B = \rho_B v^- \tag{3.14}$$

where ρ_B is the mass density. The wear velocities of the bodies A and B are defined by (3.9). The couple of friction forces acting on a single particle in the layer is given by

$$\boldsymbol{c} = \zeta_A |p_n^A| \Big(\boldsymbol{n}^+ \times \frac{\boldsymbol{V}_{AS}}{|\boldsymbol{V}_{AS}|} \Big) + \zeta_B |p_n^B| \Big(\boldsymbol{n}^- \times \frac{\boldsymbol{V}_{BS}}{|\boldsymbol{V}_{BS}|} \Big) \tag{3.15}$$

where
ζ_A, ζ_B – constants
$\boldsymbol{n}^- = -\boldsymbol{n}^+$ – unit vectors in the contact region of the body B.

The constraints of the energy dissipated in the contact region are as follows

$$\boldsymbol{p}_t^A \boldsymbol{V}_{AS} + \boldsymbol{p}_t^B \boldsymbol{V}_{BS} = \boldsymbol{c}\dot{\boldsymbol{\psi}} + q + \beta_A + \beta_B + \beta^* \tag{3.16}$$

Equation (3.16) shows that the friction force power goes into the power of the frictional couple $\boldsymbol{c}\dot{\boldsymbol{\psi}}$, the frictional heat fluxes entering into the solids and the layer q and energies spent on wear process of the solids and the layer β_A, β_B, β^*. The energies spent in the wear process are defined by: $\beta_A = -m_A \epsilon_A$, $\beta_B = -m_B \epsilon_B$, $\beta^* = -m_{cr} \epsilon_S$ where ϵ_A, ϵ_B, ϵ_S are the internal energies consumed by formation of a unit mass of wear debris, m_{cr} is the mass of the wear debris crushed into finer particles.

4. Variational formulations

4.1. Displacements in the contact system

An exact solution determining all unknowns of the contact problem cannot be found since the complex system of coupled differential equations and the boundary conditions must be taken into account. Modern methods of analysis are based on a variational approach and numerical techniques. We apply the principle of stationary total potential energy to state the equilibrium conditions of the contact system. The total potential energy Π defined in a deformed configuration of the contact system consisting of the two contacting solids A, B and the interfacial layer S of wear particles with translational and rotational degrees of freedom is given with the stored energy Π^{int} and the energy of external loads Π^{ext} as

$$\Pi = \Pi(\boldsymbol{u}_A, \boldsymbol{u}_B, \boldsymbol{u}_S, \boldsymbol{\psi}) = \Pi^{int} + \Pi^{ext} \rightarrow \text{stationary point}$$

$$\Pi(\boldsymbol{u}_A, \boldsymbol{u}_B, \boldsymbol{u}_S, \boldsymbol{\psi}) = \frac{1}{2} \int\limits_{V_A} \boldsymbol{\sigma}_A : \boldsymbol{\varepsilon}_A \, dV + \frac{1}{2} \int\limits_{V_B} \boldsymbol{\sigma}_B : \boldsymbol{\varepsilon}_B \, dV +$$

$$
\begin{aligned}
&+\frac{1}{2}\int\limits_S (\boldsymbol{\sigma}_S : \boldsymbol{\varepsilon}_S + \boldsymbol{\mu} : \boldsymbol{\kappa})\, dS - \int\limits_S (m_A \boldsymbol{V}_{AS} + m_B \boldsymbol{V}_{BS})\boldsymbol{u}_S\, dS - \\
&- \int\limits_S [(m_A + m_B)\boldsymbol{j}_S \dot{\boldsymbol{\psi}}]\boldsymbol{\psi}\, dS - \int\limits_{V_A} \widehat{\boldsymbol{f}}_A \boldsymbol{u}_A\, dV - \int\limits_{V_B} \widehat{\boldsymbol{f}}_B \boldsymbol{u}_B\, dV - \\
&- \int\limits_S (\widehat{\boldsymbol{f}}_S \boldsymbol{u}_S + \boldsymbol{c}\boldsymbol{\psi})\, dS - \int\limits_{\Omega_{qA}} \widehat{\boldsymbol{q}}_A \boldsymbol{u}_A\, dA - \int\limits_{\Omega_{qB}} \widehat{\boldsymbol{q}}_B \boldsymbol{u}_B\, dA - \int\limits_{L_q} \widehat{\boldsymbol{q}}_S \boldsymbol{u}_S\, dL - \\
&- \int\limits_{L_m} \widehat{\boldsymbol{m}}\boldsymbol{\psi}\, dL - \int\limits_{\Omega_{cA}} \boldsymbol{p}_A \boldsymbol{u}_A\, dA - \int\limits_{\Omega_{cB}} \boldsymbol{p}_B \boldsymbol{u}_B\, dA - \int\limits_S (\boldsymbol{p}_t^A + \boldsymbol{p}_t^B)\boldsymbol{u}_S\, dS
\end{aligned}
\tag{4.1}
$$

The principal difficulty in contact problems is characterized by unknown potential contact regions Ω_{cA}, Ω_{cB} and unknown contact forces $\boldsymbol{p}_A$, $\boldsymbol{p}_B$ – the quantities which depend on the solution to the problem. To solve the problem, step-by-step solution procedures must be applied (incremental formulation, iterative procedures). We search for such $\boldsymbol{u}_A$, $\boldsymbol{u}_B$, $\boldsymbol{u}_S$ and $\boldsymbol{\psi}$ which guarantee stationarity of Π at the given step of an incremental approach and iteration process, then the current contact regions and current contact forces are assumed to be given. This is a simplified local weak formulation of the contact problem.

We search for such $\boldsymbol{u}_A$, $\boldsymbol{u}_B$, $\boldsymbol{u}_S$ and $\boldsymbol{\psi}$ which guarantee stationarity of Π, i.e.

$$
\begin{aligned}
&\delta\Pi(\boldsymbol{u}_A, \boldsymbol{u}_B, \boldsymbol{u}_S, \boldsymbol{\psi}) = 0 \\
&\delta\Pi = \Big[\frac{\partial \Pi}{\partial \boldsymbol{u}_A}, \delta\boldsymbol{u}_A\Big]_A + \Big[\frac{\partial \Pi}{\partial \boldsymbol{u}_B}, \delta\boldsymbol{u}_B\Big]_B + \Big[\frac{\partial \Pi}{\partial \boldsymbol{u}_S}, \delta\boldsymbol{u}_S\Big]_S + \Big[\frac{\partial \Pi}{\partial \boldsymbol{\psi}}, \delta\boldsymbol{\psi}\Big]_S \\
&\frac{\partial \Pi}{\partial \boldsymbol{u}_A} = \boldsymbol{0} \qquad \frac{\partial \Pi}{\partial \boldsymbol{u}_B} = \boldsymbol{0} \\
&\frac{\partial \Pi}{\partial \boldsymbol{u}_S} = \boldsymbol{0} \qquad \frac{\partial \Pi}{\partial \boldsymbol{\psi}} = \boldsymbol{0}
\end{aligned}
\tag{4.2}
$$

where $[\cdot,\cdot]_A$, etc. mean scalar products (represented by integrals over the corresponding volumes and surface parts, respectively) and $\partial\Pi/\partial\boldsymbol{u}_A$, etc. are gradients of the functional Π, see Bufler (1979).

We derive equations which are a basis of approximated solutions to the contact problem. Let us consider the discretization by means of finite elements. Finite element approximations of displacements, strain and stress fields over an element e in A are as follows

$$
\begin{aligned}
&\boldsymbol{u}_A(\boldsymbol{x}_A) = \mathbf{N}_A(\boldsymbol{x}_A)\boldsymbol{u}_A^e \qquad\qquad \boldsymbol{\varepsilon}_A(\boldsymbol{x}_A) = \mathbf{B}_A(\boldsymbol{x}_A)\boldsymbol{u}_A^e \\
&\boldsymbol{\sigma}_A(\boldsymbol{x}_A) = \mathbf{E}_A\mathbf{B}_A(\boldsymbol{x}_A)\boldsymbol{u}_A^e
\end{aligned}
\tag{4.3}
$$

where

- $\mathbf{N}_A$ – shape functions
- $\mathbf{B}_A$ – derivatives of the shape functions
- $\boldsymbol{u}_A^e$ – displacements at the nodes of the finite element.

Furthermore, we approximate the local fields $\boldsymbol{u}_B$, $\boldsymbol{\varepsilon}_B$, $\boldsymbol{\sigma}_B$ in the body B and $\boldsymbol{u}_S$, $\boldsymbol{\psi}$, $\boldsymbol{\varepsilon}_S$, $\boldsymbol{\kappa}$, $\boldsymbol{\sigma}_S$, $\boldsymbol{\mu}$ in the layer S. The total potential energy $\Pi^e(\boldsymbol{u}_A, \boldsymbol{u}_B, \boldsymbol{u}_S, \boldsymbol{\psi})$ defined for the finite elements has the similar form as $(4.1)_2$. The stationarity condition of the total potential energy for any finite element leads to equations of the displacements in the body A

$$
\frac{\partial \Pi^e}{\partial \boldsymbol{u}_A^e} = \mathbf{K}_A^e\boldsymbol{u}_A^e + \widetilde{\boldsymbol{q}}_A^e \tag{4.4}
$$

where

$$
\begin{aligned}
\mathbf{K}_A^e &= \int\limits_{V_A^e} \mathbf{B}_A^\top\mathbf{E}_A\mathbf{B}_A\; dV \\
\widetilde{\boldsymbol{q}}_A^e &= -\int\limits_{V_A^e} \mathbf{N}_A^\top\widehat{\boldsymbol{f}}_A\; dA - \int\limits_{\Omega_{qA}^e} \mathbf{N}_A^\top\widehat{\boldsymbol{q}}_A\; dA - \int\limits_{\Omega_{cA}^e} \mathbf{N}_A^\top\boldsymbol{p}_A\; dA
\end{aligned}
$$

the displacements in the layer S

$$
\frac{\partial \Pi^e}{\partial \boldsymbol{u}_S^e} = \mathbf{K}_{S1}^e\boldsymbol{u}_S^e + \widetilde{\boldsymbol{q}}_S^e + \widetilde{\boldsymbol{m}}_1^e \tag{4.5}
$$

where

$$
\begin{aligned}
\mathbf{K}_{S1}^e &= \int\limits_{S^e} \mathbf{B}_S^\top\mathbf{E}_S\boldsymbol{B}_S\; dS \\
\widetilde{\boldsymbol{m}}_1^e &= -\int\limits_{S^e} \mathbf{N}_S^\top(m_A\boldsymbol{V}_{AS} + m_B\boldsymbol{V}_{BS})\; dS \\
\widetilde{\boldsymbol{q}}_S^e &= -\int\limits_{S^e} \mathbf{N}_S^\top\widehat{\boldsymbol{f}}_S\; dS - \int\limits_{S^e} \mathbf{N}_S^\top(\boldsymbol{p}_t^A + \boldsymbol{p}_t^B)\; dS - \int\limits_{L_q^e} \mathbf{N}_S^\top\widehat{\boldsymbol{q}}_S\; dL
\end{aligned}
$$

and the microrotations in the layer S

$$
\frac{\partial \Pi^e}{\partial \boldsymbol{\psi}^e} = \mathbf{K}_{S2}^e\boldsymbol{\psi}^e + \widetilde{\boldsymbol{c}}^e + \widetilde{\boldsymbol{m}}_2^e \tag{4.6}
$$

where

$$\mathbf{K}_{S2}^e = \int\limits_{S^e} \mathbf{B}_S^\top \mathbf{D}_S \mathbf{B}_S \, dS$$

$$\widetilde{\boldsymbol{m}}_2^e = \int\limits_{S^e} \mathbf{N}_S^\top (m_A + m_B)(\boldsymbol{j}_S \dot{\boldsymbol{\psi}}) \, dS$$

$$\widetilde{\boldsymbol{c}}^e = -\int\limits_{S^e} \mathbf{N}_S^\top \boldsymbol{c} \, dS - \int\limits_{L_m^e} \mathbf{N}_S^\top \widehat{\boldsymbol{m}} \, dL$$

and $\widetilde{\boldsymbol{m}}_1^e$, $\widetilde{\boldsymbol{m}}_2^e$ define the momentum and moment of momentum of the wear debris supplied to the layer S; $\widetilde{\boldsymbol{c}}^e$ is the frictional moment and the boundary moment in the layer.

Notice, that discretizing the deformation and the load history, the velocities are replaced by increments and the subsequent integrals by sums.

4.2. Mass continuity in the wear debris layer

A mass continuity functional of the layer S can be written as

$$\widehat{\Pi}(\rho_S) = \int\limits_S \Big[\frac{1}{2} \operatorname{div}_s(\rho_S \dot{\boldsymbol{u}}_S) - m_A - m_B + \frac{\partial \rho_S}{\partial t}\Big] \rho_S \, dS \tag{4.7}$$

The mass intensity function $\rho_S(\boldsymbol{l}, t)$ satisfies the following relation

$$\delta \widehat{\Pi} = \Big[\frac{\partial \widehat{\Pi}}{\partial \rho_S}, \delta \rho_S\Big]_S = 0 \tag{4.8}$$

The finite element approximation of the mass intensity field in S is given by the function

$$\rho_S(\boldsymbol{l}, t) = \mathbf{N}_S(\boldsymbol{l}) \boldsymbol{\rho}_S^e(t) \tag{4.9}$$

Let the functional $\widehat{\Pi}^e$ be defined for any finite element, then the stationarity condition is as follows

$$\frac{\partial \widehat{\Pi}^e}{\partial \boldsymbol{\rho}_S^e} = \mathbf{C}_\rho^e \dot{\boldsymbol{\rho}}_S^e + \mathbf{K}_\rho^e \boldsymbol{\rho}_S^e + \widetilde{\boldsymbol{m}}_3^e \tag{4.10}$$

where

$$\mathbf{C}_\rho^e = \int\limits_{S^e} \mathbf{N}_S^\top \mathbf{N}_S \, dS \qquad \mathbf{K}_\rho^e = \int\limits_{S^e} \mathbf{N}_S^\top \dot{\boldsymbol{u}}_S^e \mathbf{B}_S \, dS$$

$$\widetilde{\boldsymbol{m}}_3^e = -\int\limits_{S^e} (m_A + m_B) \mathbf{N}_S \, dS$$

and $\boldsymbol{\rho}_S^e$ is the mass density at the nodes of the finite element. The vector $\widetilde{\boldsymbol{m}}_3^e$ defines mass of the wear debris supplied to the layer.

4.3. Active solution strategies

The variational formulation may be restricted not so strongly as we have assumed in $(4.1)_2$. A powerful formulation of contact problems can be obtained by making use of the principle of stationary potential energy or the principle of virtual work to derive variational formulations and so the called constraint methods, e.g. Lagrange multiplier methods and their generalizations. They are called the active strategies.

The principle of virtual work in the case of a discretized contacting single body A (neglecting friction) has the following form

$$\delta L(\boldsymbol{u}_A) = \delta \boldsymbol{u}_A^\top(\mathbf{K}_A \boldsymbol{u}_A + \overline{\boldsymbol{q}}_A) = \mathbf{0} \tag{4.11}$$

where $\overline{\boldsymbol{q}}_A$ is the global external load vector for simplicity the body and boundary loads are denoted by this letter. The functional (Lagrangian) is given by

$$L(\boldsymbol{u}_A) = \frac{1}{2}\boldsymbol{u}_A^\top \mathbf{K}_A \boldsymbol{u}_A + \boldsymbol{u}_A^\top \overline{\boldsymbol{q}}_A \tag{4.12}$$

Kinematic contact condition (3.7) is a constraint on the displacements of the body in the contact region. Let us assume that the contact takes place in m nodes of the considered discretization. Then, the discretized kinematic contact condition, Eq (3.7), in the contact region (reduced to the single body) can be written as

$$\mathbf{A}\boldsymbol{u}_A - \boldsymbol{g} \leqslant \mathbf{0} \tag{4.13}$$

where

- $\mathbf{A}$ – compliance matrix
- $\boldsymbol{g}$ – initial gap vector in the nodes being in contact.

Notice, that any component of the constraint vector (4.13) is nonpositive. The equality means that condition (4.13) is related to the nodes being currently in the state of sticking. Then, the contact problem reduces to the minimization of functional (4.12) with the kinematic contact constraint on the displacements of the bodies, see (4.13). This method is designed to fulfill the constraint equation in the direction normal to the contact area.

An approach to friction problems can be introduced using the tangential and normal Lagrange multipliers, see Wriggers (1995). We distinguish the tangential gap $(\boldsymbol{g}_t^e)^\top = [g_{t1}, g_{t2}]$ and the normal gap $g_n^e = g_n + u_n^+$, where $\boldsymbol{g}_t$ is the relative sliding displacement. Then, the discretized kinematic contact conditions (in the equality version) can be written as

$$\mathbf{A}_1 \boldsymbol{u}_A - \boldsymbol{g}_n = \mathbf{0} \qquad\qquad \mathbf{A}_2 \boldsymbol{u}_A - \boldsymbol{g}_t = \mathbf{0} \tag{4.14}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ are the compliance matrices.

Due to satisfying of contact conditions (4.14), we extend the definition of the Lagrangian (4.12). Then, the Lagrange multiplier method in the equality version is to search for a saddle point of the Lagrangian (4.12) changed as follows

$$L(\boldsymbol{u}_A, \boldsymbol{\lambda}_n, \boldsymbol{\lambda}_t) = \frac{1}{2}\boldsymbol{u}_A^\top \mathbf{K}_A \boldsymbol{u}_A + \boldsymbol{u}_A^\top \overline{\boldsymbol{q}}_A + (\mathbf{A}_1 \boldsymbol{u}_A - \boldsymbol{g}_n)^\top \boldsymbol{\lambda}_n + (\mathbf{A}_2 \boldsymbol{u}_A - \boldsymbol{g}_t)^\top \boldsymbol{\lambda}_t \tag{4.15}$$

The Lagrange multipliers contain the following additional unknowns associated with sticking nodes: $(\boldsymbol{\lambda}_t^e)^\top = [\lambda_{t1}, \lambda_{t2}]$ and $\lambda_n^e = \lambda_n$. λ_n denotes the Lagrangian multiplier which can be identified as the contact pressure p_n. λ_t is associated with the tangential forces in stick or slip motion p_t. In the stick the relative tangential slip g_t is equal to zero and λ_t is the reaction. In the case of sliding λ_t is determined by the friction constitutive law.

We calculate the minimum of the Lagrangian (4.15) with respect to three independent variables, i.e. we search for such $\boldsymbol{u}_A$, $\boldsymbol{\lambda}_n$, $\boldsymbol{\lambda}_t$ which guarantee stationarity of $L(\boldsymbol{u}_A, \boldsymbol{\lambda}_n, \boldsymbol{\lambda}_t)$

$$\begin{aligned} &\delta L(\boldsymbol{u}_A, \boldsymbol{\lambda}_n, \boldsymbol{\lambda}_t) = 0 \\ &\delta L = \frac{\partial L}{\partial \boldsymbol{u}_A}\delta \boldsymbol{u}_A + \frac{\partial L}{\partial \boldsymbol{\lambda}_n}\delta \boldsymbol{\lambda}_n + \frac{\partial L}{\partial \boldsymbol{\lambda}_t}\delta \boldsymbol{\lambda}_t \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} &\frac{\partial L}{\partial \boldsymbol{u}_A} = \mathbf{K}_A \boldsymbol{u}_A + \overline{\boldsymbol{q}}_A + \mathbf{A}_1^\top \boldsymbol{\lambda}_n + \mathbf{A}_2^\top \boldsymbol{\lambda}_t = \mathbf{0} \\ &\frac{\partial L}{\partial \boldsymbol{\lambda}_n} = \mathbf{A}_1 \boldsymbol{u}_A - \boldsymbol{g}_n = \mathbf{0} \qquad \frac{\partial L}{\partial \boldsymbol{\lambda}_t} = \mathbf{A}_2 \boldsymbol{u}_A - \boldsymbol{g}_t = \mathbf{0} \end{aligned} \tag{4.17}$$

Then (4.17) yield the following system of linear equations for the displacements and the Lagrange multipliers

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{A}_1^\top & \mathbf{A}_2^\top \\ \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_2 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_A \\ \boldsymbol{\lambda}_n \\ \boldsymbol{\lambda}_t \end{bmatrix} = \begin{bmatrix} -\overline{\boldsymbol{q}}_A \\ \boldsymbol{g}_n \\ \boldsymbol{g}_t \end{bmatrix} \tag{4.18}$$

In the active solution strategies, the contact stresses are determined together with the displacements of the contact system during solving the problem, see (4.18). The contact region can be calculated in an iterative process.

5. Conclusions

- Variational formulations of displacements in rubbing and wearing solids are derived with the aid of the principle of stationary potential energy in a deformed configuration of a contact system. The definition of the gap between two contacting bodies is modified taking into account wear profiles.
- Classical variational formulations of contact problems are extended by including an interfacial layer of wear debris between wearing bodies. Final equations are presented in a discretized form using the finite element method.
- Friction and wear phenomena are very sensitive to changes of sliding conditions. Wear particles between sliding surfaces can affect frictional and wear behaviour very significantly. In everyday life, one can easily observe wear profiles and wear debris during abrasion, e.g. pencil drawing marks on a paper, a piece of chalk writing on a blackboard, a rubber eraser rubbing out pencil marks on a paper, etc.

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Wariacyjne opisy w mechanice kontaktu zużywających się ciał stałych i cząstek zużycia

Streszczenie

Badania doświadczalne wskazały na istotną rolę, którą odgrywają cząstki zużycia znajdujące się w obszarach styku. W pracy uogólniono klasyczne sformułowania wariacyjne zagadnień kontaktowych na przypadek zużywających się ciał stałych uwzględniając warstwę cząstek zużycia między stykającymi się ciałami. Opisy wariacyjne wyprowadzono wychodząc z różniczkowych postaci równań stanu dla zużywających się ciał i warstwy cząstek zużycia. Przedstawiono implementację do metody elementów skończonych. Wykorzystano metodę mnożników Lagrange'a w celu określenia sił kontaktowych oraz spełnienia kinematycznych więzów styku. Dyskretne postacie funkcjonałów wariacyjnych prowadzą do równań, które mogą być użyteczne w analizie numerycznej.

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