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INTERFACE INCLUSION PROBLEMS IN LAMINATED MEDIUM

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A moduli perturbation method is used to construct the solution for the contact stiffness in the inclusion problem of a composite laminate under a state of torsional deformation. The inclusion is considered to be embedded at the interface of the laminate. The following solutions have been obtained: (i) the exact solution for the inclusion in a medium consisting of two layers and two half- spaces, (ii) a first-order accurate solution for a layered medium consisting of $(n+m)$ constituents, (iii) an approximative solution for a physically inhomogeneous medium as the limiting case of the layered medium.

Key words: anisotropy, layered media, inclusion, perturbation method, torsion interface

1. Introduction

The stress analysis of laminated fiber-reinforced composite materials has been a subject of increasing importance due to the expanded use of such materials in diverse modern engineering applications. This analysis is not easy because composite laminates very often contain interlaminar inclusions, cracks or delaminations, which has been observed as a common and unavoidable occurrence in many practical situations. It is well known that studies of the inclusions problems for a bonded multiphase medium require special physical and analytical considerations that are not encountered in those corresponding to their homogeneous counterpars. In the multiphase system, the solution arises not only from a geometric discontinuity but also from the material discontinuity. The plane inclusion problem for two isotropic planes solved by means of singular integral equations methods was considered by Grilickiĭ and Sulim

(1975). Some inclusion problems and a comprehensive list of references up to the year 1982 are presented in the book by Aleksandrov and Michitarian (1983). Jevtushenko et al. (1995) presented some plane contact problems for a layered medium with an interface inclusion in the framework of the homogenisation theory. The problem of an interface inclusion between dissimilar orthotropic half-spaces was considered by Rogowski (1993).

In this paper, the problem of an interface inclusion in a layered composite or in a physically nonhomogeneous medium is considered in the framework of the modulus perturbation approach. This method was applied by Gao (1991), Fan ey al. (1992) to the solution to some fracture and inhomogeneity problems, respectively. The analysis differs from the previous studies in a number of aspects. First, by using the modulus perturbation approach a closed-form solution for the torsional contact stiffness of a composite which consists of two dissimilar layers and two dissimilar half-spaces is obtained. This solution has also an advantage over the previous analyses by simplicity and analytical form of the results. Further, we show that the perturbation analysis actually provides the first-order accuracy solution to the contact stiffness of an elastically nonhomogeneous medium with arbitrary, piecewise constant (layered) or continuously varying elastic moduli in the depthwise direction.

2. Reference state

We use cylindrical coordinates and denote them by (r,ϑ,z). Consider an N-layered composite laminate containing an interfacial rigid circular thin inclusion of the radius a located between the two layers and twisted by a small angle φ by means of the torque T applied to the disc. The inclusion may represent the resinous or cementing material, which is used to transfer the anchoring loads to the geological medium, for instance.

As the reference state we choose the solution to the two-phase counterpart problem obtained by Rogowski (1993). If the circumferential axisymmetric displacement of the rigid disc is

$$\nu(r,0) = \varphi_0 r \qquad r \leqslant a \tag{2.1}$$

then the contact stresses are

$$\sigma_{z\vartheta_0}(r,0) = \frac{4}{\pi}\varphi_0(-1)^i\mu_i\frac{r}{\sqrt{a^2-r^2}} \qquad r < a \qquad (-1)^i z \leqslant 0 \tag{2.2}$$

and the rigid rotation φ_0 of the disc is given by

$$\varphi_0 = \frac{3T}{16(\mu_1 + \mu_2)a^3} \tag{2.3}$$

In equations (2.2) and (2.3) μ_i is the average shear modulus of the orthotropic material, i.e. $\mu_i = \sqrt{G_{ri}G_{zi}}$, G_{ri} and G_{zi} are the material shear moduli, and $i = 1, 2$ refers to bodies 1 and 2, respectively. The stress $\sigma_{z\vartheta_0}(r, z)$ and the displacement $\nu_0(r, z)$ inside the bimaterial medium are given by formulae (Rogowski, 1993)

$$\begin{aligned}
\nu_0^{(i)}(\xi_i, \eta_i) &= \frac{2}{\pi}\varphi_0 r\Big(\frac{\pi}{2} - \tan^{-1}\xi_i - \frac{\xi_i}{1 + \xi_i^2}\Big) \\
\sigma_{z\vartheta_0}^{(i)}(\xi_i, \eta_i) &= -\frac{4}{\pi}\varphi_0\mu_i\frac{\eta_i}{\xi_i^2 + \eta_i^2}\sqrt{\frac{1 - \eta_i^2}{1 + \xi_i^2}}
\end{aligned} \tag{2.4}$$

The two sets of the oblate spheroidal coordinates $\xi_i, \eta_i)$ are related to the cylindrical coordinates (r, z) by the equations

$$r^2 = a^2(1 + \xi_i^2)(1 - \eta_i^2) \qquad s_i z = a\xi_i\eta_i \qquad \begin{array}{c} \xi_i \geqslant 0 \\ -1 \leqslant \eta_i \leqslant 1 \end{array} \tag{2.5}$$

where $s_i = \sqrt{G_{ri}/G_{zi}}$ is the measure of the orthotropy and $s_i = 1$ represents an isotropic material. Solutions (2.1)-(2.4) correspond to the bimaterial infinite medium, which is chosen as the reference state or the 0th order solution.

3. Moduli perturbation analysis

3.1. Composite consisting of two dissimilar layers and two dissimilar half-spaces

Now the question is how the torque T and the rotation φ will be related with each other in the case of a layered medium. The exact answer to this question would require the exact solution. A review of the existing literature reveals that the exact analytical solution to this problem is not available. In this paper we obtain a closed-form solution to the problem without restrictions on the range of applicability, using the moduli perturbation analysis. Consider an interface inclusion in a system of two layers bonded to the inclusion and to two half-spaces. The range of convergence of the perturbation solution will be without an restriction if we choose a bimaterial infinite medium with two average moduli corresponding to the upper and the lower film as the reference state, i.e. $\mu_i = (\mu_f^{(i)} + \mu_s^{(i)})/2$, where $\mu_f^{(i)}$ and $\mu_s^{(i)}$ are the average shear moduli of the ith film and of the ith substrate, respectively. Then introducing material parameter

$$\kappa_i = \frac{\mu_f^{(i)} - \mu_s^{(i)}}{\mu_f^{(1)} + \mu_s^{(1)} + \mu_f^{(2)} + \mu_s^{(2)}} \tag{3.1}$$

we have $\kappa_i \in (-1, 1)$. The parameter κ_i denotes the ratio of the change of the average shear modulus of the ith layer (film) to the modulus of the reference bimaterial medium. Alternatively $-\kappa_i$ denotes the ratio of the change of the average shear modulus of the ith substrate to the modulus of the reference medium. During the transformation, the applied torque T is kept constant but the twist angle φ and the strain energy are allowed to change. The energy conservation law requires that the extra work done by the torque T be equal to the energy change in the whole body. To the first-order accuracy in the moduli variations κ_i for $0 < |z| < h_i$ and $-\kappa_i$ for $|z| > h_i$ the equation of the energy conservation reads

$$\frac{1}{2} T \delta\varphi_0 = -\frac{1}{2} \sum_{i=1}^{2} \kappa_i \Big(\int\limits_{A_f^{(i)}} \sigma_{z\vartheta_0}^{(i)} \nu_0^{(i)} \, dA_i + \int\limits_{A_s^{(i)}} \sigma_{z\vartheta_0}^{(i)} \nu_0^{(i)} \, dA_i \Big) \tag{3.2}$$

where $\delta\varphi_0$ is the rotation change, $\sigma_{z\vartheta_0}^{(i)}$ and $\nu_0^{(i)}$ are the known stresses and displacements for the reference bimaterial medium, and $A_f^{(i)}$, $A_s^{(i)}$ denote the boundary surfaces of the transforming regions. For the present of geometry $A_f^{(i)}$ consists of the three horizontal planes: $z = 0^+$ and $z = h_1^-$, $z = -h_2^-$ and $A_s^{(i)}$ consists of the two planes: $z = h + 1^+$ and $z = -h_2^+$. The area integral on the plane $z = 0$ is equal $T\varphi_0$. The right-hand side of Eq. (3.2) denotes the first order energy variation due to the moduli transformation. Eq. (3.2) reduces to the following expression for $\delta\varphi_0/\varphi_0$

$$\frac{\delta\varphi_0}{\varphi_0} = -\sum_{i=1}^{2} \kappa_i \Big[1 + \frac{2\pi}{T\varphi_0} \int\limits_0^\infty (\sigma_{z\vartheta_0}^{(i)} \nu_0^{(i)})_{z=\pm h_i^\pm} \, r \, dr \Big] \tag{3.3}$$

Substituting Eq. $(2.4)_1$ and (2.5) with $\mu_i = (\mu_f^{(i)} + \mu_s^{(i)})/2$ and noticing that $s_i z$ is equal $s_f^{(1)}$ for $0 < z \leqslant h_1$ and $s_s^{(1)} z + h_1(s_f^{(1)} - s_s^{(1)})$ for $z \geqslant h_1$ i.e. $\xi_0^{(1)} = s_f^{(1)} h_1/a$, and alternatively $\xi_0^{(2)} = s_f^{(2)} h_2/a$ for the planes $z = h_1$

and $z=-h_2$, respectively, and then integrating we arrive at the following expression for $\delta\varphi_0/\varphi_0$

$$\frac{\delta\varphi_0}{\varphi_0}=-\kappa_1\Big[2I\Big(\frac{s_f^{(1)}h_1}{a}\Big)-1\Big]-\kappa_2\Big[2I\Big(\frac{s_f^{(2)}h_2}{a}\Big)-1\Big] \tag{3.4}$$

where the integral $I(\xi_0^{(i)}$, $\xi_0^{(i)}=s_f^{(i)}h_i/a$, is found to be

$$\begin{aligned}&1+\frac{2\pi}{T\varphi_0}\int\limits_0^\infty(\sigma_{z\vartheta_0}^{(i)}\nu_0^{(i)})_{z=\pm h_i^\pm}\;r\;dr=\\&=1-\frac{6}{\pi}\xi_0^{(i)}\int\limits_{\xi_0^{(i)}}^\infty\Big(\frac{\pi}{2}-\tan^{-1}\xi_i-\frac{\xi_i}{1+\xi_i^2}\Big)\Big(\frac{1}{\xi_i^2}-\frac{(\xi_0^{(i)})^2}{\xi_i^4}\Big)\;d\xi_i=2I(\xi_0^{(i)})-1\end{aligned} \tag{3.5}$$

$$I(\xi_0^{(i)})=\frac{2}{\pi}\tan^{-1}\xi_0^{(i)}+\frac{\xi_0^{(i)}}{\pi}\Big[\Big(3+2(\xi_0^{(i)})^2\Big)\ln\Big(\frac{1+(\xi_0^{(i)})^2}{(\xi_0^{(i)})^2}\Big)-2\Big]$$

$$\xi_0^{(i)}=\frac{s_f^{(i)}h_i}{a}$$

The first order approximation of $\varphi,\varphi_1=\varphi_0+\delta\varphi_0$ is $0(\kappa_i)$, 1st order solution

$$\varphi_1=\varphi_0\Big[1-\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)-\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)\Big] \tag{3.6}$$

The corresponding higher order solutions are obtained recursively from the previous solutions, so that $\varphi_2=\varphi_0+\delta\varphi_1$, is $0(\kappa_i^2)$, 2nd order solution

$$\begin{aligned}\varphi_2\;&=\;\varphi_0\Big\{1-\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)-\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)+\\&+\;\Big[\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)+\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)\Big]^2\Big\}\end{aligned} \tag{3.7}$$

In general, one has $0(\kappa_i^n)$, nth order solution, $n>1$

$$\varphi_n=\varphi_0\left[1+\sum_{k=1}^n(-1)^k\Big[\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)+\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)\Big]^k\right] \tag{3.8}$$

It is seen that the sum converges to the exact solution as $n\to\infty$ in the range of convergence, i.e.

$$\begin{aligned}&\varphi=\lim_{n\to\infty}\varphi_n=\varphi_0\Big[1+\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)+\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)\Big]^{-1}=\\&=\frac{3T}{16a^3}\Big[\mu_s^{(1)}\Big(1-I(\xi_0^{(1)})\Big)+\mu_f^{(1)}I(\xi_0^{(1)})+\mu_s^{(2)}\Big(1-I(\xi_0^{(2)})\Big)+\mu_f^{(2)}I(\xi_0^{(2)})\Big]^{-1}\end{aligned} \tag{3.9}$$

if

$$\Big|\kappa_1\Big(2I(\xi_0^{(1)})-1\Big)+\kappa_2\Big(2I(\xi_0^{(2)})-1\Big)\Big|<1 \tag{3.10}$$

The weighting functions $I(\xi_0^{(i)})$, $\xi_0^{(i)}=s_f^{(i)}h_i/a$, vanish as $h_i\to 0$ and approach unity as $h_i\to\infty$, and monotonically increase with $\xi_0^{(i)}$. The inequality in Eq. (3.10) is satisfied since $|\kappa_1+\kappa_2|<1$ and $2I(\xi_0^{(i)})-1\in(-1,1)$ for any values of the material and geometric parameters.

The perturbation solution for φ in Eq. (3.9) perfectly matches the exact solution in both limiting bimaterial cases, i.e. as $s_f^{(i)}h_i/a\to 0$ and $s_f^{(i)}h_i/a\to\infty$, corresponding to the cases of the bimaterial infinite medium, with the interface inclusion, made entirely of the half-space materials or, alternatively, of two layers by transforming to the two half space regions $|z|>h_i$, respectively. It can be seen from Eqs (3.5) and (3.9) that the weighting functions $I(\xi_0^{(i)})$ and $1-I(\xi_0^{(i)})$ give the ratios of the strain energy stored in the transforming regions of the films and of the substrata, respectively, to the total strain energy stored in the reference bimaterial medium. The effective contact stiffness should have the form

$$\mu_{eff}=\mu_s^{(1)}\Big(1-I(\xi_0^{(1)})\Big)+\mu_f^{(1)}I(\xi_0^{(1)})+\mu_s^{(2)}\Big(1-I(\xi_0^{(2)})\Big)+\mu_f^{(2)}I(\xi_0^{(2)}) \tag{3.11}$$

Hence, it can be said that μ_{eff} is the average shear modulus weighted by the strain energy density distribution.

3.2. Nonhomogeneous medium with piecewise constant moduli

The modulus perturbation approach provides a natural channel for extension to more complicated problems. Using the previously presented analysis, it can be shown that the twist angle of the interface inclusion in the $(n+m)$th layer of the composite laminate can be obtained as

$$\varphi=\frac{T}{16a^3\mu_{eff}} \tag{3.12}$$

with the effective stiffness

$$\begin{aligned}\mu_{eff}&=\mu_s^{(1)}\Big[1-I\Big(\frac{z_ns_n^{(1)}}{a}\Big)\Big]+\mu_s^{(2)}\Big[1-I\Big(\frac{z_ms_m^{(2)}}{a}\Big)\Big]+\\&+\sum_{i=1}^{n}\mu_i^{(1)}\Big[I\Big(\frac{z_is_i^{(1)}}{a}\Big)-I\Big(\frac{z_{i-1}s_{i-1}^{(1)}}{a}\Big)\Big]+\sum_{i=1}^{n}\mu_i^{(2)}\Big[I\Big(\frac{|z_i|s_i^{(2)}}{a}\Big)-I\Big(\frac{|z_{i-1}|s_{i-1}^{(2)}}{a}\Big)\Big]\end{aligned} \tag{3.13}$$

Here $I(\cdot)$ is defined by Eq. (3.5), z_i is the z coordinate of the interface between the ith and $(i+1)$th layer (film), so that the layer thickness is given by $h_i = z_i - z_{i-1}$, $i = 1, 2, ...n$, with $z_0 = 0$. A rigorous analysis of the error bounds for the perturbation formula in Eq. (3.13) is not yet available. Hence, it is necessary to examine the following inequality

$$\begin{aligned} \Big| &\frac{1}{\mu_s^{(1)} + \mu_s^{(2)}} \Big\{ \sum_{i=1}^{n} \mu_i^{(1)} \Big[I\Big(\frac{z_i s_i^{(1)}}{a}\Big) - I\Big(\frac{z_{i-1} s_{i-1}^{(1)}}{a}\Big) \Big] + \\ &+ \sum_{i=1}^{n} \mu_i^{(2)} \Big[I\Big(\frac{|z_i| s_i^{(2)}}{a}\Big) - I\Big(\frac{|z_{i-1}| s_{i-1}^{(2)}}{a}\Big) \Big] \Big\} - \\ &- \frac{\mu_s^{(1)}}{\mu_s^{(1)} + \mu_s^{(2)}} I\Big(\frac{z_n s_n^{(1)}}{a}\Big) - \frac{\mu_s^{(2)}}{\mu_s^{(1)} + \mu_s^{(2)}} I\Big(\frac{|z_m| s_m^{(1)}}{a}\Big) \Big| < 1 \end{aligned} \tag{3.14}$$

3.3. Nonhomogeneous medium with continuously varying moduli

When the elastic modulus is a function of spatial coordinates then the problem becomes more complicated compared to the homogeneous case. It is noted that the perturbation solution can be obtained in that case under some restrictions. The solution to the problem of a rigid interface disc in the bimaterial nonhomogeneous medium with continuously varying moduli $G_z^{(i)}(z)$, $i = 1, 2$, and constant parameters $s^{(i)}$, $G_r^{(i)}(z) = s_i^2 G_z^{(i)}(z)$, can be constructed by taking the limits as $z_i \to z_{i-1}$ and $|z_i| \to |z_{i-1}|$ and $n \to \infty$ in Eq. (3.13). The effective contact stiffness has then the integral form

$$\mu_{eff} = s^{(1)} \int_0^\infty \frac{dI(s^{(1)} z/a)}{dz} G_z^{(1)}(z)\, dz + s^{(2)} \int_0^\infty \frac{dI(s^{(2)}|z|/a)}{d|z|} G_z^{(2)}(z)\, dz \tag{3.15}$$

where

$$\frac{dI(s^{(1)} z/a)}{dz} = \frac{3s^{(i)}}{\pi a} \Big[\Big(1 + 2\frac{(s^{(i)})^2 z^2}{a^2}\Big) \ln\Big(\frac{a^2 + (s^{(i)})^2 z^2}{(s^{(i)})^2 z^2}\Big) - 2 \Big] \tag{3.16}$$

When the elastic medium consists of two nonhomogeneous layers of the thickness h_1 and h_2 and the shear moduli $G_{zf}^{(1)}(z)$ and $G_{zf}^{(2)}(z')$ with $G_{rf}^{(i)}(z) = (s_f^{(i)})^2 G_{zf}^{(i)}(z)$, $s_f^{(i)} = \text{const}$ bonded to two homogeneous half-spaces with the moduli $G_{zs}^{(1)}$, $G_{rs}^{(i)} = (s_s^{(i)})^2 G_{zs}^{(i)}$ then the solution for the contact stiffness can be obtained as

$$\mu_{eff} = \mu_s^{(1)}\Big[1 - I\Big(\frac{s_f^{(1)}h_1}{a}\Big)\Big] + \mu_s^{(2)}\Big[1 - I\Big(\frac{s_f^{(2)}h_1}{a}\Big)\Big] +$$
$$+ s_f^{(1)}\int\limits_0^{h_1} \frac{dI(s_f^{(1)}z/a)}{dz} G_{zf}^{(1)}(z)\,dz + s_f^{(2)}\int\limits_0^{h_2} \frac{dI(s_f^{(2)}|z'|/a)}{d|z'|} G_{zf}^{(2)}(z')\,dz' \tag{3.17}$$

When a rigid inclusion is embedded in a nonhomogeneous infinite medium with the shear moduli $G_z(z)$ and $G_r(z) = s^2 G_z(z)$ then the contact stiffness is given by the equation

$$\mu_{eff} = \frac{3s^2}{\pi a}\int\limits_{-\infty}^{\infty} \Big[\Big(1 + 2\frac{s^2z^2}{a^2}\Big)\ln\Big(1 + \frac{a^2}{s^2z^2}\Big) - 2\Big] G_z(z)\,dz \tag{3.18}$$

The difficulty with the perturbation method is that in special cases the expression for the contact stiffness given by Eqs (3.15) or (3.18) may reduce to divergent integrals. We expect that these formulae have similar ranges of validity as Eq. (3.13).

3.4. Two-constituent composites

Rogowski (1995) performed an integral equation analysis to study an axisymmetric torsion interface inclusion, which appears between a boundary layer and dissimilar half-space (Fig. 2). In this analysis the solution of the first-order accuracy has the form of Eq. (3.12), where (in our notations)

$$\mu_{eff} = (\mu_1 + \mu_2)\Big[1 - \frac{\eta(3)}{3\pi\xi_0^3} + \frac{\eta(5)}{5\pi\xi_0^5} + ...\Big] \qquad \xi_0 = \frac{hs_1}{a}$$
$$\eta(2m+1) = \frac{\mu_1}{\mu_1 + \mu_2}\sum_{n=1}^{\infty}\Big(\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\Big)^{n-1}\frac{1}{n^{2m+1}} \tag{3.19}$$

In particular, if $\mu_1 = \mu_2$ then Eq. (3.19) yields

$$\mu_{eff} = 2\mu_1\Big[1 - \frac{a^3}{6\pi h^3 s_1^3} + \frac{a^5}{10\pi h^5 s_1^5} + ...\Big] \tag{3.20}$$

Solutions (3.18) and (3.19) are valid if $hs_1/a > 1$. The moduli perturbation approach, see Eq. (3.11), gives

$$\mu_{eff} = \mu_2 + \mu_1 I\Big(\frac{s_1 h}{a}\Big) \tag{3.21}$$

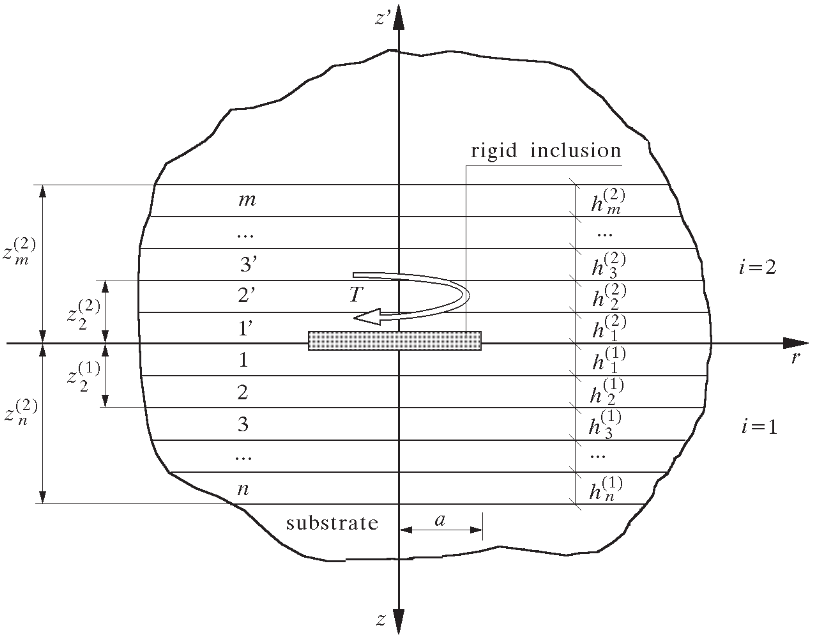

Fig. 1. Inclusions geometry, coordinates and notation

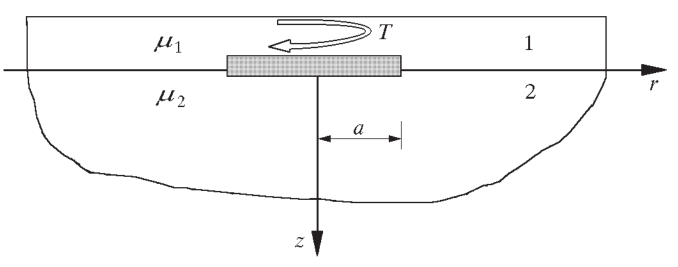

Fig. 2. Two-phase counterpart of the interface inclusion problem

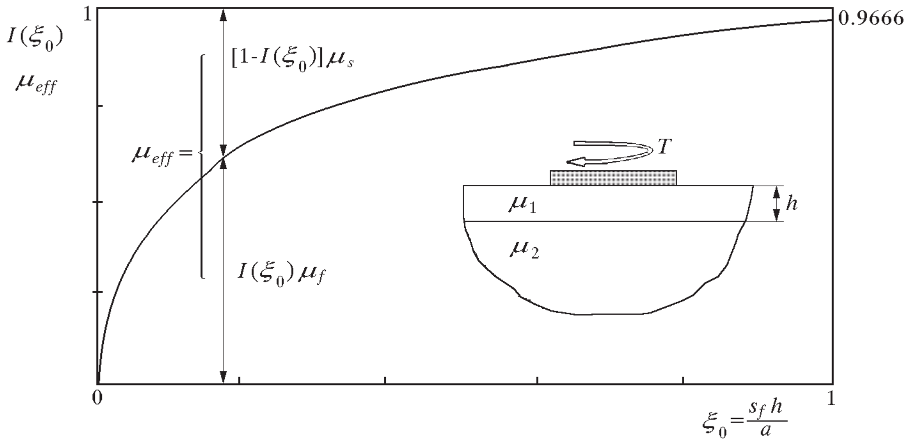

Fig. 3. Torsional stiffness of the rigid disc on a two-phase orthotropic half-space; Eq. (3.11) for $\mu_s^{(2)} = 0 = \mu_f^{(2)},\ \mu_f^{(1)} = \mu_f,\ \mu_s^{(1)} = \mu_s,\ \xi_0^{(1)} = \xi_0 = s_f h/a$

or for $\mu_1 = \mu_2$

$$\mu_{eff} = \mu_1\Big[1 + I\Big(\frac{s_1 h}{a}\Big)\Big] \tag{3.22}$$

Equations (3.21) and (3.22) can be rewritten as follows

$$\mu_{eff} = (\mu_1 + \mu_2)\Big\{1 - \frac{\mu_1}{\mu_1 + \mu_2}\Big[1 - I\Big(\frac{s_1 h}{a}\Big)\Big]\Big\} \tag{3.23}$$

and for $\mu_1 = \mu_2$

$$\mu_{eff} = 2\mu_1\Big\{1 - \frac{1}{2}\Big[1 - I\Big(\frac{s_1 h}{a}\Big)\Big]\Big\} \tag{3.24}$$

For a thick boundary layer $(hs_1/a) > 1$ Eqs (3.23) and (3.19) or (3.24) and (3.20) give the results which differ less than one percent. Perturbation solutions (3.21), (3.22) are valid also for small values of hs_1/a. For example, Eq. (3.21) yields

$$\mu_{eff} = \begin{cases} \mu_2 + 0.0880\mu_1 & \text{for} \quad s_1 h/a = 0.01 \\ \mu_2 + 0.4434\mu_1 & \text{for} \quad s_1 h/a = 0.10 \\ \mu_2 + 0.9666\mu_1 & \text{for} \quad s_1 h/a = 1.00 \end{cases} \tag{3.25}$$

If the inclusion is embedded at the finite distance h from the bimaterial interface of the infinite medium with the shear moduli μ_1 and μ_2 (in the material "1") then Eq. (3.11) yields

$$\mu_{eff} = \mu_1\Big[1 + I\Big(\frac{s_1 h}{a}\Big)\Big] + \mu_2\Big[1 - I\Big(\frac{s_1 h}{a}\Big)\Big] \tag{3.26}$$

This equation yields

$$\mu_{eff} = \begin{cases} 1.0880\mu_1 + 0.9120\mu_2 & \text{for} \quad s_1h/a = 0.01 \\ 1.4434\mu_1 + 0.5566\mu_2 & \text{for} \quad s_1h/a = 0.10 \\ 1.9666\mu_1 + 0.0334\mu_2 & \text{for} \quad s_1h/a = 1.00 \end{cases} \tag{3.27}$$

4. Concluding remarks

We have presented a modulus perturbation scheme for determining elastic contact stiffness distributed by inhomogeneities. Although finite element methods, see Laursen and Simo (1992) for instance, or boundary element methods, see e.g. Telles and Brebbia (1981), can handle these types of inhomogeneity problems, the present perturbation procedure still shows its advantages by simplicity and analytical form of the results. The presented closed-form perturbation solution (3.11) may be applicable without any restrictions, i.e. for any combinations of the elastic constants of material constituents, while the first-order accurate solutions (3.13), (3.15), and (3.17) may be applied in a moderate range of material combinations of practical significance. The effective elastic constants of the laminated medium are given by Achenbach (1975) and Christensen (1979). If we consider a laminated composite consisting of alternating plane parallel layers of two homogeneous isotropic materials, then Achenbach's and Christensen's results give

$$G_z = \frac{G_1G_2}{G_2\delta_1 + G_1(1-\delta_1)} \qquad\qquad G_r = G_1\delta_1 + G_2(1-\delta_1)$$

where $\delta_1 = h_1/(h_1+h_2)$ and where h_1, h_2 are the thicknesses and G_1, G_2 the shear moduli of the two elastic layers. Parallel to the above studies of elastic contact problems a homogenized model with microlocal parameters has also been developed to evaluate the effective stiffness of a layered body. The related works are given by Kaczyński and Matysiak (1988, 1992), Matysiak and Woźniak (1987).

From the solutions presented in this paper we conclude, that elastic properties of the boundary layer influence strongly the effective stiffness of a layered body and the solution depends also on the ratio of the layer thickness to the radius of contact region. The formulae mentioned in the literature do not describe these effects. By replacing the shear modulus with the effective shear modulus in equations (2.1), (2.2) and (2.3) we obtain the solution.

The mechanical torsion fields of a laminated orthotropic medium under and above the interface inclusion are given by:
— rotation

$$\varphi = \frac{3T}{16(\mu_{1\,eff} + \mu_{2\,eff})a^3} \tag{4.1}$$

— displacement

$$\begin{aligned}\nu^{(i)}(\xi_i,\eta_i) &= \varphi r\Big[1 - \frac{2}{\pi}\Big(\tan^{-1}\xi_i + \frac{\xi_i}{1+\xi_i^2}\Big)\Big] = \\ &= \begin{cases} \varphi & z = 0^{\pm} \quad r \leqslant a \\ \dfrac{2}{\pi}\varphi r\Big(\sin^{-1}\dfrac{a}{r} - \dfrac{a}{r}\sqrt{1-\dfrac{a^2}{r^2}}\Big) & z = 0^{\pm} \quad r \geqslant a \end{cases}\end{aligned} \tag{4.2}$$

— stresses

$$\begin{aligned}\sigma_{z\vartheta}^{(i)}(\xi_i,\eta_i) &= -\frac{4}{\pi}\varphi\mu_{i\,eff}\frac{\eta_i}{\xi_i^2+\eta_i^2}\sqrt{\frac{1-\eta_i^2}{1+\xi_i^2}} = \\ &= \begin{cases} \dfrac{4}{\pi}(-1)^i\varphi\mu_{i\,eff}\dfrac{r}{\sqrt{a^2-r^2}} & z = 0^{\pm} \quad r < a \\ 0 & z = 0^{\pm} \quad r > a \end{cases}\end{aligned} \tag{4.3}$$

$$\begin{aligned}\sigma_{r\vartheta}^{(i)}(\xi_i,\eta_i) &= -\frac{4}{\pi}\varphi G_r^{(i)}\frac{r^2}{a^2}\frac{\xi_i}{(1+\xi_i^2)^2 + (\xi_i^2+\eta_i^2)^2} = \\ &= \begin{cases} 0 & z = 0^{\pm} \quad r < a \\ -\dfrac{4}{\pi}\varphi G_r^{(i)}\dfrac{a^3}{r^2\sqrt{r^2-a^2}} & z = 0^{\pm} \quad r > a \end{cases}\end{aligned} \tag{4.4}$$

— stress concentration factors $(i = 1, 2)$

$$\begin{aligned} L_{z\vartheta}^{(I)} &= \frac{3T}{4\pi\sqrt{a^5}}(-1)^i\frac{\mu_{i\,eff}}{\mu_{1\,eff}+\mu_{2\,eff}} \qquad z = 0^{\pm} \quad r \to a^- \\ L_{r\vartheta}^{(I)} &= -\frac{3T}{4\pi\sqrt{a^5}}\frac{G_r^{(i)}}{\mu_{1\,eff}+\mu_{2\,eff}} \qquad z = 0^{\pm} \quad r \to a^+ \end{aligned} \tag{4.5}$$

In the above solutions $\mu_{i\,eff}$, $i = 1, 2$, is the effective contact stiffness of the lower and upper nonhomogeneous half-spaces, respectively. The oblate spheroidal coordinates associated with the material parameters s_i by Eq. (2.5) are continuous at the interfaces since the plane $z = z_i$, $r \geqslant 0$ is given by

$$\xi_i \geqslant \xi_i' = \frac{s_i z_i}{a} \qquad\qquad \eta_i = \frac{\xi_i'}{\xi_i} \tag{4.6}$$

The displacement ν and the stress $\sigma_{z\vartheta}$ are continuous in the laminated medium, but the stress $\sigma_{r\vartheta}$ has jumps in the interfaces, which are given by

$$[\![\sigma_{r\vartheta}^{(i)}]\!] = -\frac{4}{\pi}\varphi[G_r^{(i+1)} - G_r^{(i)}]\frac{r^2}{a^2}\frac{\xi_i^3}{(1+\xi_i^2)^2 + (\xi_i^4 + \xi_i'^2)} \qquad \xi_i \geqslant \xi_i' = \frac{s_i z_i}{a} \tag{4.7}$$

We observe that both stress components have square root singularities as $r \to a^-$ or $r \to a^+$, respectively. These singularities are presented by the stress concentration factors, see Eq. (4.5).

When the torsional forces are distributed along the circle $(r = r', z = z')$ in the interior of the ith layer $(z_{i-1} < z \leqslant z_i)$ then the rotation φ of the rigid inclusion will be

$$\varphi = \frac{3T}{16(\mu_{1\,eff} + \mu_{2\,eff})a^3}\Big[1 - \frac{2}{\pi}\Big(\tan^{-1}\xi_i + \frac{\xi_i}{1+\xi_i^2}\Big)\Big] \tag{4.8}$$

where ξ_1, η_i, $i = 1, 2, ..., n$ are related to $r = r'$ and $z = z'$ and s_i, as shows Eqs (2.5), and T is the resultant torque of the applied torsional forces. The similarity between formulae (4.2) and (4.8) result from Betti's reciprocal theorem (compare the solution in the paper by Rogowski (1998)). If the upper layered half-space is also loaded in the same manner (symmetrically) then the rotation φ of the rigid inclusion will be

$$\varphi = \frac{3T}{16(\mu_{1\,eff} + \mu_{2\,eff})a^3}\Big[2 - \frac{2}{\pi}\Big(\tan^{-1}\xi_i + \frac{\xi_i}{1+\xi_i^2} + \tan^{-1}\xi_j + \frac{\xi_j}{1+\xi_j^2}\Big)\Big] \tag{4.9}$$

where ξ_j is associated with s_i, $i = 1, 2, ..., n$, and ξ_j is associated with s_j, $j = 1, 2, ..., m$.

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Zagadnienia międzypowierzchniowej inkluzji w ośrodkach warstwowych

Streszczenie

Za pomocą metody perturbacji modułów znaleziono rozwiązania określające kontaktową sztywność w zagadnieniach inkluzji umieszczonej na powierzchni "sklejenia" warstwowego kompozytu bedącego w stanie skrętnej deformacji. Otrzymano następujące rozwiązania: (i) dokładne rozwiązanie dla inkluzji w ośrodku składającym się z dwóch warstw i dwóch półprzestrzeni, (ii) rozwiązanie o dokładności pierwszego rzędu dla ośrodka $(n+m)$-warstwowego, (iii) przybliżone rozwiązanie dla ośrodka fizycznie niejednorodnego z modułami ścinania zmieniającymi się w sposób ciągły, otrzymane jako przejście graniczne w rozwiązaniu dla ośrodka warstwowego.

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