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AUGMENTED LAGRANGIAN METHODS FOR A CLASS OF CONVEX AND NONCONVEX CONTACT PROBLEMS

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The aim of this contribution is threefold. First, we formulate unilateral contact problems for three models of plates and the Koiter shell model. Contact conditions have been formulated on the face being in contact with an obstacle and not on the mid-plane of the plate or the middle surface of the shell. Such a rigorous approach results in nonconvex minimization problems even in the case of thin, geometrically linear plates. Existence theorems are formulated for each model considered. Second, the Ito and Kunisch (1990, 1995) augmented Lagrangians methods have been extended to nonconvex problems. Third, nonconvex duality theory by Rockafellar and Wets (1998), valid for finite-degree-of-freedom systems has been extended to continuous systems. Specific examples have also been provided.

Key words: unilateral contact problems without friction, plates, Koiter's shell model, augmented Lagrangian methods, nonconvex duality

1. Introduction

Contact conditions for thin structures like plates and shells are usually posed on the mid-plane of the plate or the middle surface of the shell, cf Duvaut and Lions (1972, 1974), Panagiotopoulos (1985), Telega (1987). Such an approach is unacceptable in the case of moderately thick structures and in the case of friction beetween the structure and obstacle. Also, rigorously formulated contact problems should be formulated on the face being in contact with the obstacle.

Our considerations are confined to static frictionless contact problems. It is then possible to formulate relevant boundary value problems in the form of corresponding minimization problems. Since we are interested in the contact conditions imposed on the face being in contact with an obstacle, the resulting minimization problem is in general nonconvex even in the case of a geometrically linear structure. We shall consider two geometrically linear elastic plates, the von Kármán plate model and the linear Koiter shell model. A moderately thick nonlinear plate was studied by Bielski and Telega (1998), cf also Bielski and Telega (1992, 1996). Other models of plates and shells, including geometrically nonlinear models, can be studied similarly.

Ito and Kunisch (1990, 1995) developed mathematically rigorous augmented Lagrangian methods valid for convex problems. We propose an extension to the nonconvex contact problems, combining the approach of these two authors with iterative procedures, cf also Bielski et al. (2000). An example has also been provided. The papers by Telega and Gałka (1998, 2001) provide many examples of usefulness of the method of the augmented Lagrangian.

The third topic studied in this paper concerns duality theory in the case of nonconvex primal problems. In a series of papers we have shown that the so-called Rockafellar's theory of duality, as presented in the book by Ekeland and Temam (1976), imposes restrictions on dual variables, cf Bielski and Telega (1992, 1985a-d, 1986, 1996), Bielski et al. (1988, 1989), Telega et al. (1988), Gałka et al. (1989), Telega (1989), Gałka and Telega (1990, 1992, 1995). For instance, in the case of von Kármán's plates, the matrix of membrane forces has to be positive semi-definite, thus precluding compressed plates. Otherwise the primal and dual problems will be characterised by a duality gap. Another possibility is offered by so-called anomalous dual variational principles, cf Gałka and Telega (1995), Telega (1995). However, their usefulness seems to be of limited value, as prove the examples of compressed beems, studied in these two papers. Recently, Rockafellar and Wets (1998) proposed a novel approach to the formulation of dual problems in the nonconvex case where the duality gap is possible. Their approach is confined to finite-degree-of-freedom systems (discrete or discretized). In essence, this new approach exploits properly chosen augmented Lagrangians. We succeeded to extend the Rockafellar and Wets (1998) duality theory to continuous systems.

2. Geometrically linear plates

In Section 2 we shall formulate minimization problems in the case of the obstacle problem for the linear Kirchhoff plate and Reissner plate model. The obstacle is rigid and the contact occurs through the lower face of the plate. An extension to the case where both the lower and upper faces may come into contact with rigid obstacles is straightforward.

2.1. Thin plates

Let $\Omega \subset \mathbb{R}^2$ be a sufficiently smooth domain and $\Gamma = \partial\Omega$ its boundary. Ω denotes the mid-plane of an undeformed plate. The plate occupies the region $\Omega \times (-h, h) \subset \mathbb{R}^3$. The boundary Γ is decomposed into two parts: Γ_0 and Γ_1 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $\boldsymbol{v} = [v_i]$, $v_i = v_i(x_\alpha, z)$ be the displacement vector of a point $(x_\alpha, z) \in \Omega \times (-h, h)$, $\alpha = 1, 2$; $i = 1, 2, 3$. The axis z is directed downwards. We assume the classical Kirchhoff-Love kinematical hypothesis

$$v_\alpha(x_\beta, z) = u_\alpha(x_\beta) - z w,_\alpha(x_\beta) \qquad\qquad v_3(x_\beta, z) \equiv w(x_\beta) \tag{2.1}$$

Here $\boldsymbol{u} = (u_\alpha)$ stands for the in-plane displacement vector whilst w denotes the transverse displacement. By c_{ijkl} we denote the elasticity tensor of the material of the plate. We assume that the plane $z = 0$ is the plane of the material symmetry; hence $c_{\alpha\beta\gamma 3} = c_{333\alpha} = 0$. For a thin elastic plate the constitutive relationship takes the form

$$\sigma_{\alpha\beta} = C_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\boldsymbol{u}) \qquad\qquad \sigma_{\alpha 3} = 2c_{\alpha 3\lambda 3}\varepsilon_{\lambda 3}(\boldsymbol{u}) \qquad\qquad \sigma_{33} = 0 \tag{2.2}$$

where σ_{ij} are components of the stress tensor, and

$$C_{\alpha\beta\lambda\mu} = c_{\alpha\beta\lambda\mu} - c_{\alpha\beta 33}c_{33\lambda\mu}c_{3333}^{-1}$$

Here (c_{ijkl}) denotes the elasticity tensor of the material of the plate. As usual, the strain-displacement relation is given by

$$\varepsilon_{ij}(\boldsymbol{u}) = u_{(i,j)} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{2.3}$$

Let $\mathbf{N} = (N_{\alpha\beta})$ and $\mathbf{M} = (M_{\alpha\beta})$ be the membrane force tensor and moment tensor, respectively, defined by

$$N_{\alpha\beta} = \int\limits_{-h}^{h} \sigma_{\alpha\beta}\, dz \qquad\qquad M_{\alpha\beta} = \int\limits_{-h}^{h} z\sigma_{\alpha\beta}\, dz \tag{2.4}$$

The constitutive relations are given by

$$N_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\boldsymbol{u}) \qquad\qquad M_{\alpha\beta} = B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(w) \tag{2.5}$$

Here $\varepsilon_{\alpha\beta}(\boldsymbol{u})$ and $\kappa_{\alpha\beta}(w)$ are the strain measures defined by

$$\varepsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \qquad\qquad \kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}$$

and

$$A_{\alpha\beta\lambda\mu} = \int\limits_{-h}^{h} C_{\alpha\beta\lambda\mu}\, dz \qquad\qquad B_{\alpha\beta\lambda\mu} = \int\limits_{-h}^{h} z^2 C_{\alpha\beta\lambda\mu}\, dz$$

The equilibrium equations of the plate (in the absence of the obstacle) are

$$N_{\alpha\beta,\beta} + p_\alpha = 0 \qquad\qquad M_{\alpha\beta,\beta\alpha} + p = 0 \qquad \text{in} \quad \Omega \tag{2.6}$$

Let the continuous function

$$f:\ \Omega_1 \to \mathbb{R} \qquad\qquad z = f(x_\alpha) \qquad\qquad \Omega \subset \Omega_1$$

determines a rigid obstacle. The unilateral condition is specified by, cf Bielski and Telega (1998), Dhia (1989)

$$w(x_\alpha) + h \leqslant f\big(x_\alpha + u_\alpha(x_\beta) - hw_{,\alpha}(x_\beta)\big) \tag{2.7}$$

The lower face of the plate may come into contact with the rigid obstacle. We introduce the set

$$K = \Big\{(\boldsymbol{u}, w) \in H^1(\Omega)^2 \times H^2(\Omega) \ \Big|\ (2.7) \text{ is satisfied for } (x_\alpha) \in \Omega\Big\}$$

Remark 2.1. If K is non-empty, then, in general, it is a *non-convex* set. K is a *convex* set provided that f is a concave function. □

The boundary conditions are assumed in the form

$$w = 0 \qquad\qquad \frac{\partial w}{\partial \boldsymbol{n}} = 0 \qquad\qquad \boldsymbol{u} = 0 \quad \text{on } \ \Gamma_0 \qquad\qquad \text{meas}\, \Gamma_0 > 0$$

Here $\boldsymbol{n}$ denotes the outer unit vector normal to Γ. We set

$$V = \Big\{(\boldsymbol{u}, w) \in H^1(\Omega)^2 \times H^2(\Omega) \ \Big|\ \boldsymbol{u} = 0,\ w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0\Big\}$$

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &= \int\limits_{\Omega} A_{\alpha\beta\lambda\mu}(\boldsymbol{x})\varepsilon_{\alpha\beta}(\boldsymbol{u})\varepsilon_{\lambda\mu}(\boldsymbol{v})\,dx \\
b(w,t) &= \int\limits_{\Omega} B_{\alpha\beta\lambda\mu}(\boldsymbol{x})\kappa_{\alpha\beta}(w)\kappa_{\lambda\mu}(t)\,dx
\end{aligned}
\tag{2.8}
$$

where $\boldsymbol{u},\boldsymbol{v} \in H^1(\Omega)^2$ and $t,w \in H^2(\Omega)$.

The functional of the external loading is assumed in the form

$$
L(\boldsymbol{u},w) = \int\limits_{\Omega} (p_\alpha u_\alpha + pw)\,dx + \int\limits_{\Gamma_1} \Big(r_\alpha u_\alpha + qw - \overline{M}\frac{\partial w}{\partial \boldsymbol{n}}\Big)\,d\Gamma \tag{2.9}
$$

where $r_\alpha, q, \overline{M} \in L^2(\Gamma_1)$, and $p_\alpha, p \in L^2(\Omega)$. The functional of the total potential energy is given by

$$
J(\boldsymbol{u},w) = \frac{1}{2}a(\boldsymbol{u},\boldsymbol{u}) + \frac{1}{2}b(w,w) - L(\boldsymbol{u},w) \tag{2.10}
$$

Now we are in a position to formulate the first, in general a *nonconvex*, minimization problem.

Problem (P)

Find

$$
\inf\Big\{J(\boldsymbol{u},w)\ \Big|\ (\boldsymbol{u},w) \in K \cap V\Big\}
$$

We observe that on account of unilateral condition (2.7) the in-plane and transverse displacements are interrelated. Consequently, the problem (P) cannot be decomposed into membrane and plate problems. We recall that if the contact condition is imposed on the mid-plane of the plate then both problems are independent and only the bending problem is of a unilateral type.

Theorem 2.2. The problem (P) possesses at least one solution $(\widetilde{\boldsymbol{u}},\widetilde{w}) \in K \cap V$, provided that $K \neq \emptyset$. □

For the proof the reader is referred to Bielski and Telega (1998).

Remark 2.3. The linearization of the r.h.s. of (2.7) was considered by Bielski and Telega (1998). In the same paper the linearization of the r.h.s. of (2.7) has also been carried out. □

2.2. Reissner's plate model

In a simple model of moderately thick plates accounting for transverse shear deformations it is assumed that, cf Jemielita (1991), Lewiński (1987), Reissner (1985)

$$
\begin{aligned}
&v_\alpha(\boldsymbol{x},z) = u_\alpha(\boldsymbol{x}) + z\varphi_\alpha(\boldsymbol{x}) \qquad\qquad (\boldsymbol{x},z) \in \Omega \times (-h,h)\\
&v_3(x_\beta,z) \equiv w(x_\beta)
\end{aligned}
$$

Here φ_α $(\alpha = 1,2)$ denote the rotations of the plate transverse cross-sections. The strain measures are given by

$$
\begin{aligned}
&\varepsilon_{\alpha\beta}(\boldsymbol{u}) = u_{(\alpha,\beta)} = \frac{1}{2}\Big(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\Big) \qquad\qquad \rho_{\alpha\beta}(\boldsymbol{\varphi}) = \varphi_{(\alpha,\beta)}\\
&d_\alpha(w,\boldsymbol{\varphi}) = w_{,\alpha} + \varphi_\alpha
\end{aligned}
\tag{2.11}
$$

Let us denote by $\boldsymbol{T} = (T_\alpha)$ the transverse shear force vector. The constitutive relationships are given by

$$
\begin{aligned}
&N_{\alpha\beta} = A_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\boldsymbol{u}) \qquad\qquad M_{\alpha\beta} = B_{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(\boldsymbol{\varphi})\\
&T_\alpha = H_{\alpha\beta} d_\beta(w,\boldsymbol{\varphi})
\end{aligned}
\tag{2.12}
$$

where the elastic moduli $A_{\alpha\beta\lambda\mu}$ and $B_{\alpha\beta\lambda\mu}$ are specified in Section 2.1, and

$$
H_{\alpha\beta} = \int\limits_{-h}^{h} c_{\alpha 3\beta 3}\; dz
$$

The equilibrium equations have now the form

$$
N_{\alpha\beta,\beta} + p_\alpha = 0 \qquad\qquad M_{\alpha\beta,\beta} - T_\alpha + m_\alpha = 0 \qquad\qquad T_{\alpha,\alpha} + p = 0 \tag{2.13}
$$

provided that the obstacle is absent. The boundary conditions are

$$
\boldsymbol{u} = \boldsymbol{0} \qquad\qquad \boldsymbol{\varphi} = \boldsymbol{0} \qquad\qquad w = 0 \quad \text{on} \ \ \Gamma_0
$$

where meas $\Gamma_0 > 0$.

We set

$$
V_1 = \Big\{(\boldsymbol{u},w,\boldsymbol{\varphi}) \in H^1(\Omega)^2 \times H^1(\Omega) \times H^2(\Omega)^2 \ \Big| \ \boldsymbol{u} = 0,\ \boldsymbol{\varphi} = 0,\ w = 0 \text{ on } \Gamma_0\Big\} \tag{2.14}
$$

Now, the impenetrability condition is given by, cf (2.7)

$$w(\boldsymbol{x}) + h \leqslant f\big(x_\alpha + u_\alpha(\boldsymbol{x}) + h\varphi_\alpha(\boldsymbol{x})\big) \qquad \boldsymbol{x} \in \Omega \tag{2.15}$$

Consequently, the set of kinematically admissible displacements is defined by

$$K_1 = \Big\{(\boldsymbol{u}, w, \boldsymbol{\varphi}) \in V_1 \;\Big|\; w(\boldsymbol{x}) + h \leqslant f(x_\alpha + u_\alpha(\boldsymbol{x}) + h\varphi_\alpha(\boldsymbol{x})) \qquad \boldsymbol{x} \in \Omega\Big\} \tag{2.16}$$

We assume that $K_1 \neq \emptyset$. The functional of the total potential energy is expressed by

$$J_1(\boldsymbol{u}, w, \boldsymbol{\varphi}) = \frac{1}{2}\int\limits_\Omega \Big[A_{\alpha\beta\lambda\mu}\varepsilon_{\alpha\beta}(\boldsymbol{u})\varepsilon_{\lambda\mu}(\boldsymbol{u}) + B_{\alpha\beta\lambda\mu}\rho_{\alpha\beta}(\boldsymbol{\varphi})\rho_{\lambda\mu}(\boldsymbol{\varphi}) + \tag{2.17}$$
$$+H_{\alpha\beta}d_\alpha(w, \boldsymbol{\varphi})d_\beta(w, \boldsymbol{\varphi})\Big]\, d\boldsymbol{x} - L_1(\boldsymbol{u}, w, \boldsymbol{\varphi})$$

where

$$L_1(\boldsymbol{u}, w, \boldsymbol{\varphi}) = \int\limits_\Omega (p_\alpha u_\alpha + pw + m_\alpha\varphi_\alpha)\, dx + \int\limits_{\Gamma_1} (r_\alpha u_\alpha + qw + \overline{M}_\alpha\varphi_\alpha)\, d\Gamma \tag{2.18}$$

We formulate the second minimization problem.

Problem (P_1)

$$\left|\begin{array}{l}\text{Find}\\ \qquad \inf\Big\{J_1(\boldsymbol{u}, w, \boldsymbol{\varphi}) \;\Big|\; (\boldsymbol{u}, w, \boldsymbol{\varphi}) \in K_1\Big\}\end{array}\right.$$

In general, this problem is also *nonconvex*. The following existence results are formulated as follows.

Theorem 2.4. The problem (P_1) possesses at least one minimizer $(\widetilde{\boldsymbol{u}}, \widetilde{w}, \widetilde{\boldsymbol{\varphi}}) \in K_1$. □

For the proof the reader is referred to Bielski and Telega (1998).

3. Von Kármán's plates

This model is still based on the Kirchhoff-Love kinematical hypotheses. The strain measures are defined by, cf Fung (1965), Ciarlet and Rabier (1980), Lewiński and Telega (2000)

$$e_{\alpha\beta}(\boldsymbol{u}, w) = \varepsilon_{\alpha\beta}(\boldsymbol{u}) + \frac{1}{2}w_{,\alpha}\, w_{,\beta} \qquad\qquad \kappa_{\alpha\beta}(w) = -w_{,\alpha\beta} \tag{3.1}$$

where $\varepsilon_{\alpha\beta}(\boldsymbol{u}) = u_{(\alpha,\beta)}$. We note that only the first strain measure is *nonlinear*.

The constitutive equations have the form

$$N_{\alpha\beta} = A_{\alpha\beta\lambda\mu} e_{\lambda\mu}(\boldsymbol{u}, w) \qquad\qquad M_{\alpha\beta} = B_{\alpha\beta\lambda\mu} \kappa_{\lambda\mu}(w) \tag{3.2}$$

As previously, $\mathbf{N}$ and $\mathbf{M}$ are the membrane forces tensor and moments tensor, respectively. In the absence of the obstacle the equilibrium equations are given by

$$N_{\alpha\beta,\beta} + p_\alpha = 0 \qquad\qquad M_{\alpha\beta,\beta\alpha} + (N_{\alpha\beta} w,_{\beta}),_{\alpha} + p = 0 \qquad \text{in } \Omega \tag{3.3}$$

We impose the following boundary conditions

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_0 \qquad\qquad w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma$$

An appropriate space for displacements is

$$V_2 = \Big\{ (\boldsymbol{u}, w) \in H^1(\Omega)^2 \times H^2_0(\Omega) \ \Big|\ \boldsymbol{u} = 0 \ \text{ on } \ \Gamma_0 \Big\} \tag{3.4}$$

The functional of the total potential energy is now given by

$$\begin{aligned} J_2(\boldsymbol{u}, w) = \frac{1}{2} \int\limits_{\Omega} \Big[A_{\alpha\beta\lambda\mu} \Big(\varepsilon_{\alpha\beta}(\boldsymbol{u}) + \frac{1}{2} w,_{\alpha} w,_{\beta} \Big) \Big(\varepsilon_{\lambda\mu}(\boldsymbol{u}) + \frac{1}{2} w,_{\lambda} w,_{\mu} \Big) + \\ + B_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta}(w) \kappa_{\lambda\mu}(w) \Big] \, d\boldsymbol{x} - \int\limits_{\Omega} (p_\alpha u_\alpha + p w) \, d\boldsymbol{x} - \int\limits_{\Gamma_1} r_\alpha u_\alpha \, d\Gamma \end{aligned} \tag{3.5}$$

The nonlinear strain measure renders the functional J_2 nonconvex on $H^1(\Omega)^2 \times H^2_0(\Omega)$, and particularly on V_2. This functional is weakly lower semicontinuous and bounded from below, cf Bielski and Telega (1996), Ciarlet and Rabier (1980). For the obstacle problem the set of kinematically admissible fields is specified by

$$K_2 = \Big\{ (\boldsymbol{u}, w) \in V_2 \ \Big|\ w(\boldsymbol{x}) + h \leqslant f\big(x_\alpha + u_\alpha(\boldsymbol{x}) - h w,_{\alpha}(\boldsymbol{x})\big), \ \boldsymbol{x} \in \Omega \Big\}$$

We assume that $K_2 \neq \emptyset$. We can now formulate the obstacle contact problem.

Problem (P_2)

Find

$$\inf \Big\{ J_2(\boldsymbol{u}, w) \ \Big|\ (\boldsymbol{u}, w) \in K_2 \Big\}$$

The existence to the solution to the Problem (P_2) is ensured by the following result.

Theorem 3.1. The functional J_2 has at least one minimizer on the set K_2. $\square$

For the proof the reader is referred to Bielski and Telega (1998).

4. Obstacle problem for linear Koiter's shell

Consider a shell of the thickness $2h$. Let the middle surface S of the shell be specified by the equation

$$\begin{aligned} &\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{\xi}) \qquad && \boldsymbol{x} = (x_i) \in S \qquad && i = 1,2,3 \qquad && S = \boldsymbol{\Phi}(\Omega) \\ &\boldsymbol{\xi} = (\xi_\alpha) \in \Omega \qquad && \alpha = 1,2 \end{aligned} \tag{4.1}$$

where Ω is a bounded sufficiently regular domain in $\mathbb{R}^2$ in the Cartesian coordinate system with the base $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Let (u_α, w) be the displacement vector of a point belonging to S. Let $\boldsymbol{r}^h(\xi_\alpha)$ denote the position vector of a point lying on the lower face of the deformed shell, cf Fig. 1.

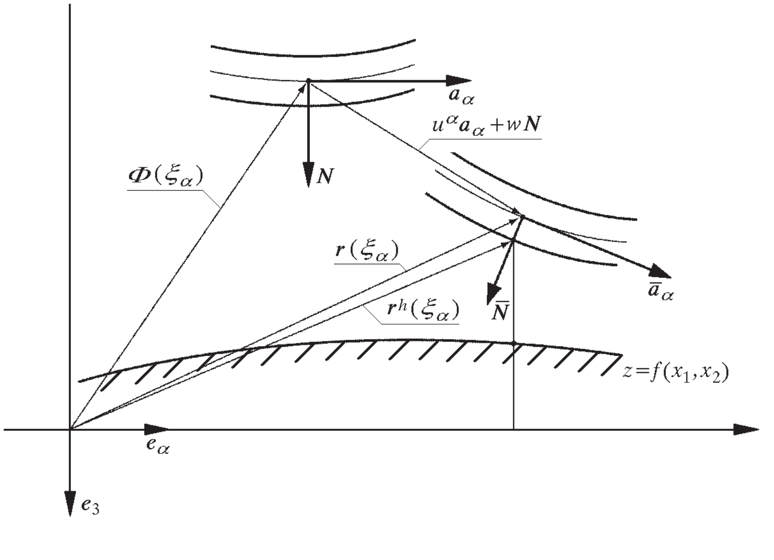

Fig. 1. Unilateral contact of a shell with a rigid obstacle

We have

$$\boldsymbol{r}^h(\xi_\alpha) = \boldsymbol{r}(\xi_\alpha) + h\overline{\boldsymbol{N}} \tag{4.2}$$

where

$$\boldsymbol{r}(\xi_\alpha) = \boldsymbol{\Phi}(\xi_\alpha) + w\boldsymbol{N} + u_\alpha \boldsymbol{a}_\alpha$$

is the placement vector of a point lying on the middle surface of the deformed shell. Here $(\boldsymbol{a}_\alpha, \boldsymbol{N})$ forms a local base for the middle surface of the undeformed shell and $\overline{\boldsymbol{N}}$ is given by, cf Koiter (1965)

$$\overline{\boldsymbol{N}} = \sqrt{\frac{a}{\overline{a}}}\Big[(\varphi_\lambda l^\lambda_\mu - \varphi_\mu l^\lambda_\lambda)a^{\mu\nu}\boldsymbol{a}_\nu + \frac{1}{2}(l^\lambda_\lambda l^\mu_\mu - l^\lambda_\mu l^\mu_\lambda)\boldsymbol{N}\Big]$$

where

$$l^\kappa_\alpha = \delta^\kappa_\alpha + u^\kappa{}_{|\alpha} - b^\kappa_\alpha w \qquad\qquad \varphi_\alpha = w_{,\alpha} + b^\kappa_\alpha u_\kappa$$

Here a and $\overline{a}$ are the determinants of the first quadratic forms of the middle surfaces of the undeformed and deformed shells, respectively. After linearization we get

$$\frac{\overline{a}}{a} = 1 + 2\varepsilon^\alpha_\alpha$$

Let $\Omega_1 \subset \mathbb{R}^2$ be such that $\Omega \subset \Omega_1$. As previously, $z = f(x_\alpha)$, $(x_\alpha) \in \Omega_1$, defines a rigid obstacle. The impenetrability condition is now given by

$$\boldsymbol{r}(\xi_\alpha)\cdot\boldsymbol{e}_3 + h\overline{\boldsymbol{N}}\cdot\boldsymbol{e}_3 \leqslant f(\boldsymbol{r}\cdot\boldsymbol{e}_\alpha) \tag{4.3}$$

After the linearization of $\overline{\boldsymbol{N}}$ we get

$$\overline{\boldsymbol{n}} = -(w_{,\mu} + b_{\mu\sigma}u^\sigma)\boldsymbol{a}^\mu + \boldsymbol{N}$$

For the linear Koiter shell model the strain measures are

$$\varepsilon_{\alpha\beta}(\boldsymbol{u}, w) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w$$

$$\kappa_{\alpha\beta}(\boldsymbol{u}, w) = -w_{|\alpha\beta} - b^\gamma_{\alpha|\beta}u_\gamma - b^\gamma_\alpha u_{\gamma|\beta} - b^\gamma_\beta u_{\gamma|\alpha} + b_{\alpha\beta}w$$

Here $\boldsymbol{b} = (b_{\alpha\beta})$ is the second quadratic form of the middle surface.

The total potential energy of the shell is expressed by

$$\begin{aligned} J(\boldsymbol{u}, w) &= L(\boldsymbol{u}, w) + \\ &+\frac{1}{2}\int\limits_\Omega \Big[A_{\alpha\beta\lambda\mu}\varepsilon_{\alpha\beta}(\boldsymbol{u}, w)\varepsilon_{\lambda\mu}(\boldsymbol{u}, w) + D_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(\boldsymbol{u}, w)\kappa_{\lambda\mu}(\boldsymbol{u}, w)\Big]\sqrt{a}\,dx \end{aligned} \tag{4.4}$$

where $A_{\alpha\beta\lambda\mu} \in L^\infty(\Omega)$, $D_{\alpha\beta\lambda\mu} \in L^\infty(\Omega)$ and $L(\boldsymbol{u}, w)$ is the functional of external loadings. The precise form of L is not needed, it suffices to assume that it is weakly continuous in the topology of $H^1(\Omega)^2 \times H^2(\Omega)$.

For the linear Koiter shell being in unilateral contact with the obstacle the set of constraints is given by

$$\mathcal{K}_s = \left\{(\boldsymbol{u}, w) \in H(\Omega)^2 \times H^2(\Omega) \;\middle|\; \boldsymbol{r}(\xi_\alpha) \cdot \boldsymbol{e}_3 + h\overline{\boldsymbol{n}} \cdot \boldsymbol{e}_3 \leqslant f(\boldsymbol{r} \cdot \boldsymbol{e}_\alpha), \quad (\xi_\alpha) \in \Omega\right\}$$

For a discussion of Sobolev's spaces of functions defined on the middle surface of the shell the reader is referred to Bernadou (1996) and Lewiński and Telega (2000). The set $\mathcal{K}_s$ is weakly closed. The proof is similar to the one given by Bielski and Telega (1998) for plates, cf also Baiocchi et al. (1988).

Let the shell be clamped along $\partial S_0 \subset \partial S$. Now we formulate the minimization problem.

Problem (P_s)

Find

$$\inf\left\{J(\boldsymbol{u}, w) \;\middle|\; (\boldsymbol{u}, w) \in \mathcal{K}_s, \quad \boldsymbol{u} = \boldsymbol{0}, \quad w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \text{ on } \partial S_0\right\}$$

Now we are in a position to formulate the existence theorem.

Theorem. The problem (P_s) has at least one solution. □

Remark 4.1. The function $f(\boldsymbol{r} \cdot \boldsymbol{e}_\alpha)$ can be linearized, compare the linearization in the case of plates by Bielski and Telega (1998). The constraints set $\mathcal{K}_s$ is then convex. We observe that only partial results concerning unilateral contact problems for shells are available in the literature, cf Floss and Ulbricht (1994), Telega (1987). □

5. Augmented Lagrangian methods for nonconvex problems

In this section we propose augmented Lagrangian methods applicable to nonconvex contact problems. To this end we extend the approach developed by Ito and Kunisch (1990, 1995), cf also Bielski et al. (2000).

5.1. Nonconvex set of constraints

Ito and Kunisch (1990) carefully studied the augmented Lagrangian method directly applicable to geometrically linear problems in the case of convex sets of constraints, cf also Cea (1971).

This approach is now extended to geometrically nonlinear contact problems in the presence of nonconvex constraints. First, we consider the case where only the set of constraints is nonconvex. The algorithm, we are going to present, is applicable to geometrically linear structures where constraints are nonconvex.

The problem under investigation is

$$(\mathcal{P}) \qquad \min\Big\{\frac{1}{2}a(u,u) - l(u) \;\Big|\; g(u) \leqslant 0, \quad u \in B\Big\}$$

Here the following spaces and mappings are used: V is a Hilbert space; B is a reflexive Banach space continuously embedded into V; H is a Hilbert lattice with the inner product $\langle\cdot,\cdot\rangle$; $a(\cdot,\cdot) : V \times V$ is a bilinear and continuous, V-eliptic form, with $a(u,u) \geqslant C_0\|u\|_V^2$, for some $C_0 > 0$; $l : V \to \mathbb{R}$ is a continuous linear functional; $g : B \to H$ is in general a *nonconvex*, continuous, Gâteaux's differentiable mapping.

From the practical point of view, the expression "Hilbert lattice" merely means that the constraint $g(u) \leqslant 0$ appearing in problem $(\mathcal{P})$ is meaningful. For a general definition of spaces being lattices the reader is referred to Yosida (1978). We assume that

$$g(u) = G(u) - G_1(u) \tag{5.1}$$

where the mapping G is convex whilst G_1 is nonconvex. The Ito and Kunisch (1990) procedure can be extended by combining their augmented Lagrangian technique with an iterative procedure:
— the mth step

$$G(u) \leqslant G_1(u^{m-1}) \qquad m = 1, 2, \ldots \tag{5.2}$$

Then the set

$$\mathcal{K}_m = \Big\{u \in B \;\Big|\; G(u) \leqslant G_1(u^{m-1})\Big\} \tag{5.3}$$

is convex. At each step m we define a family of augmented Lagrangian problems by

$$(\mathcal{P})_{m,c,\lambda} \qquad L_{m,c}(u^m, \lambda^m) = \min\Big\{L_{m,c}(u,\lambda) \;\Big|\; u \in B\Big\}$$

where

$$L_{m,c}(u,\lambda) = \frac{1}{2}a(u,u) - l(u) + \langle\lambda, \widehat{g}_m(u,\lambda,c)\rangle + \frac{c}{2}\|\widehat{g}_m(u,\lambda,c)\|_H^2$$

and $\lambda \in H$, $c > 0$, $c \in \mathbb{R}^+$. Moreover

$$\widehat{g}_m(u,\lambda,c) = \sup\Big(g_m(u), -\frac{\lambda}{c}\Big)$$

The mapping g_m is defined by

$$g_m(u) = G(u) - G(u^{m-1}) \tag{5.4}$$

The Algorithm

(1) Choose $\lambda_1^m \in H$, $\lambda_1^m \geqslant 0$, and $c > 0$
(2) put $n = 1$
(3) solve $(\mathcal{P})_{m,c,\lambda_n^m}$ for u_n^m
(4) put $\lambda_{n+1}^m = \lambda_n^m + c\widehat{g}(u_n^m, \lambda_n^m, c) = \sup(0, \lambda_n^m + cg(u_n^m))$
(5) put $n = n+1$ and return to (3).

We observe that the parameter c may also depend on m.

Applying Ito and Kunisch's (1990) results we get

$$C_0 \sum_{n=1}^{\infty} \|u_n^m - u^m\|_V^2 \leqslant \frac{1}{2c}\|\lambda_1^m - \lambda^m\|_H^2 \leqslant \sup_{m\geqslant 1} \frac{1}{2c}\|\lambda_1^m - \lambda^{*m}\|_H^2 < \infty \tag{5.5}$$

since c can be taken sufficiently large, such that for each $m \in \mathbb{N}$ we have

$$\frac{1}{2c}\|\lambda_1^m - \lambda^m\|_H^2 < C_1 \qquad\qquad C_1 > 0 \tag{5.6}$$

Let us pass to examples.

Example 5.1. As we already know, the sets of constraints given by K, K_1, and K_2 are, in general, nonconvex. We can easily introduce sequences of *convex* sets of constraints by

$$\begin{aligned}\mathcal{K}^m &= \Big\{(\boldsymbol{u}, w) \in H^1(\Omega)^2 \times H^2(\Omega) \;\Big| \\ &\qquad \Big|\; w(\boldsymbol{x}) + h \leqslant f\big(\boldsymbol{x} + \boldsymbol{u}^{m-1}(\boldsymbol{x}) - h\nabla w^{m-1}(\boldsymbol{x})\big),\;\; \boldsymbol{x} \in \Omega\Big\}\end{aligned}$$

and

$$\mathcal{K}_1^m = \Big\{(\boldsymbol{u}, w, \varphi) \in V_1 \;\Big|\; w(\boldsymbol{x}) + h \leqslant f\big(\boldsymbol{x} + \boldsymbol{u}^{m-1}(\boldsymbol{x}) + h\varphi^{m-1}(\boldsymbol{x})\big),\;\; \boldsymbol{x} \in \Omega\Big\}$$

and similarly for the set $\mathcal{K}_2^m$, $m = 1, 2, \ldots$

Example 5.2. To cope with geometrically nonlinear plates we aditionally introduce a sequence of bilinear forms. For instance, in the case of von Kármán's plates we take

$$a_m(\boldsymbol{u}, w; \boldsymbol{u}, w) = \int_{\Omega} \Big[A_{\alpha\beta\lambda\mu}\Big(\varepsilon_{\alpha\beta}(\boldsymbol{u}) + \frac{1}{2}w^{m-1}_{,\alpha}w^{m-1}_{,\beta}\Big) \cdot$$
$$\cdot\Big(\varepsilon_{\lambda\mu}(\boldsymbol{u}) + \frac{1}{2}w^{m-1}_{,\lambda}w^{m-1}_{,\mu}\Big) + B_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(w)\kappa_{\lambda\mu}(w)\Big]\, dx \qquad m = 1, 2, \ldots$$

Another possibility is to introduce the following sequence of the bilinear forms

$$\widetilde{a}_m(\boldsymbol{u}, w; \boldsymbol{u}, w) = \int_{\Omega} \Big[A_{\alpha\beta\lambda\mu}\Big(\varepsilon_{\alpha\beta}(\boldsymbol{u}) + \frac{1}{2}w_{,\alpha}w^{m-1}_{,\beta}\Big) \cdot$$
$$\cdot\Big(\varepsilon_{\lambda\mu}(\boldsymbol{u}) + \frac{1}{2}w_{,\lambda}w^{m-1}_{,\mu}\Big) + B_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(w)\kappa_{\lambda\mu}(w)\Big]\, dx \qquad m = 1, 2, \ldots$$

Then, instead of the problem $(\mathcal{P})$, we have a sequence of the following problems

$$(\mathcal{P}_m) \qquad \min\Big\{\frac{1}{2}a_m(\boldsymbol{u}, w; \boldsymbol{u}, w) - l(\boldsymbol{u}, w) \;\Big|\; g(\boldsymbol{u}, w) \leqslant 0, \;\; (\boldsymbol{u}, w) \in \mathcal{K}_2\Big\} \tag{5.7}$$

$m = 1, 2, \ldots$, and similarly in the case of $\widetilde{a}_m$.

Here $l(\boldsymbol{u}, w)$ is the loading functional. If K_2 is a nonconvex set, in order to use the previouly outlined augmented Lagrangian method, we have to replace K_2 by a sequence of convex sets of the constraints $\mathcal{K}_2^m$.

5.2. Nonconvex extension of Ito and Kunisch's (1995) augmented Lagrangian method

Ito and Kunisch (1995) investigated an augmented Lagrangian method for a significant class of nonsmooth convex optimization problems in infinite dimensional Hilbert spaces. More precisely, let X, H be real Hilbert spaces and $\mathcal{K}$ a closed convex subset of X. Consider the minimization problem

$$(\mathcal{Q}) \qquad \min\Big\{J(u) + \varphi(\Lambda u) \;\Big|\; u \in \mathcal{K}\Big\}$$

where $J : X \to \mathbb{R}$ is a lower, semicontinuous differentiable, convex function, $\Lambda \in L(X, H)$ and $\varphi : X \to \mathbb{R}$ is a proper, lower semicontinuous convex function. The convex functional φ is not necessarily smooth; in applications it can be an indicator function of a closed convex set. Several examples of the linear and continuous operator Λ are provided by Ito and Kunisch (1995).

For instance, in unilateral contact problems with constraints inposed on the boundary, Λ is a trace operator (in the sense of value of a function on the boundary).

A smooth approximation of φ yields the following problem:

$$(\mathcal{Q}) \qquad \min\Big\{ L_c(\boldsymbol{u},\lambda) \;\Big|\; \boldsymbol{u} \in \mathcal{K} \Big\}$$

where

$$\begin{aligned} L_c(\boldsymbol{u},\lambda) &= J(\boldsymbol{u}) + \varphi_c(\Lambda u,\lambda) \\ \varphi_c(\boldsymbol{v},\lambda) &= \inf\Big\{ \varphi(\boldsymbol{v}-\boldsymbol{u}) + \langle \lambda, u\rangle_H + \frac{c}{2}\|u\|_H^2 \Big\} \end{aligned} \tag{5.8}$$

Here $(c,\lambda) \in \mathbb{R}^+ \times H$. We observe that $\varphi(\cdot,\lambda)$ is (Lipschitz) continuously Fréchet differentiable.

Ito and Kunisch (1995) developed the following augmented Lagrangian method involving a sequential minimization:

Augmented Lagrangian Algorithm

Step 1: Choose a starting value $\lambda_1 \in H$, a positive number c and set $k=1$.

Step 2: Having given $\lambda_k \in H$ find $u_k \in \mathcal{K}$ by

$$L_c(u_k,\lambda_k) = \min\Big\{ L_c(u,\lambda_k) \;\Big|\; u \in \mathcal{K} \Big\}$$

Step 3: Update λ_k by $\lambda_{k+1} = \varphi_c'(\Lambda u_k,\lambda_k)$, where φ' denotes the Fréchet derivative of the functional $\varphi(\cdot,\lambda)$.

Step 4: If the convergence criterion is not satisfied then set $k=k+1$ and go to Step 2.

Under suitable, physically plausible assumptions, the just sketched augmented Lagrangian algorithm converges.

Obviously, this algorithm is not directly applicable to nonconvex contact problems of, say, finitely deformed elastic bodies and geometrically nonlinear structures. There are three basic sources of nonconvexity:

(i) a nonconvex functional J,

(ii) a nonconvex functional φ,

(iii) nonlinear operator appearing in the functional φ.

Such an operator is denoted by N. Obviously, in practice, various combinations of the cases (i)-(iii) are important.

For geometrically nonlinear problems the functional φ is usually an indicator function of a (weakly) closed and nonconvex set, cf Examples (5.1), (5.2) and He et al. (1996). We already know how to generate a sequence of convex sets of constraints. A large class of geometrically nonlinear problems leads to the functional J of the form, cf Bielski and Telega (1985b), Gałka and Telega (1992)

$$J(u) = G(\widetilde{\Lambda}u) + F(u) \tag{5.9}$$

where G represents the functional of the total internal energy whilst F is the loading functional, usually a linear one. The functional G is nonconvex. For nonlinear structures it can often be written as follows

$$G(\widetilde{\Lambda}u) = G(\Lambda_1 u, \Lambda_2 u) \tag{5.10}$$

where the functional $G(\cdot, \Lambda_2 u)$ is *convex* whilst $G(\Lambda_1 u, \cdot)$ is nonconvex. To use the augmented Lagrangian method we combine the approach by Ito and Kunisch (1995) with the iterative procedure. To this end we set

$$G_m(\widetilde{\Lambda}u) = G(\Lambda_1 u, \Lambda_2 u^{m-1}) \qquad m = 1, 2... \tag{5.11}$$

and consider a sequence of regularized minimization problems

$$(\mathcal{Q})_{m,c,\lambda} \qquad \min\Big\{ G_m(\widetilde{\Lambda}u) + F(u) + \varphi_c(\Lambda u, \lambda) \;\Big|\; u \in \mathcal{K} \Big\}$$

Now we have a sequence of the convex problems $(\mathcal{Q})_{m,c,\lambda}$, $m = 1, 2, \ldots$, to which we can apply the augmented Lagrangian method developed by Ito and Kunisch (1995).

Consider now a more specific case of a body made of the Saint-Venant Kirchhoff material, cf Benaouda and Telega (1997). Let $\boldsymbol{F}$ stand for the deformation gradient, $\boldsymbol{F} = \nabla \boldsymbol{\chi}$, $\boldsymbol{\chi} = (\chi_i)$, $i = 1, 2, 3$. The stored energy function of isotropic Saint-Venant Kirchhoff material is expressed by, see Ciarlet (1988), Benaouda and Telega (1997)

$$W(\boldsymbol{F}) = \frac{\mu}{4}\|\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}\|^2 + \frac{\lambda}{8}(\|\boldsymbol{F}\|^2 - 3)^2 \tag{5.12}$$

where λ and μ are the Lamé moduli. The function W is not of even rank-one convex, consequently it is neither quasiconvex nor polyconvex. However, defining the sequence of the convex function

$$W_m(\boldsymbol{F}) = \frac{\mu}{4}\|\boldsymbol{F}^\top \boldsymbol{F}_{m-1} - \boldsymbol{I}\|^2 + \frac{\lambda}{8}(\|\boldsymbol{F}_{m-1}\|^2 - 3)^2 \tag{5.13}$$

where $\boldsymbol{F}_{m-1} = \nabla \boldsymbol{\chi}^{m-1}$, $m = 1, 2, \ldots$, one can apply the outlined augmented Lagrangian method to various frictionless contact problems for bodies made of the Saint-Venant Kirchhoff material.

Remark 5.1.

- **(i)** It seems possible to apply the approach sketched for the Saint-Venant Kirchhoff stored energy functions to other, well-known hyperelastic materials. Such stored energy functions were discussed by Ciarlet (1988) and Ogden (1984). Obviously, the choice of the sequence W_m, $m = 1, 2, \ldots$, is not unique and depends on the particular case.
- **(ii)** The paper by Telega and Gałka (2001) reviews various applications of augmented Lagrangian methods, including contact problems, cf also Telega and Gałka (1998), Telega and Jemioło (2001). However, the presented approach seems to be novel.
- **(iii)** Our study is confined to frictionless contact problems. Unilateral contact problems with friction are still more complicated. It seems possible to extend the Ito and Kunisch (1990, 1995) augmented Lagrangian methods to contact problems with friction by combining these methods with time discretization. □

6. Specific one-dimensional nonconvex contact problem

In this section we are going to study a simple one-dimensional nonconvex contact problem. Consider the following minimization problem. Find $u \in K$ such that

$$J(u) = \inf_{v\in K} J(v)$$

where

$$J(u) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} a\Big(u_{,x} + \frac{1}{2}u_{,x}u_{,x}\Big)^2\;dx + \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} bu(x)\;dx \qquad\qquad a > 0$$

$$K = \Big\{u \in W^{1,4}(0,1)\;\Big|\;u\Big(-\frac{1}{2}\Big) = u\Big(\frac{1}{2}\Big) = 0, \quad g(u) \leqslant 0\Big\}$$

Particular forms of the function g are given below. Anyway, we assume that the set K is convex. To solve this problem we introduce the sequence of functionals, cf the previous section

$$J_m(u) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} a\Big(u_{,x} + \frac{1}{2}u_{,x}u_{,x}^{m-1}\Big)^2\;dx + \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} bu(x)\;dx \qquad\qquad m = 1, 2, \ldots$$

and the family of augmented Lagrangians

$$L_{m,c,\lambda}(u) = J_m(u) + \frac{1}{2c} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left[(\sup\{0,\ \lambda + cg(u)\})^2 - \lambda^2 \right] dx$$

To apply the augmented Lagrangian method we consider two cases of constraints

$$g(u) = \begin{cases} -u(x) - \dfrac{1}{64} & \text{case (a)} \\ -u(x) - \dfrac{65}{64} - \sqrt{1-x^2} & \text{case (b)} \end{cases}$$

In the first case we put: $b = -1$, $u^0(x) = 0$, $\lambda_0 = 1$, $c = 200$.

In case (b) we take $b = -1$, $u^0(x) = 0$, $\lambda_0 = 1$, $c = 50$.

The results of calculations are presented in Fig. 2 -Fig. 4. They have been obtained by using FEM.

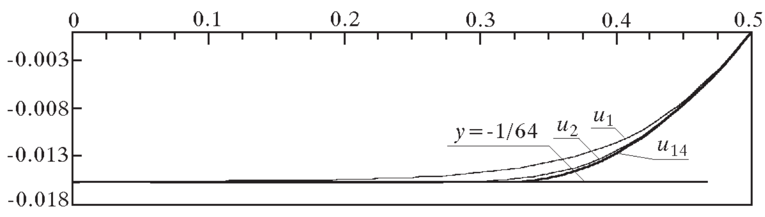

Fig. 2. The function $u(x)$ in case (a), steps 1,2 and 14; $c = 200$, c – the parameter in the augmented Lagrangian

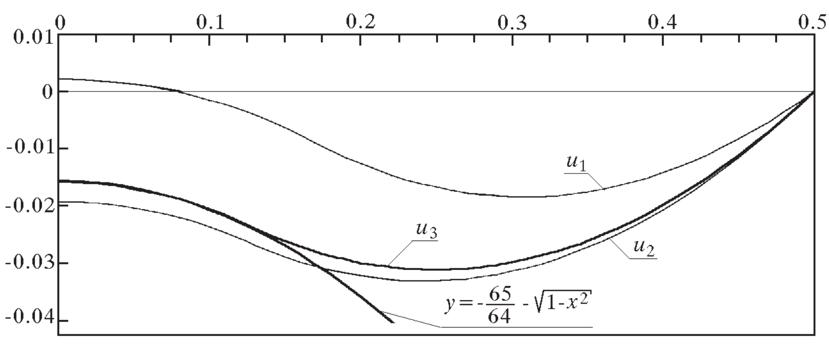

Fig. 3. The function $u(x)$ in case (b), steps 1,2 and 3; $c = 50$

We observe that the Lagrangian multiplier λ represents the contact forces. The augmented Lagrangian solutions tend to the problem with the obstacle.

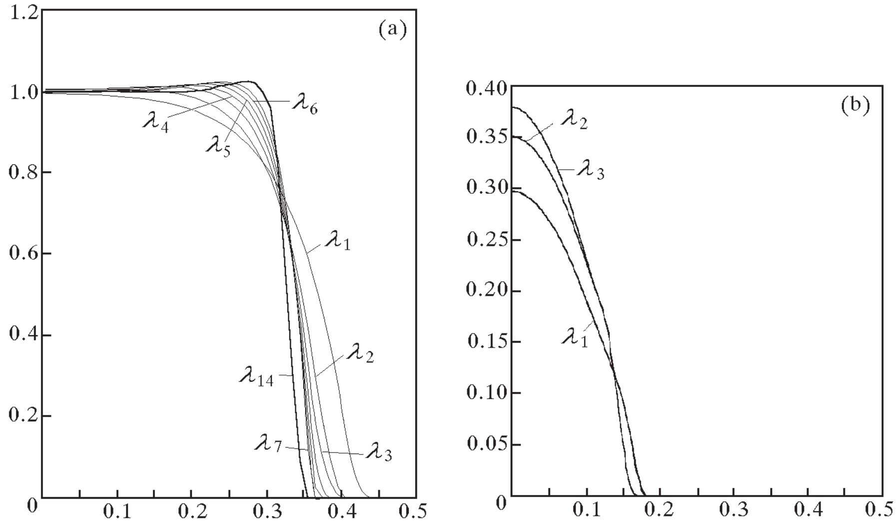

Fig. 4. Lagrangian multiplier λ for cases (a) and (b)

7. Augmented Lagrangians and nonconvex duality

In a series of papers we studied dual problems for nonlinear elastic solids and structures, cf Bielski and Telega (1985a,c,d, 1986, 1988, 1992, 1996), Bielski et al. (1988, 1989), Telega et al. (1988), Gałka et al. (1989), Telega (1989, 1995), Gałka and Telega (1990, 1992, 1995). We derived the dual problems by using the duality theory expounded by Ekeland and Temam (1976). Unfortunately, this theory is more appropriate for convex problems, since in the case of nonconvex problems it imposes restrictions on dual variables. For instance, in the case of von Kármán's plates the matrix formed of the membrane forces $\mathbf{N} = (N_{\alpha\beta})$, $\alpha, \beta = 1, 2$, has to be positive semi-definite. Without this type of restriction the duality gap

$$\inf \mathcal{P} > \sup \mathcal{P}^* \tag{7.1}$$

arises. Here $(\mathcal{P})$ denotes the primal problem and $(\mathcal{P}^*)$ is its dual.

Rockafellar and Wets (1998) developed the duality theory which avoids the duality gap like that given by inequality (7.1). This duality theory, however, is confined to finite dimensional spaces. It means that its applicability is restricted to discrete or discretized problems, including contact problems of this type.

The aim of this section is to extend the nonconvex duality theory by Rockafellar and Wets (1998) to infinite dimensional spaces. Consequently, it will be possible to apply it to nonlinear solids and structures, thus extending the range of applicability of our previous results concerning the duality. Our approach combines some results presented by Ekeland and Temam (1976) with the developments of Rockafellar and Wets (1976).

Let V and Y be locally convex topological spaces, and V^*, Y^* their duals, cf Ekeland and Temam (1976). One may think of Sobolev's spaces and L^p-spaces. The space V is usually the space of kinematically admissible displacements. The primal problem means evaluating

$$(\mathcal{P}) \qquad \inf \mathcal{F}(u) = \inf\Big\{\Phi(u,0) \;\Big|\; u \in V\Big\}$$

where

$$\Phi(u,p) = J(u, \Lambda u - p) \tag{7.2}$$

and $\Lambda : V \to Y$ is a linear and continuous operator.

Definition 7.1. For a primal problem of minimizing $\mathcal{F}(u)$ over $u \in V$ and any dualizing parametrization $\mathcal{F} = \Phi(\cdot, 0)$ for a choice of $\Phi : V \times Y \to \overline{\mathbb{R}} = [-\infty, +\infty]$, consider any *augmenting functional* f; by which a proper, lower semicontinuos, convex functional is meant

$$f:\ Y \to \overline{\mathbb{R}} \qquad \text{with} \;\; \min f = 0 \quad \arg\,\min f = \{0\}$$

The corresponding augmented Lagrangian with the penalty parameter $c > 0$ is then the functional

$$\overline{L}(u, p^*, c) := \inf_{p\in Y}\Big\{\Phi(u,p) + cf(p) - \langle p^*, p\rangle\Big\} \tag{7.3}$$

The corresponding dual problem consists of maximizing over all $(p^*, c) \in Y^* \times (0,\infty)$ the functional

$$\overline{G}(p^*, c) := \inf\Big\{\Phi(u,p) + cf(p) - \langle p^*, p\rangle \;\Big|\; (u,p) \in V \times Y\Big\} \tag{7.4}$$

Here $\langle \cdot, \cdot \rangle :\ Y^* \times Y \to \overline{\mathbb{R}}$ denotes the *duality pairing*, cf Ekeland and Temam (1976).

To formulate the duality theorem we set

$$\begin{aligned} h(p) &:= \inf\Big\{\Phi(u,p) \;\Big|\; u \in V\Big\} \\ h_{c,f}(p) &:= \inf\Big\{\Phi_{c,f}(u,p) \;\Big|\; u \in V\Big\} \end{aligned} \tag{7.5}$$

where

$$\Phi_{c,f}(u,p) := \Phi(u,p) + cf(p) \tag{7.6}$$

The notion of the augmented Lagrangian in the nonconvex duality arises from the idea of replacing the known inequality in the convex duality

$$\left.\begin{array}{l}\inf \mathcal{P} = \sup \mathcal{P}^* \\ \overline{p}^* \in \arg\ \max(P^*)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} h(p) \geqslant h(0) + \langle \overline{p}^*, p\rangle \ \ \forall p, \\ \text{with } p(0) \neq -\infty \end{array}\right. \tag{7.7}$$

with one of the form

$$h(p) \geqslant h(0) + \langle \overline{p}^*, p\rangle - cf(p) \qquad \forall p$$

What makes the approach successful in modifying the dual problem to get rid of the duality gap is that the last inequality is identical to

$$h_{c,f}(p) \geqslant h_{c,f}(0) + \langle \overline{p}^*, p\rangle \qquad \forall p$$

Indeed, $h_{c,f}(p) = h(p) + cf(p)$ and $h_{c,f}(0) = h(0)$, because $f(0) = 0$.

The Lagrangian associated with $\Phi_{c,f}$ is $L_{c,f}(u, p^*) = \overline{L}(u, p^*, c)$, where $\overline{L}$ is defined by (7.3). The resulting dual problem consists of maximizing $G_{c,f} = -\Phi^*_{c,f}(0, \cdot)$ over $p^* \in Y^*$. We have

$$G_{c,f}(p^*) = \overline{G}(p^*, c)$$

We can apply the theory developed by Ekeland and Temam (1976) to this modified formulation, where $\Phi_{c,f}$ replaces Φ, and in that way capture new powerful features.

Theorem 7.1 (duality without convexity).

For the problem of minimizing $\mathcal{F}$ on V consider the augmented Lagrangian $\overline{L}(u, p^*, c)$ associated with the dualizing parametrization $\mathcal{F} = \Phi(\cdot, 0)$, $\Phi : V \times Y \to \overline{\mathbb{R}}$, and a certain augmented functional $f : Y \to \overline{\mathbb{R}}$. Suppose that $\Phi(u, p)$ is level-bounded in u locally uniformly in p, and let $h(p) := \inf\{\Phi(u, p) \mid u \in V\}$. Suppose further that $\inf_u \overline{L}(u, p, c) > -\infty$ for at least one $(u, c) \in V \times (0, \infty)$. Then

$$\mathcal{F}(u) = \sup_{p^*, c} \overline{L}(u, p^*, c) \qquad\qquad \overline{G}(p^*, c) = \inf_u \overline{L}(u, p^*, c)$$

where actually $\mathcal{F}(u) = \sup_{p^*} \overline{L}(u, p^*, c)$ for every $c > 0$, and in fact

$$\inf_{u \in V} \mathcal{F}(u) = \inf_u [\sup_{p^*, c} \overline{L}(u, p^*, c)] = \sup_{p^*, c} [\inf_u \overline{L}(u, p^*, c)] = \sup_{p^*, c} \overline{G}(p^*, c)$$

Moreover, the optimal solutions to the primal and *augmented dual problems* are characterized as saddle points of the augmented Lagrangian

$$\left.\begin{array}{l}\overline{u} \in \arg\ \min \mathcal{F}(u)\\ (\overline{p}^*, c) \in \arg\ \max\limits_{p^*,c} \overline{G}(p^*, c)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\inf\limits_{u} \overline{L}(u, \overline{p}^*, \overline{c}) = \overline{L}(\overline{u}, \overline{p}^*, \overline{c}) =\\ = \sup\limits_{p^*,c} \overline{L}(\overline{u}, p^*, c)\end{array}\right.$$

the elements of $\arg\ \max_{p^*,c} \overline{G}(p^*, c)$ being precisely the pairs $(\overline{p}^*, \overline{c})$ with the property that

$$h(p) \geqslant h(0) + \langle \overline{p}^*, p\rangle - \overline{c} f(p) \qquad \forall p \qquad \square$$

The proof will be given elsewhere.

Let us recall the definition of the *level boundedness*, cf Rockafellar and Wets (1998).

A functional $g : V \to \overline{\mathbb{R}}$ is (lower) level bounded if for every $\alpha \in \mathbb{R}$ the set $\mathrm{level}_{\leqslant \alpha} g := \{\alpha \in V \mid g(u) \leqslant \alpha\}$ is bounded (possibly empty). This requirement can be replaced by *coercivity*.

Specific case

Consider now the case where

$$f(p) = \frac{1}{2}\|p\|^2 = \frac{1}{2}\|p\|^2_{L^2} \tag{7.8}$$

Then, since f is finite we have

$$\overline{L}(u, p^*, c) = \sup_{q^*}\Big\{L(u, q^*) - \frac{1}{2c}\|q^* - p^*\|^2\Big\} = \sup_{q^*}\Big\{L(u, p^* - q^*) - \frac{1}{2c}\|q^*\|^2\Big\}$$

where L is the standard Lagrangian

$$L(u, q^*) = \inf\Big\{\varPhi(u, p) - \langle p^*, p\rangle \ \Big|\ p \in Y\Big\} = -\sup\Big\{\langle p^*, p\rangle - \varPhi(u, p) \ \Big|\ p \in Y\Big\}$$

For $\varPhi$ being given by (7.2) we get, cf Ekeland and Temam (1976)

$$L(u, q^*) = -\langle q^*, \varLambda u\rangle - J^*_u(-q^*)$$

where J_u denotes the functional $p \to J(u, p)$ and J^*_u is its dual defined by

$$J^*_u(q^*) = \sup\Big\{\langle q^*, q\rangle - J(\cdot, q) \ \Big|\ q \in Y\Big\}$$

After some calculations we obtain, cf (7.4)

$$\overline{G}(p^*, c) = \inf_{u} \overline{L}(u, p^*, c) = \sup_{q^*}\Big\{-J^*(\varLambda^* q^*, -q^*) - \frac{1}{2c}\|q^* - p^*\|^2\Big\} \tag{7.9}$$

Here Λ^* is the adjoint (dual) operator of Λ.

For the practically important case where

$$J(u, \Lambda u) = G(\Lambda u) + F(u)$$

we calculate

$$\begin{aligned} \overline{G}(p^*, c) &= \sup_{q^*} \Big\{ -G^*(-q^*) - F^*(\Lambda^* q^*) - \frac{1}{2c} \|q^* - p^*\|^2 \Big\} = \\ &= -\inf \Big\{ G^*(-q^*) + F^*(\Lambda^* q^*) + \frac{1}{2c} \|q^* - p^*\|^2 \Big\} \end{aligned} \tag{7.10}$$

Remark 7.1. From relations (7.9) we conclude that, at least for the augmented functional given by Eq. (7.8), the dual functional $\overline{G}$ consists of the standard term $J^*(\Lambda^* q^*, -q^*)$ and the regularizing term $\|q^* - p^*\|^2/(2c)$. According to the terminology given by Rockafellar and Wets (1998) the dual function $\overline{G}(p^*, c)$ is then the minus of the Moreau envelope of $J^*(\Lambda^* q^*, -q^*)$. □

Remark 7.2. Indicator functions of a set determining constraints can be included into the functional F. □

Example 7.1. Consider a simple case of the nonconvex functional G in the one-dimensional case of an elastic nonlinear rod. Then

$$G(\Lambda u) = \frac{1}{2} \int_0^l a \Big(u_{,x} + \frac{1}{2} u_{,x}^2 \Big)^2 \, dx \qquad a > 0$$

and

$$\|q^* - p^*\|^2 = \int_0^l (q^* - p^*)^2 \, dx$$

The primal problem means evaluating

$$(cP) \qquad \inf \Big\{ G(\Lambda u) - \int_0^l r(x) u(x) \, dx \;\Big|\; u \in W^{1,4}(0, l), \;\; u(0) = u(l) = 0 \Big\}$$

Now $\Lambda u = (u_{,x}, u_{,x})$, and

$$G(q_1, q_2) = \int_0^l W(q_1, q_2) \, dx$$

We recall that the operator Λ has to be linear. Standard calculation yields, cf Bielski and Telega (1985b), Gałka and Telega (1995)

$$W^*(q_1^*, q_2^*) = \frac{1}{2a}(q_1^*)^2 + \begin{cases} 0 & \text{if} \quad q_2^* = 0 \ \wedge \ q_1^* \geqslant 0 \\ \dfrac{1}{2q_1^*}(q_2^*)^2 & \text{if} \quad q_1^* > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{7.11}$$

The physical meaning of the dual variable q_1^*, q_2^* is: $q_1^* = N := \sigma_x$, $q_2^* = Nu_{,x}$, where σ_x is the normal stress. From Eq. (7.11) we conclude that N has to be non-negative, i.e., the classical duality theory admits only tension. To include compression we use the developed nonconvex duality theory. Now we have

$$\begin{aligned} \overline{G}(N,Q,c) = \inf_{(\widetilde{N},\widetilde{Q})\in[L^{4/3}(0,l)]^2} \Big\{ \int\limits_0^l \Big[W^*(\widetilde{N}(x), \widetilde{Q}(x)) + \\ + \frac{1}{2c}[(\widetilde{N}-N)^2 + (\widetilde{Q}-Q)^2]\Big]\, dx + I_{\mathcal{S}}(\widetilde{N},\widetilde{Q})\Big\} \end{aligned} \tag{7.12}$$

where

$$\begin{aligned} \mathcal{S} \ = \ & \Big\{(N,Q) \in [L^{4/3}(0,l)]^2 \ \Big| \ (N+Q)_{,x} \in L^{4/3}(0,l), \\ & (N+Q)_{,x} + r = 0, \quad x \in (0,l)\Big\} \end{aligned}$$

provided that the rod is clamped at $x = 0$ and $x = l$. Here $r(x)$ $(x \in (0,l))$ denotes the loading distributed along the rod. It can be shown that

$$\overline{G}(N,Q,c) = -\int\limits_0^l W_c^*(N,Q)\, dx - I_{\mathcal{S}}(N,Q) \tag{7.13}$$

where

$$W_c^*(N,Q) = \begin{cases} W^*(N,Q) & \text{if} \quad Q = 0 \ \wedge \ N \geqslant 0 \text{ or } N > 0 \\ \dfrac{1}{2c}(N^2 + Q^2) & \text{otherwise} \end{cases} \tag{7.14}$$

Now we conclude that the normal force N in the problem (P^*) *is not necessarily non-negative*, due to the regularization given by Eq. (7.14).

The augmented dual problem takes eventually the form

$$\sup\Big\{\overline{G}(N,Q,c) \ \Big| \ (N,Q) \in \mathcal{S}, \quad c > 0\Big\}$$

Remark 7.3. The augmented dual problem is by no means unique. There are a lot of problems related to the augmenting functionals $f(p)$, satisfying the conditions specified in Theorem 7.1. □

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Metody rozszerzonego lagranżianu dla pewnej klasy wypukłych i niewypukłych zagadnień kontaktowych

Streszczenie

Cel pracy jest trojaki. Po pierwsze, sformułowane zostały jednostronne zagadnienia kontaktowe dla trzech modeli płyt oraz liniowego modelu powłok Koitera. Warunki kontaktu zostały sformułowane na powierzchni będącej w kontakcie z podłożem, a nie na powierzchni środkowej płyty lub powłoki. Takie ścisłe podejście prowadzi do niewypukłych zadań minimalizacji, nawet w przypadku płyt cienkich. Dla każdego zagadnienia sformułowano twierdzenie o istnieniu rozwiązań. Po drugie, metody rozszerzonego lagranżianu Ito i Kunischa (1990, 1995) uogólnione zostały na przypadek zagadnień niewypukłych. Po trzecie, teoria dualności Rockafellara i Wetsa (1998), opracowana dla skończenie wymiarowych zagadnień niewypukłych, została rozszerzona na przypadek układów ciągłych. Podano również kilka przykładów.

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