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CONTACT OF A RIGID FLAT PUNCH WITH A WEDGE SUPPORTED BY THE WINKLER FOUNDATION

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The contribution deals with the new class of contact problems related with an elastic wedge. It is supposed that the wedge rests on the Winkler foundation. The wedge is in the plane frictionless contact with a rigid flat plate (punch). The problem is solved using the Mellin integral transforms method and is reduced to an integral equation for unknown contact pressure, which was solved numerically. The results concerning the contact pressure distribution and the punch displacement and slope are presented for different values of mechanical and geometrical parameters.

Key words: contact problem, elastic wedge, rigid punch, Winkler foundation

1. Introduction

Solutions to contact problems involving a deformable subgrade and a rigid plate (punch) have many applications, particularly in soil mechanics, geotechnical engineering and foundation design. Deformable subgrades are generally considered as an elastic half-space or a layer, see for example Gladwell (1980). But many geotechnical applications prove that the subgrade soil has the shape of a wedge. Previous investigations of contact problems related with the elastic wedge, see e.g. Aleksandrov (1967), Aleksandrov and Pozharski (1988) were done on the assumption that the wedge rests without friction on a rigid base. In this paper we propose new formulation of the contact problem for the elastic wedge assuming that the wedge is underlain by a deformable base of the Winkler type.

We investigate the contact problem for an elastic, homogeneous, and isotropic wedge supported by the Winkler foundation (Fig. 1). The wedge is planar and cuts out an infinite sector of θ_0. The upper surface of the wedge is in tensionless smooth contact with the rigid flat punch. The problem is assumed to be planar and stationary.

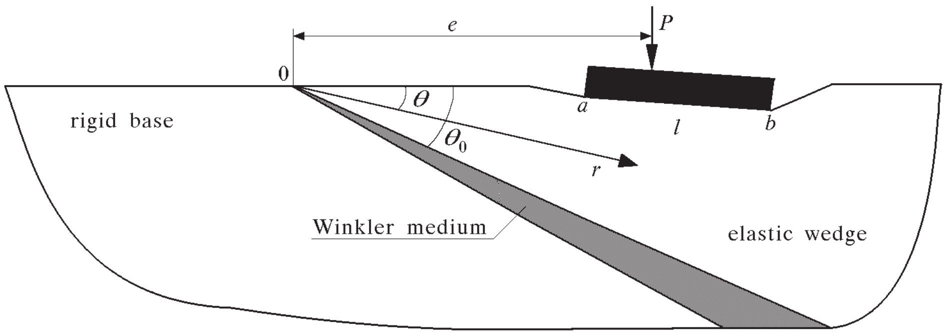

Fig. 1. Geometry of contact

Mathematically, the above formulated contact problem is reduced to solving the elasticity equations in the wedge (Timoshenko and Goodier, 1951)

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ &\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} = 0 \end{aligned} \tag{1.1}$$

with the following boundary conditions on the wedge surfaces

$$\begin{array}{lll} \tau_{r\theta}(r,0) = 0 & r \geqslant 0 & \\ \sigma_\theta(r,0) = 0 & 0 \leqslant r < a & r > b \\ u_\theta(r,0) = g_0 + rg_1 & a \leqslant r \leqslant b & \\ \sigma_\theta(r,\theta_0) = r^{-1}k_\theta u_\theta(r,\theta_0) & r \geqslant 0 & \\ \tau_{r\theta}(r,\theta_0) = r^{-1}k_r u_r(r,\theta_0) & r \geqslant 0 & \end{array} \tag{1.2}$$

where u_r, u_θ and σ_r, σ_θ, $\tau_{r\theta}$ are displacements and stresses in the polar coordinates system $0r\theta$, respectively; k_r, k_θ are the Winkler medium stiffnesses in the radial and angular directions; (a,b) is the contact area, which is given for the flat punch. The unknown parameters g_0, g_1 define the rigid displacement and slope of the punch, respectively.

2. General solutions

Differential equations (1.1) in the polar coordinate system can be solved by the method of Mellin's integral transforms (Sneddon, 1951). The fields of stresses and displacements in the wedge have the forms of contour integrals

$$\begin{aligned}\sigma_r(r,\theta) &= -\frac{1}{2\pi \mathrm{i}}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s-1}s\Big\{(s-1)[A\sin(s-1)\theta + B\cos(s-1)\theta]+\\ &+ (s+3)[C\sin(s+1)\theta + D\cos(s+1)\theta]\Big\}\,ds\end{aligned} \tag{2.1}$$

$$\begin{aligned}\sigma_\theta(r,\theta) &= \frac{1}{2\pi \mathrm{i}}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s-1}s(s-1)\Big\{[A\sin(s-1)\theta + B\cos(s-1)\theta]+\\ &+ [C\sin(s+1)\theta + D\cos(s+1)\theta]\Big\}\,ds\end{aligned} \tag{2.2}$$

$$\begin{aligned}\tau_{r\theta}(r,\theta) &= \frac{1}{2\pi \mathrm{i}}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s-1}s\Big\{(s-1)[A\cos(s-1)\theta - B\sin(s-1)\theta]+\\ &+ (s+1)[C\cos(s+1)\theta - D\sin(s+1)\theta]\Big\}\,ds\end{aligned} \tag{2.3}$$

$$\begin{aligned}u_\theta(r,\theta) &= -\frac{1+\nu}{2\pi \mathrm{i}E}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\Big\{(s-1)[A\cos(s-1)\theta - B\sin(s-1)\theta]+\\ &+ (s-\kappa)[C\cos(s+1)\theta - D\sin(s+1)\theta]\Big\}\,ds\end{aligned} \tag{2.4}$$

$$\begin{aligned}u_r(r,\theta) &= \frac{1+\nu}{2\pi \mathrm{i}E}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\Big\{(s-1)[A\sin(s-1)\theta + B\cos(s-1)\theta]+\\ &+ (s+\kappa)[C\sin(s+1)\theta + D\cos(s+1)\theta]\Big\}\,ds\end{aligned} \tag{2.5}$$

where A, B, C, D are the unknown functions of s and c is the real number which makes the integrands in (2.1)-(2.5) regular. Moreover, ν and E are Poisson's ratio and Young's modulus, respectively, and $\kappa = 3-4\nu$ is Kolosov's constant.

3. Point load solution

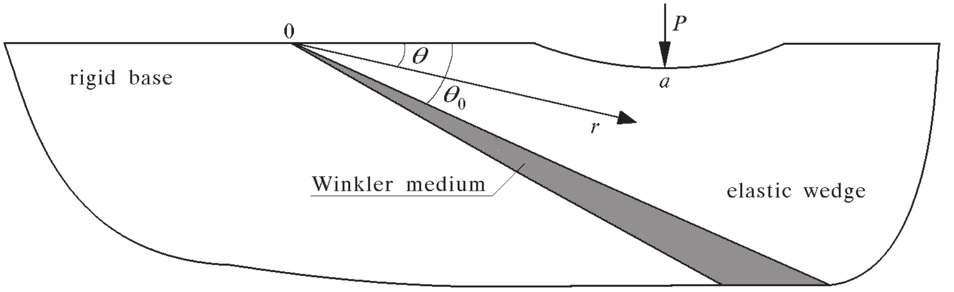

Fig. 2. Scheme of point load solution

First, we consider the point load problem for a wedge as shown in Fig. 2. Satisfying the following point load boundary conditions by equations (2.1)-(2.5)

$$
\begin{aligned}
&\tau_{r\theta}(r,0) = 0 && r \geqslant 0\\
&\sigma_\theta(r,0) = P\delta(r-a) && r \geqslant 0\\
&\sigma_\theta(r,\theta_0) = r^{-1}k_\theta u_\theta(r,\theta_0) && r \geqslant 0\\
&\tau_{r\theta}(r,\theta_0) = r^{-1}k_r u_r(r,\theta_0) && r \geqslant 0
\end{aligned}
\tag{3.1}
$$

we obtain a system of four algebraic equations for A, B, C, D, which has the solutions

$$
\begin{aligned}
A(s) &= -\frac{s+1}{s-1}C(s)\\
B(s) &= -D(s) - P\frac{a^s}{s(s-1)}\\
C(s) &= P\frac{U_0(s)+\alpha_r U_1(s)+\alpha_\theta U_2(s)+\alpha_r\alpha_\theta U_3(s)}{\varDelta_0(s)+\alpha_r\varDelta_1(s)+\alpha_\theta\varDelta_2(s)+\alpha_r\alpha_\theta\varDelta_3(s)}\frac{a^s}{s}\\
D(s) &= P\frac{V_0(s)+\alpha_r V_1(s)+\alpha_\theta V_2(s)+\alpha_r\alpha_\theta V_3(s)}{\varDelta_0(s)+\alpha_r\varDelta_1(s)+\alpha_\theta\varDelta_2(s)+\alpha_r\alpha_\theta\varDelta_3(s)}\frac{a^s}{s}
\end{aligned}
\tag{3.2}
$$

where

$$\varDelta_0(s) = 2s^2(s^2-1+\cos 2s\theta_0 - s^2\cos 2\theta_0)$$

$$\varDelta_1(s) = s(\kappa+1)(s\sin 2\theta_0 - \sin 2s\theta_0)$$

$$\Delta_2(s) = -s(\kappa+1)(s\sin 2\theta_0 + \sin 2s\theta_0)$$

$$\Delta_3(s) = 2s^2 - 1 - \kappa^2 - 2s^2\cos 2\theta_0 - 2\kappa\cos 2s\theta_0$$

$$U_0(s) = s^2(s\sin 2\theta_0 + \sin 2s\theta_0)$$

$$U_1(s) = 0.5s(\kappa+1)(\cos 2\theta_0 + \cos 2s\theta_0)$$

$$U_2(s) = -0.5s(\kappa+1)(\cos 2\theta_0 - \cos 2s\theta_0)$$

$$U_3(s) = s\sin 2\theta_0 - \kappa\sin 2s\theta_0$$

$$V_0(s) = s^2(s\cos 2\theta_0 + \cos 2s\theta_0 - s - 1)$$

$$V_1(s) = 0.5s(\kappa+1)(\sin 2\theta_0 + \sin 2s\theta_0)$$

$$V_2(s) = -s(\kappa+1)(\sin 2\theta_0 + \sin 2s\theta_0)$$

$$V_3(s) = s\cos 2\theta_0 - \kappa\cos 2s\theta_0 - s - 1$$

and

$$\alpha_r = \frac{1+\nu}{E}k_r \qquad\qquad \alpha_\theta = \frac{1+\nu}{E}k_\theta$$

are dimensionless stiffnesses of the Winkler medium.

To satisfy contact boundary condition $(1.2)_3$ we need a normal deflection of the wedge upper surface. Substituting solutions (3.2) into formula (2.4) we obtain

$$u_\theta(r,0) = \frac{2(1-\nu^2)}{\pi \mathrm{i} E}P\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\Big(\frac{a}{r}\Big)^{-s}\frac{L(s)}{s}\,ds \tag{3.3}$$

where the kernel of this equation has the form

$$L(s) = \frac{U_0(s) + \alpha_r U_1(s) + \alpha_\theta U_2(s) + \alpha_r\alpha_\theta U_3(s)}{\Delta_0(s) + \alpha_r\Delta_1(s) + \alpha_\theta\Delta_2(s) + \alpha_r\alpha_\theta\Delta_3(s)} \tag{3.4}$$

Let us observe the following properties of the kernel $L(s)$

(i) $L(-s) = -L(s)$

(ii) $$L(s) \sim \frac{a_0s^3 + \alpha_r a_1 s + \alpha_\theta a_2 s + \alpha_r\alpha_\theta a_3 s}{b_0s^4 + \alpha_r b_1 s^2 + \alpha_\theta b_2 s^2 + \alpha_r\alpha_\theta b_3} \qquad \text{for} \quad s \to 0$$

where a_i, b_i, $i = 0,1,2,3$ are some known constants.

Taking $c = 0$ in integral (3.3) and using methods of contour integration the normal deflection of the wedge upper surface can be obtained in the form

$$u_\theta(r,0) = \int\limits_0^\infty \frac{L^*(t)}{t}\cos(tR)\,dt \tag{3.5}$$

where

$$\delta = \frac{2(1-\nu^2)}{E} \qquad L^*(t) = \frac{2L(\mathrm{i}t)}{\mathrm{i}t} \qquad R = \ln\frac{a}{r}$$

Assuming now that the loading $p(r)$ is distributed over the region (a,b) we obtain from (3.5) the normal deflection

$$u_\theta(r,0) = \frac{\delta}{\pi}\int\limits_a^b p(\rho)K\Big(\ln\frac{\rho}{r}\Big)\,d\rho \qquad r \geqslant 0 \tag{3.6}$$

where the kernel $K(\cdot)$ has the form of the integral

$$K(R) = \int\limits_0^\infty \frac{L^*(t)}{t}\cos(tR)\,dt \tag{3.7}$$

Using the value of the integral (Gradshteyn and Ryzhik, 1965)

$$\int\limits_0^\infty \frac{1-\mathrm{e}^{-t}}{t}\cos(tR)\,dt = -\ln|R| \tag{3.8}$$

we can present the kernel $K(\cdot)$ in the following form

$$K(R) = -\ln|R| + \Phi(R) \tag{3.9}$$

where

$$\Phi(R) = \int\limits_0^\infty \frac{L^*(t)-1+\mathrm{e}^{-t}}{t}\cos(tR)\,dt \tag{3.10}$$

is a regular function.

Let us note here that the well known result for the elastic wedge resting on a rigid base (see Aleksandrov, 1967), can be obtained directly from (3.6), (3.4) for $\alpha_r,\alpha_\theta \to \infty$.

4. Integral equation of the contact problem

Satisfying boundary condition $(1.2)_3$ by formula (3.6) we arrive at the integral equation of the considered contact problem

$$\frac{\delta}{\pi}\int\limits_a^b p(\rho)K\Big(\ln\frac{\rho}{r}\Big)\,d\rho = g_0 + rg_1 \qquad r \in (a,b) \tag{4.1}$$

This equation must be solved together with the two equilibrium conditions

$$\int\limits_a^b p(r)\,dr = P \qquad\qquad \int\limits_a^b rp(r)\,dr = \mathrm{e}P \tag{4.2}$$

The distribution of the contact pressure $p(r)$, rigid displacement g_0 and slope g_1 of the punch are unknown in the system of integral equations (4.1) and (4.2).

Introducing dimensionless variables and functions

$$\begin{aligned} &\tau = \lambda \ln\frac{\rho}{a} - 1 && t = \lambda \ln\frac{r}{a} - 1 && r = a\exp\Big(\frac{t+1}{\lambda}\Big) \\ &q(\tau) = \frac{\rho}{\lambda P}p(\rho) && \lambda = 2\Big(\ln\frac{b}{a}\Big)^{-1} && \end{aligned} \tag{4.3}$$

the system of integral equations (4.1) and (4.2) can be rewritten into the new form

$$\begin{aligned} &\frac{1}{\pi}\int\limits_{-1}^{1} q(\tau)K\Big(\frac{\tau-t}{\lambda}\Big)\,d\tau = G_0 + (L-1)G_1\exp\Big(\frac{t+1}{\lambda}\Big) \qquad t \in (-1,1) \\ &\int\limits_{-1}^{1} q(t)\,dt = 1 \qquad\qquad \int\limits_{-1}^{1} \exp\Big(\frac{t+1}{\lambda}\Big)q(t)\,dt = \varepsilon \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &G_0 = \frac{g_0}{\delta P} && G_1 = \frac{g_1 c}{\delta P} && \varepsilon = \frac{\mathrm{e}}{a} \\ &L = \frac{l}{c} && l = \frac{b+a}{2} && c = \frac{b-a}{2} \end{aligned}$$

5. Numerical solutions to the system of integral equations

Introducing collocation points

$$\begin{aligned} &\tau_i = -1 + (i-1)dt && i = 1,...,n+1 && \\ &t_i = -1 + \Big(i - \frac{1}{2}\Big)dt && i = 1,...,n && dt = \frac{2}{n} \end{aligned} \tag{5.1}$$

and using rectangular quadratic formulae we obtain the discretized form of the system of integral equations (4.4)

$$\begin{aligned} &\frac{1}{\pi}\sum_{1=1}^{n} q(\tau_i)A_{im} - G_0 - (L-1)G_1\exp\Big(\frac{t_m+1}{\lambda}\Big) = 0 \qquad m=1,...,n \\ &dt\sum_{1=1}^{n} q(t_i) = 1 \qquad \lambda\sum_{1=1}^{n} q(t_i)\Big[\exp\Big(\frac{t_{i+1}+1}{\lambda}\Big) - \exp\Big(\frac{t_i+1}{\lambda}\Big)\Big] = \varepsilon \end{aligned} \tag{5.2}$$

where the matrices $\{A_{im}\}$ have the forms

$$A_{im} = \int\limits_{\tau_i}^{\tau_{i+1}} K\Big(\frac{\tau - t_m}{\lambda}\Big)\,d\tau \tag{5.3}$$

and using the formulae (3.9), (3.10) can be calculated as

$$\begin{aligned} A_{im} &= -\int\limits_{\tau_i}^{\tau_{i+1}} \ln\Big|\frac{\tau - t_m}{\lambda}\Big|\,d\tau + \int\limits_{\tau_i}^{\tau_{i+1}} \varPhi\Big(\frac{\tau - t_m}{\lambda}\Big)\,d\tau = \\ &= \lambda[Z_2\ln|Z_2| - Z_1\ln|Z_1| + \varPhi(Z_2) - \varPhi(Z_1)] \end{aligned} \tag{5.4}$$

where

$$Z_1 = \frac{\tau_i - t_m}{\lambda} \qquad Z_2 = \frac{\tau_{i+1} - t_m}{\lambda} \qquad i,m = 1,...,n \tag{5.5}$$

and

$$\varPhi_1(Z) = \int\limits_0^{\infty} \frac{L^*(t) - 1 + \mathrm{e}^{-t}}{t^2}\sin(Zt)\,dt \tag{5.6}$$

is the regular integral which was calculated numerically.

The set of $n+2$ linear algebraic equations (5.2) is sufficient to find $n+2$ unknowns: the dimensionless rigid displacement G_0 and slope G_1 of the punch and the distribution of the dimensionless contact pressure $q(t_i)$, $i = 1,...,n$.

6. Numerical results

The system of algebraic equations (5.2) was solved numerically. The input parameters for the calculations were: ν – Poisson's ratio, θ_0 – wedge angle,

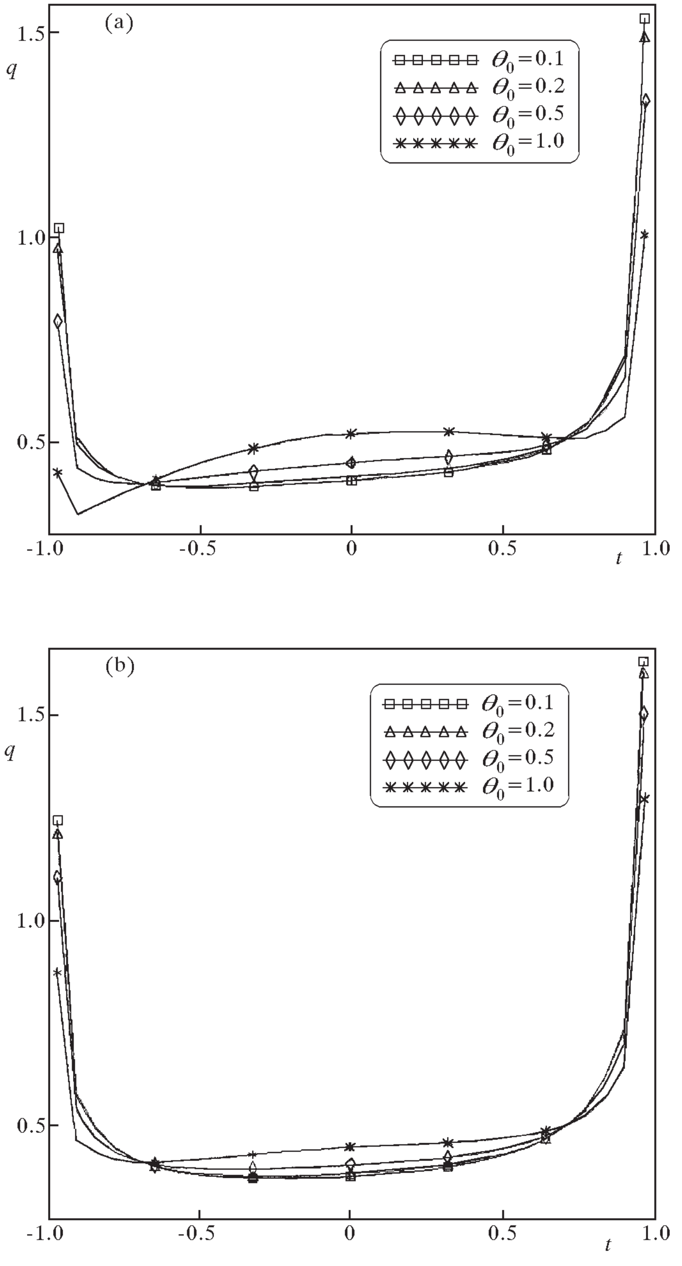

Fig. 3. Distribution of dimensionless contact pressure for $L = 2$ (a) and $L = 5$ (b)

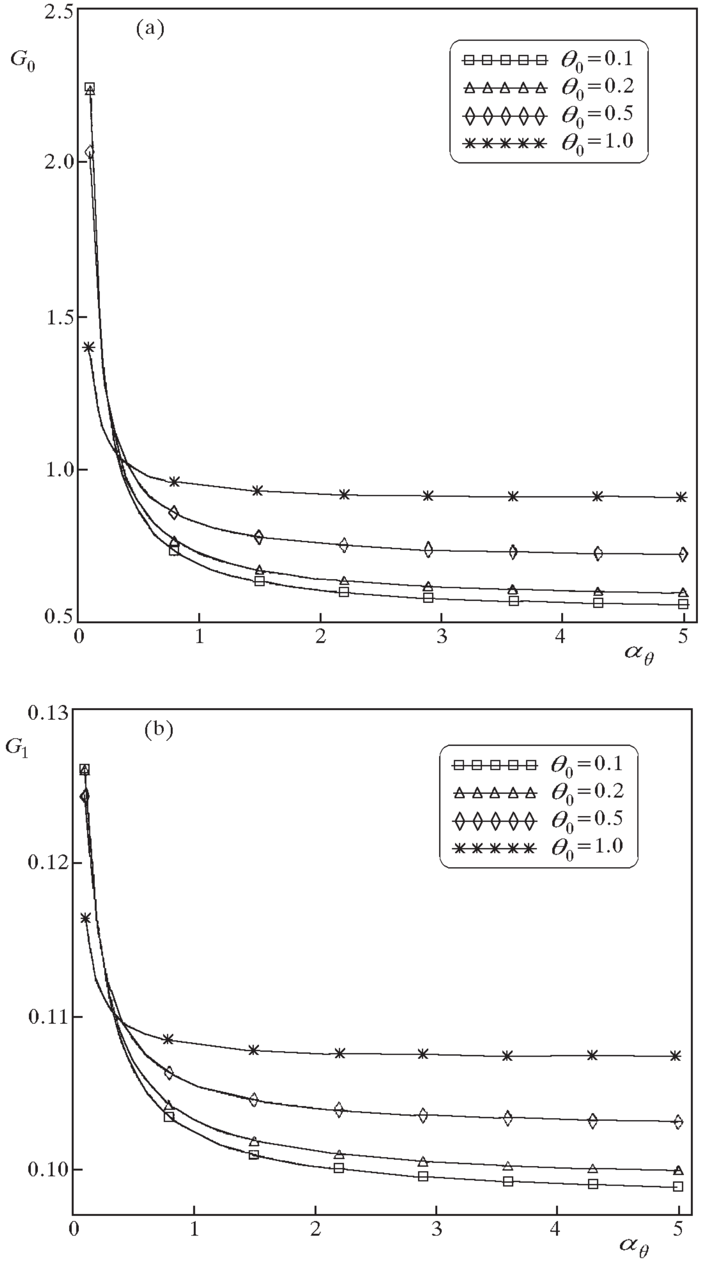

Fig. 4. Dimensionless rigid vertical displacement (a) and slope (b) of the punch versus the stiffness α_θ

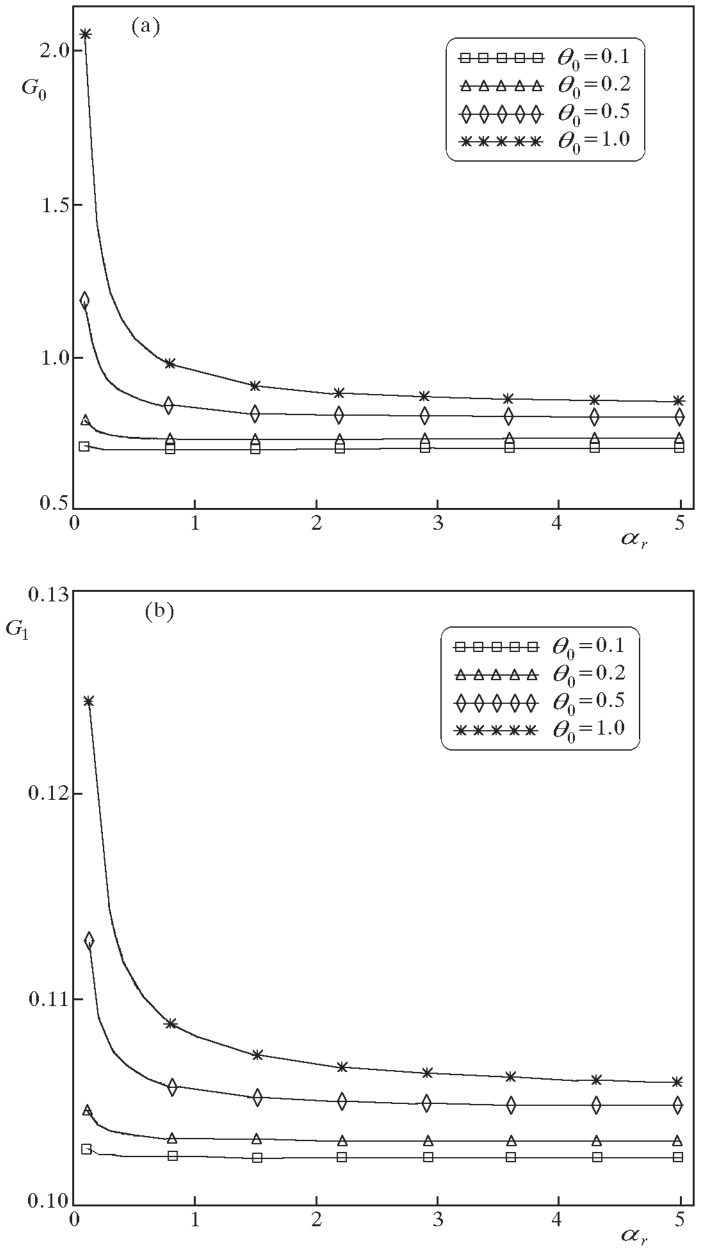

Fig. 5. Dimensionless rigid vertical displacement (a) and slope (b) of the punch versus the stiffness α_r

$\varepsilon > 1$ – dimensionless eccentricity, $L > 1$ – dimensionless location of the plate center, α_r, α_θ – dimensionless stiffnesses of the Winkler medium. The parameter λ can be calculated as $\lambda = 2\Big(\ln \frac{L+1}{L-1}\Big)^{-1}$. For numerical calculations we put $\nu = 0.3$ and $\varepsilon = L$, which means that the load P is applied to the center of the rigid plate.

The calculations were performed to display complex effects of the Winkler medium, wedge angle θ_0 and distance L to the punch on the distribution of the contact pressure $q(t)$, rigid vertical displacement G_0 and slope G_1 of the punch.

The distributions of the dimensionless contact pressure $q(t)$ are presented in Fig. 3 for some values of the wedge angle θ_0. The curves in Fig. 3a were found for the punch situated near to the wedge corner $(L = 2)$ but the results presented in Fig. 3b were obtained for a larger distance $(L = 5)$. The effect of the angle θ_0 is greater for small values of L. The distributions of the contact pressure shown in Fig. 3b are almost symmetrical. These results were obtained for $\alpha_r = \alpha_\theta = 1.0$, and our investigation displayed that the Winkler medium had small effect on the contact pressure. This does not mean that the boundary conditions on the lower surface of the wedge have no effect on the solution to the contact problem. The stiffnesses α_r, α_θ have great effect on the values of the rigid displacement G_0 and slope G_1 of the punch. These effects are shown in Fig. 4 and Fig. 5 for some values of the wedge angle and for the fixed distance $L = 2$. The diagrams presented in Fig. 4 are found for $\alpha_r = 1.0$ and those in Fig. 5 for $\alpha_\theta = 1.0$. The parameters G_0 and G_1 decrease with the growth of stiffnesses and tend to constant values for α_θ or α_r equal to 5. The comparison of the results presented in Fig. 4 and Fig. 5 displays that the angular stiffness α_θ plays a greater role than the radial one α_r. The main effects are observed for small values of the stiffnesses and the large values correspond to the problem of the wedge resting on the rigid base.

References

1. Aleksandrov V.M., 1967, Contact Problems for an Elastic Wedge, *Izvestia AN SSSR, MTT*, **2**, 120-131

2. Aleksandrov V.M., Pozharski D.A., 1988, On the Contact Problem for an Elastic Wedge, *PMM J. Appl. Math. Mech.*, **52**, 4, 651-656

3. Gladwell G.M.L., 1980, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff, Alphen aan den Rijn

4. Gradshteyn I.S., Ryzhik I.M., 1965, *Tables of Integrals, Series and Products*, Academic Press, New York
5. Sneddon I., 1951, *Fourier Transforms*, McGraw Hill, New York
6. Timoshenko S., Goodier J.N., 1951, *Theory of Elasticity*, McGraw Hill, New York

Współpraca sztywnego płaskiego stempla z klinem opartym na podłożu Winklera

Streszczenie

Praca dotyczy nowej klasy zagadnień kontaktowych dla sprężystego klina spoczywającego na podłożu Winklera. Klin ten znajduje się w płaskim kontakcie ze sztywną płytą (stemplem). Używając transformacji całkowych Mellina, zagadnienie sprowadzono do równania całkowego względem funkcji nacisków kontaktowych, które rozwiązywano numerycznie. Przedstawiono wyniki dla ciśnienia kontaktowego, osiadania i przechylenia stempla w zależności od różnych mechanicznych i geometrycznych parametrów zagadnienia.

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