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# A KINEMATIC INTERNAL VARIABLE APPROACH TO DYNAMICS OF BEAMS WITH A PERIODIC – LIKE STRUCTURE

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In the paper a new averaged model for the dynamic response of a periodic-like straight beam with a variable cross-section is proposed. The beam under consideration is assumed to interact with Winkler's subsoil having periodic-like properties. Assumptions of the Bernoulli-Euler beam theory as well as the influence of axial forces on deflection of the beam are taken into account. The approach adapted in the paper is based on concepts of the tolerance averaging approach by Woźniak (1999). In this way it is possible to formulate the averaged equations of the structured beams, which describe the length-scale effect.

*Key words:* dynamics, beam, periodic-like structure, homogenization

## 1. Introduction

This paper deals with dynamics of a periodic-like straight beam with a highly oscillating variable cross section: in a special case it can be composed of a very large number of repeated line segments. It means that the length dimensions of these segments are sufficiently small compared to the smallest wavelength of a dynamic deformation pattern.

In most cases an exact dynamic analysis of these systems, based on solid or structural mechanics equations, is difficult even using numerical methods. However, restricting the analysis to the dominant signal wavelengths large compared to the length of elements, a number of various approximate approaches to dynamics of periodic systems have been proposed. The best known are those related to periodic systems and based on the homogenization of differential equations with highly oscillating periodic coefficients (see Bensoussan et al., 1978; Jikov et al., 1994; Sanchez-Palencia, 1980). Using this approach a periodic heterogeneous system is approximated by a certain homogeneous medium characterized by effective medullae, which describe the averaged properties of a periodic system. However, the homogenization methods are incapable of describing dispersion and attenuation effects. To remove this drawback different alternative approaches mainly for investigations of waves propagating in periodic composite materials have been proposed.

It is known that the waves moving through a highly heterogeneous composite medium are continuously refracted and reflected at the interfaces between constituents. At the same time there exist certain stable wave trains (Floquet's waves), which retain their form (Sun et al., 1968a,b). The Floquet wave theory in elastic periodic composites deals with exact equations of the elasticity theory in every typical cell of a periodic medium. However, in most cases the required solutions have to be obtained using approximate methods. The direct application of Floquet's equations leads to rather troublesome numerical calculations.

The simplified analysis of waves in periodic composites, which allows the description of the dispersion phenomena, can be carried out in the framework of the effective stiffness theories (see Achenbach et al., 1968; Achenbach and Sun, 1972; Herrmann et al., 1976; Lee, 1972). This approach was mainly related to dynamics of laminates or directionally reinforced composites. The effective stiffness theories as well as similar approaches (such as the theory of interacting continua (see Bedford and Stern, 1971; Christensen, 1979; Hegemeier, 1972; Lee, 1972; Maewal, 1986)) are based on transitions from the solid mechanics equations with periodic coefficients to partial differential equations with constant coefficients representing continua with microstructure (see Mindlin, 1964; Mühlhaus, 1995). Using these equations we are able to investigate the dispersion phenomena in heterogeneous solids, but mainly for laminates or directionally reinforced composites. Alternative approaches to the averaged dynamics of periodic media were proposed by using mixture theories or micromorphic continuum theories; the review of early papers can be found in Hegemeier (1972) and Lee (1972).

In the last decade a unified treatment of the dynamic phenomena both for discrete and continuum periodic structures as well as periodic composite materials was developed by Cielecka (1995), Cielecka et al. (1998), Woźniak (1970), (1993a,b,c), (1995), (1997). The proposed averaging approach introduces into the modeling procedure the concept of kinematic internal variables as certain extra-unknown fields, which together with the averaged displacement-type fields describe the class of motion we are to analyze. The characteristic feature of the internal variables is that they are governed by the ordinary differential equations involving only time derivatives of these variables. Hence the kinematic internal variables do not enter the boundary conditions. So far, this concept was used mainly in formulations of the constitutive relations (Coleman and Gurtin, 1967).

The approach adapted in the paper is based on concepts of the internal variable model of dynamics (see Woźniak, 1997), and the tolerance averaging approach by Woźniak (1999). In this way it is possible to formulate the averaged equations of periodic beams, which describe the length-scale effect.

## 2. Preliminaries

In this paper dynamics of a periodic straight heterogeneous beam with a highly oscillating variable cross-section will be carried out on the well-known assumptions of the Euler-Bernoulli linear elastic beam theory. The axis of the beam will coincide with the interval $< 0, L >$ of the $x$-axis in $0xyz$-space and the beam has the $0xy$-plane as the symmetry plane. The assumption of uniaxial stress-strain relations $\sigma = \sigma(\varepsilon)$ of the beam material will be represented by Hook's law $\sigma = E\varepsilon$ where $E = E(x, y, z)$ stands for Young's modulus. Let $v = v(x, t)$, $x \in< 0, L >$, stand for the deflection of the beam axis in the y-axis direction and $A = A(x)$ be the cross-section of the beam at $x \in< 0, L >$. The assumption that plane cross-sections of the beam remain plane leads to the well-known relation for the bending moments

$$M(x,t) = B(x)v''(x,t) \tag{2.1}$$

where

$$B(x) = \int\limits_{A(x)} y^2 E(x,y,z)\; dydz$$

stands for the flexural beam stiffness. Moreover, let $\rho(x)$ be the mass density per unit length of the beam. The beam interacts with a subsoil the properties of which are determined by Winkler's coefficient $k = k(x)$; hence the reaction of the subsoil per unit length of the beam is given by

$$f(x,t) = -k(x)v(x,t) \tag{2.2}$$

Let the beam be subjected to the known transverse loading along the $0y$-axis having the density $p(x, t)$ per unit length of the beam axis as well as subjected to the constant axial force $N$. The equation of the beam has the well-known form

$$M''(x,t) - Nv''(x,t) + k(x)v(x,t) + \rho(x)\ddot{v}(x,t) = p(x,t) \tag{2.3}$$

and together with Eqs (2.1), (2.2) it represents the system of the governing equations following the Euler-Bernoulli beam theory. Introducing the differential operator $D$ acting on the function $u = u(x,t)$, $x \in< 0, L >$, defined by

$$Du = (Bu'')'' - Nu'' + ku + \rho\ddot{u} \tag{2.4}$$

from Eqs (2.1)-(2.3) we obtain the beam equation in the operator form

$$Dv = p$$

which, together with adequate boundary conditions and initial conditions, represents a problem of the Euler-Bernoulli beam theory. The known mathematical procedures and standard methods for the analysis of dynamics of beams are effective provided that the coefficients in Eqs (2.1)-(2.3) are constant or slowly varying. If these coefficients are rapidly varying functions then most dynamic problems may become rather difficult to solve.

In this paper the considerations will be restricted to beams in which the rapidly varying functional coefficients $B(\cdot)$, $k(\cdot)$, $\rho(\cdot)$ are represented by periodic-like functions. It means that there exist a slowly varying function $l = l(x)$, $x \in< 0, L >$, where $\max l(x) \ll L$, such that in every interval $\Delta(x) = (x - l(x)/2, x + l(x)/2)$, $\Delta x \in< 0, L >$, the functions $B(\cdot)$, $k(\cdot)$, $\rho(\cdot)$ can be approximated respectively by certain $l = l(x)$-periodic functions $B_x(\xi)$, $k_x(\xi)$, $\rho_x(\xi)$, $\xi \in< x - l/2, x + l/2 >$. In the special case $l = \text{const}$ we obtain beams with the $l$-periodic structure.

Moreover, all length dimensions of an arbitrary cross-section $A(x)$ of the beam must be small compared to $l$. Beams satisfying the aforementioned requirement will be referred to as structured beams.

In the framework of the kinematic internal variable model of dynamics (Woźniak, 1997), and the tolerance averaging approach by Woźniak (1999), it is possible to formulate equations of the structured beams in the form of the system of averaged differential equations with slowly varying or constant coefficients. This approximation describes the effect of structural length parameter of the beam. In contrast, in the classical homogenization solutions, for instance in one by Achenbach et al. (1968), this effect disappears. The paper presents a generalization of the problems of bending of the beams with a periodic structure (Mazur-Śniady, 1997, 1993).

## 3. Modeling assumptions

The approach adapted in the paper is based on concepts of the tolerance averaging approach by Woźniak (1999). Define $\Omega = (0, L)$, $l = l(x)$, $\Delta(x) = (x - l/2, x + l/2)$, where $\max l \ll L$, $x \in \Omega^0$, $\Omega^0 = \{x \in \Omega : \Delta(x) \subset \Omega\}$. Hence $\Omega^0$ is the subset of $\Omega = (0, L)$ such that for every $x \in \Omega^0$ the interval $\Delta(x)$, which will be called a line segment at $x$, is a part of $\Omega$.

The functions will be averaged by means of the formula.

$$\langle\varphi\rangle(x) = \frac{1}{l} \int\limits_{x-l/2}^{x+l/2} \varphi(\xi)\, d\xi \qquad x \in \Omega^0 \tag{3.1}$$

where $\varphi(\cdot)$ is an arbitrary integrable function defined (almost everywhere) on $\Omega = (0, L)$.

If $\varphi$ is $l$-periodic then $\langle\varphi\rangle = \text{const}$ and for $\varphi$ depending also on $t$ we shall write $\langle\varphi\rangle(x,t)$; we shall also write $\langle\varphi\rangle$ instead of $< \varphi > (x,t)$.

For an arbitrary but sufficiently regular function $F(\cdot)$ defined in $\Omega$ and belonging to a certain linear normed function space, we shall write $F \in SV(l)$ provided that the approximation condition $\langle F\rangle(x) \cong F(x)$ holds in $\Omega^0$, where $\cong$ stands for a certain tolerance relation. It means that the LHS of this relation can be approximated with a sufficient accuracy by its RHS and hence $||\langle F(x)\rangle - F(x)|| \leqslant \varepsilon_F$, where $\varepsilon_F$ is the pertinent accuracy parameter. In this case $F(\cdot)$ will be referred to as the $l$-slowly varying function. It has to be remembered that if $F_1$, $F_2$ are differentiable functions belonging to a certain linear normed space then the tolerance relation $F_1 \cong F_2$ implies similar relations for all derivatives of $F_1$, $F_2$; to emphasize this fact we shall also write $\nabla F_1 \cong \nabla F_2$.

Let $\psi_x(\xi)$ be a $l$-periodic function of $\xi$ for every $x \in \Omega^0$ and be a $l$-slowly varying function of $x$ for every $\xi$. Then setting $\psi(x) = \psi_x(x)$ in $\Omega^0$ and assuming that $\psi(\cdot)$ belongs to a certain linear normed function space, we shall write $\psi \in PL(l)$ and refer $\psi(\cdot)$ to as the $l$-periodic-like function. Roughly speaking, every $l$-periodic-like function $\psi(\cdot)$ after restricting its domain to an arbitrary line segment $\Delta(x)$, $x \in \Omega^0$ can be approximated in this segment (within a certain tolerance) by the $l$-periodic function $\psi_x(\cdot)$. It means that $\langle\psi\rangle(x) \approx \langle\psi_x\rangle(x)$ and $\psi_x$ will be called the $l$-periodic approximation of $\psi(\cdot)$ in $\Delta(x)$. It follows that if $\psi \in PL(l)$ then $\langle\psi\rangle \in SV(l)$. If $\psi \in PL(l)$, $\langle\rho\psi\rangle = 0$ then $\psi$ will be referred to as the oscillating (with the weight $\rho$) $l$-periodic-like function, $\psi \in OPL(l)$; here $\langle\rho\psi\rangle = 0$ is called the normality condition.

By the tolerance averaging, we shall mean the tolerance relations (Woźniak, 1999)

$$\begin{aligned} \langle\varphi F\rangle(x) &\cong \langle\varphi\rangle(x)F(x) \qquad x \in \Omega^0 \\ \langle\varphi\psi\rangle(x) &\cong \langle\varphi\psi_x\rangle(x) \end{aligned} \tag{3.2}$$

which have to hold for every $F \in SL(l)$, $\psi \in OPL(l)$ with $\varphi$ as an arbitrary integrable function defined in $\Omega$ and which make it possible to replace the left-hand sides of (3.2) by their right-hand sides. At the same time formulae similar to (3.2) have to hold also for all derivatives of $F(\cdot)$ and $\varphi(\cdot)$. All the aforementioned concepts require specification of the tolerance relations $\cong$ in the calculations of averages (3.2), which depends on the accuracy of computations for the problem under consideration.

We base on the physical assumption that the deflection of the $l$-periodic beam axis is the $l$-periodic-like function

$$v(\cdot,t) \in PL(l) \tag{3.3}$$

It follows that $v(\xi,t) \cong v_x(\xi,t)$ for every $\xi \in \overline{\Delta}(x)$.

Let as define the averaged deflection $w(x,t)$ by means of

$$w(x,t) = \langle\rho\rangle^{-1}(x)\langle\rho v\rangle(x,t) \qquad x \in \Omega^0 \tag{3.4}$$

The total deflection of the beam can be represented by the sum

$$v(x,t) = w(x,t) + d(x,t) \qquad x \in \Omega^0 \tag{3.5}$$

where $d(x,t)$ is the function of the deflection disturbances.

Taking into account (3.4) and (3.5) we observe that $w(\cdot,t)$ is the slowly varying function $w(x,t) \in SV(l)$ and $d(\cdot,t)$ is the oscillating (with the weight $\rho$) $l$-periodic-like function.

The unknown functions $w(x,t)$, $d(x,t)$ has to satisfy the averaged equation of motion

$$\langle D(w+d)\rangle(x,t) = \langle p\rangle(x,t) \tag{3.6}$$

for every $x \in \Omega^0$ with $D(\cdot)$ given by (2.4), where the functional coefficients $B(\cdot)$, $k(\cdot)$, $\rho(\cdot)$ in (2.4) are the $l$-periodic-like functions.

Let us observe that from the above assumption and from the concept of a slowly varying function it follows that

$$\langle\rho d\rangle(x,t) = \langle\rho v\rangle(x,t) - \langle\rho w\rangle(x,t) \cong \langle\rho v\rangle(x,t) - \langle\rho\rangle(x)w(x,t) = 0 \tag{3.7}$$

holds for every $x \in \Omega^0$.

As a sequel the deflection disturbance fields $d(\cdot, t)$ will be assumed in the form of the series

$$d(x,t) = h^A(x)\psi_A(x,y) \qquad x \in \Omega^0 \tag{3.8}$$

(summation convention over $A = 1, 2, ...$ holds), where $h^A(\cdot)$ are the known *a priori* oscillating $l$-periodic-like functions having the periodic approximations $h_x^A(\xi)$, $\xi \in \overline{\Delta}(x)$, at every $x \in \Omega^0$ and $\psi_A(\cdot)$ are sufficiently regular slowly varying functions which have to satisfy the equations

$$\langle h_x^B(D(h_x^A\psi_A) + Dw)\rangle(x,t) = \langle h_x^B p\rangle(x,t) \tag{3.9}$$

for every $x \in \Omega^0$.

Consequently, the $l$-periodic-like functions $h^A(\cdot)$, $A = 1, 2, ...$ will be referred to as the global mode shape functions contrary to their periodic (local) approximations $h_x^A(\cdot)$. It is assumed that every $h^A(\cdot)$ has to be the oscillating $l$-periodic-like function; this condition is related to the fact that $\langle \rho d\rangle(x,t) \cong 0$ for every $x \in \Omega^0$ and hence $\langle \rho h^A\rangle(x) \cong 0$.

We shall seek an approximate solution to problem (3.9) taking $A = 1, 2, ..., n$. Here the positive integer $n$ is arbitrary but fixed and hence we can look for the solution to (3.6) on different levels of accuracy. In this case the functions $\psi_A$, $A = 1, 2, ..., n$, will be referred to as the kinematic internal variables for the reason which will be explained in Section 4.

## 4. Averaged equations

After taking into account modeling assumption (3.3), definition (2.4) of the operator $D$ as well as formulae (3.8),(3.9), after simple transformations we obtain the system of $n+1$ equations

$$\begin{aligned} &\langle M''\rangle + \langle \rho(\ddot{w} + h^A\ddot{\psi}_A)\rangle - N\langle w + h^A\psi_A\rangle'' + \langle k(w + h^A\psi_A)\rangle = \langle p\rangle \\ &\langle M''h^B\rangle + \langle \rho h^B(h^A\ddot{\psi}_A) + \ddot{w}\rangle - N\langle h^B((h^A)''\psi_A + w'')\rangle + \\ &+\langle kh^B(h^A\psi_A + w)\rangle = \langle h^B p\rangle \end{aligned} \tag{4.1}$$

where $M$ are bending moments related to the total deflection $v = w + d$ by means of

$$M = B(w'' + d'') \tag{4.2}$$

Let us transform equations (4.1) taking into account that $w(\cdot,t)$, $\psi_A(\cdot,t)$ are slowly varying functions, $B(\cdot)$, $k(\cdot)$, $\rho(\cdot)$ are $l$-periodic-like functions and $h^A(\cdot)$ are oscillating $l$-periodic-like functions. Denoting $m = \langle M \rangle$, taking into account the properties of the slowly varying functions and $\langle \rho h^A \rangle = 0$, the first of equations (4.1) reads

$$m'' + \langle \rho \rangle \ddot{w} - N w'' - N(\langle h^A \rangle \psi_A)'' + \langle k \rangle w + \langle k h^A \rangle \psi_A = \langle p \rangle \tag{4.3}$$

Using (3.2) we also obtain

$$m = \langle B \rangle w'' + \langle B (h^A)'' \rangle \psi_A \tag{4.4}$$

Let us transform the second equation of (4.1). From (4.2) and (3.8) it follows that $M(\cdot)$ and $M'(\cdot)$ are periodic-like functions; they have to be also continuous excluding the points where the concentrated loadings are applied. At the same time $h^B(\cdot)$ are oscillating $l$-periodic-like functions; we shall assume that their derivatives are also oscillating $l$-periodic-like functions. Hence formula (3.6) yields

$$\langle M'' h^B \rangle \cong -\langle M' (h^b)' \rangle = \langle M (h^B)'' \rangle$$

Denoting $m^B = \langle M (h^B)'' \rangle$ and using (3.2) as well as $\langle h^B (h^A)'' \rangle \cong -\langle (h^A)' (h^B)' \rangle$ we can write the second equations of (4.1) in the form

$$\begin{aligned} & m^B + \langle \rho h^B h^A \rangle \ddot{\psi}_A + N \langle (h^A)' (h^B)' \rangle \psi_A - N \langle h^B \rangle w'' + \\ & + \langle k h^B h^A \rangle \psi_A + \langle k h^B \rangle w = \langle h^B p \rangle \end{aligned} \tag{4.5}$$

where by means of (3.2)

$$m^B = \langle B (h^A)'' (h^B)'' \rangle \psi_A + \langle B (h^B)'' \rangle w'' \tag{4.6}$$

Notice, that (4.5) are the ordinary differential equations for $\psi_A$ involving only the time derivatives of $\psi_A$; hence $\psi_A$ do not enter the boundary conditions and that is why they were called the kinematic internal variables.

In formulae (4.3)-(4.6) we have replaced the tolerances by the equalities obtaining the system of averaged equations of dynamics of a structured beam. Substituting the right-hand sides of equations (4.4) and (4.6) to equations (4.3) and (4.5), respectively, we arrive at the system of $n+1$ equations for the averaged deflection $w(x,t)$, $x \in \Omega^0$, and the kinematic internal variables $\psi_A(x,t)$, $x \in \Omega^0$. All the coefficients in these equations are slowly varying functions because $B(\cdot)$, $k(\cdot)$, $\rho(\cdot)$ are periodic-like functions and $h^A(\cdot)$ are oscillating $l$-periodic-like functions.

- The first characteristic feature of averaged equations (4.3), (4.6) is that for $\psi_A(\cdot,t)$, $A=1,2,...,n$, we obtain from (4.5), (4.6) the system of $n$ ordinary differential equations given by

$$\begin{aligned}&\langle \rho h^B h^A\rangle \ddot{\psi}_A + \langle B(h^B)''(h^A)''\rangle \psi_A + \langle B(h^B)''\rangle w'' +\\ &+N\langle (h^B)'(h^A)'\rangle \psi_A - N\langle h^B\rangle w'' + \langle k h^B h^A\rangle \psi_A + \langle k h^B\rangle w = \langle h^B p\rangle\end{aligned} \tag{4.7}$$

which involve only the time derivatives of $\psi_A$.

Equations (4.7) are coupled with the equation obtained from (4.3), (4.4)

$$\begin{aligned}&\langle \rho\rangle \ddot{w} + \Big(\langle B\rangle w'' + \langle B(h^A)''\rangle \psi_A\Big)'' - Nw'' - N\Big(\langle h^A\rangle \psi_A\Big)'' +\\ &+\langle k\rangle w + \langle k h^A\rangle \psi_A = \langle p\rangle\end{aligned} \tag{4.8}$$

The boundary conditions for $w(\cdot)$ can be assumed in the form similar to that formulated in the Euler-Bernoulli beam theory. For the initial-value problem two initial conditions for $w(\cdot)$, $\psi_A(\cdot)$, $A=1,2,...,n$, should be also known.

- The second characteristic feature of the averaged equations is that they describe the effect of cell length $l$ on the dynamic behavior of a structured beam; this statement is implied by the dependence of the mode shapes $h_x^A(\cdot)$ on the segment span $l$.

Now the problem arises how to determine the form of the mode shape functions $h_x^A(\cdot)$. To this end we shall assume that they are sufficiently smooth solutions to the eigenvalue periodic problem given by the equation

$$(Bh_x'')'' - Nh_x'' + kh_x - \omega^2 \rho h_x = 0 \tag{4.9}$$

and the periodic boundary conditions at $x \pm l/2$ together with the pertinent jump conditions. Let $h_x^A(\cdot)$, $A=1,2,...,n$, be a sequence of eigenfunctions defined on $\overline{\Delta}(x)$ and related to the sequence of the eigenfrequencies $\omega_A$. Every function $h_x^A(\xi)$, $\xi \in \overline{\Delta}(x)$, represents a certain vibration mode shape assigned to the eigenfrequency $\omega_A$ and related to the free vibrations of the beam with the span $l$ and the axis bounded by the points $x \pm l/2$.

Let us introduce the oscillating $l$-periodic-like functions $g^A(\cdot)$, $\tilde{g}^A(\cdot)$, $G^A(\cdot)$ defined by

$$g^A = h^A \tilde{l}^{-2} \qquad \tilde{g}^A = (h^A)' \tilde{l}^{-1} \qquad G^A(\cdot) = (h^A)''$$

where $\tilde{l} \equiv l(x)$.

The functions $g^A(\cdot)$, $\tilde{g}^A(\cdot)$, $G^A(\cdot)$ are of the same order with respect to $\tilde{l}$ and their values be treated as independent of $\tilde{l}$. The system of averaged equations (4.7), (4.8), which hold in $\Omega^0$, is now seen to be

$$\begin{aligned}
&\langle\rho\rangle\ddot{w} + \Big(\langle B\rangle w'' + \langle BG^A\rangle\psi_A - Nw - N\tilde{l}^2\langle g^A\rangle\psi_A\Big)'' + \langle k\rangle w + \\
&+\tilde{l}^2\langle kg^A\rangle\psi_A = \langle p\rangle \\
&\tilde{l}^4\langle\rho g^B g^A\rangle\ddot{\psi}_A + \langle BG^B G^A\rangle\psi_A + \langle BG^B\rangle w'' + \tilde{l}^2 N\langle\tilde{g}^b\tilde{g}^A\rangle\psi_A + \\
&-N\tilde{l}^2\langle g^B\rangle w'' + \tilde{l}^4\langle kg^B g^A\rangle\psi_A + \tilde{l}^2\langle kg^B\rangle w = \tilde{l}^2\langle g^B p\rangle
\end{aligned} \tag{4.10}$$

and depends explicitly on the structural length parameter $\tilde{l}$ given by $\max l(x)$, $x \in \Omega^0$, which plays the role of a certain length-scale parameter. All the functional coefficients in equations (4.10) are sufficiently smooth slowly varying functions.

## 5. Determination of mode shape functions

For a beam with a periodic-like structure the mode shape functions $h^A(\cdot)$ are $l$-periodic. Hence these functions are uniquely determined by the functions $h_0^A(\xi)$, $\xi \in \langle -l/2, l/2\rangle$ where $h^A(x) = h^A(sl+\xi) = h_0^A(\xi)$, $s = 1,2,...$ with $x = sl + \xi$. In order to find $h_0^A(\cdot)$ we have to solve the eigenvalue problem, given by the equation

$$(B(\xi)h_0''(\xi))'' - \omega^2\rho(\xi)h_0(\xi) = 0 \tag{5.1}$$

where $\omega$ is the eigenfrequency.

Equation (5.1) has to hold for almost every $\xi \in (-l/2, l/2)$; where the interval functions $h_0(\xi)$, $h_0'(\xi)$, $B_0(\xi)h_0''(\xi)$, $(B_0(\xi)h_0''(\xi))'$ have to be continuous. For $\xi = -l/2$ and $\xi = l/2$ the function $h_0(\xi)$ must satisfy the periodic boundary conditions. In order to prove that

$$\langle\rho h_0\rangle = 0 \tag{5.2}$$

we get, by introducing the Hilbert space $\mathcal{H}$ of real functions within the domain $< -l/2, l/2 >$, the scalar product

$$\ll f|g \gg := \int_{-l/2}^{l/2} \rho(\xi)f(\xi)g(\xi)\, d\xi$$

and the operator $\mathcal{A}$ acting on the function $h_0(\xi)$, $\xi \in< -l/2, l/2 >$, defined by

$$(\mathcal{A}h_0)(\xi) = \frac{(B_0(\xi)h_0''(\xi))''}{\rho(\xi)} \tag{5.3}$$

for which we can write equation (5.1) as

$$(\mathcal{A}h_0)(\xi) = \kappa h_0(\xi) \tag{5.4}$$

where $\kappa = \omega^2$.

Taking into account the periodic boundary conditions we see that for any two functions $f(\xi)$, $j(\xi) \in \mathcal{H}$ the scalar product of the functions $f(\xi)$ and $(\mathcal{A}j)(\xi)$

$$\ll f|\mathcal{A}j \gg := \int_{-l/2}^{l/2} \rho(\xi)f(\xi)\frac{(B(\xi)j''(\xi))''}{\rho(\xi)}\, d\xi = \int_{-l/2}^{l/2} f''(\xi)B(\xi)j''(\xi)\, d\xi \tag{5.5}$$

and the scalar product of the functions $(\mathcal{A}f)(\xi)$ and $j(\xi)$

$$\ll \mathcal{A}f|j \gg := \int_{-l/2}^{l/2} \rho(\xi)\frac{(B(\xi)f''(\xi))''}{\rho(\xi)}j(\xi)\, d\xi = \int_{-l/2}^{l/2} f''(\xi)B(\xi)j''(\xi)\, d\xi \tag{5.6}$$

are equal, hence the operator $\mathcal{A}$ is self-adjoint.

Having the two eigenfunctions $h_0^A$ and $h_0^B$ corresponding to the eigenvalues $\kappa_A$ and $\kappa_B$, for $\kappa_A \neq \kappa_B$ we obtain

$$\kappa_A \ll h_0^A|h_0^B \gg = \ll \kappa_A h_0^A|h_0^B \gg = \ll Ah_0^A|h_0^B \gg =$$

$$= \ll h_0^A|Ah_0^B \gg = \ll h_0^A|\kappa_B h_0^B \gg = \kappa_B \ll h_0^A|h_0^B \gg$$

which leads to

$$(\kappa_A - \kappa_B) \ll h_0^A|h_0^B \gg = 0$$

hence

$$\ll h_0^A|h_0^B \gg = \int_{-l/2}^{l/2} \rho(\xi)h_0^A(\xi)h_0^B(\xi)\, d\xi = 0 \tag{5.7}$$

Since $h_0(\xi) = 1$ is the eigenfunction corresponding to the eigenvalue $\kappa = 0$, then for any eigenfunction $h_0^A$ with $\kappa_A \neq 0$ we have

$$\int_{-l/2}^{l/2} \rho(\xi)h_0^A(\xi) \cdot 1\, d\xi = 0 \tag{5.8}$$

satisfying condition (5.2), which ends the proof.

Finding the solutions to eigenvalue problem (5.1) is rather a difficult task and in most cases the eigenfunctions $h_0^A$ have to be obtained using approximate methods. But in particular cases we can obtain the exact solution.

As an example we consider the segment $\Delta(\xi)$, $\xi \in (-l/2, l/2)$ with the piecewise constant rigidity $B(\cdot)$ and the mass density $\rho(\cdot)$ as it is shown in Fig.1. For $\xi \in <-a, a>$ we have $B(\xi) = B_1$, $\rho(\xi) = \rho_1$, for $\xi \in <-l/2, -a>$ and $<a, l/2>$ we have $B(\xi) = B_2$, $\rho(\xi) = \rho_2$, where $B_1$, $B_2$, $\rho_1$, $\rho_2$ are constant.

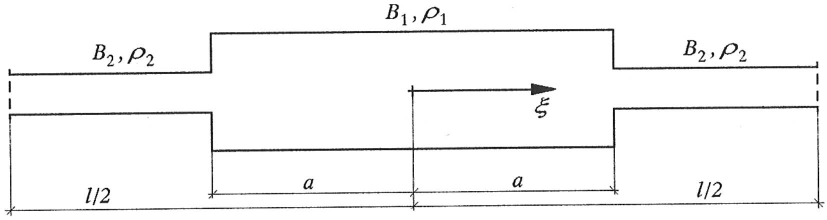


Fig. 1. The segment $\Delta(\xi)$ with the piecewise constant rigidity $B(\cdot)$ and the mass density $\rho(\cdot)$

In this case we have to solve equation (5.1) in the form

$$\begin{aligned} h_1^{IV} - \lambda_1^4 h_1 &= 0 \qquad \text{for} \quad \xi \in (-a, a) \\ h_2^{IV} - \lambda_2^4 h_2 &= 0 \qquad \text{for} \quad \xi \in (-l/2, -a) \wedge (a, l/2) \end{aligned} \tag{5.9}$$

with the eigenvalues $\lambda_\beta^4 = \omega^2 \rho_\beta / B_i$, $\beta = 1, 2$.

The obtained solution to equations (5.9) ought to be regular, i.e., the mode function, its first derivative, the bending moment and the transverse force have to be continuous. For the segment under consideration, the function $h_2(\cdot)$ must satisfy the following periodic boundary conditions

$$\begin{aligned} h_2(l/2) &= h_2(-l/2) \qquad & h_2''(l/2) &= h_2''(-l/2) \\ h_2'(l/2) &= h_2'(-l/2) \qquad & h_2'''(l/2) &= h_2'''(-l/2) \end{aligned} \tag{5.10}$$

The continuity conditions at the points where $B(\cdot)$ and $\rho(\cdot)$ suffer jumps will be called the jump conditions. They have the following form

$$\begin{aligned} h_1(a) &= h_2(a) \qquad & h_1(-a) &= h_2(-a) \\ h_1'(a) &= h_2'(a) \qquad & h_1'(-a) &= h_2'(-a) \\ B_1 h_1''(a) &= B_2 h_2''(a) \qquad & B_1 h_1''(-a) &= B_2 h_2''(-a) \\ B_1 h_1'''(a) &= B_2 h_2'''(a) \qquad & B_1 h_1'''(-a) &= B_2 h_2'''(-a) \end{aligned} \tag{5.11}$$

The obtained mode function has to satisfy condition (5.2), namely

$$l^{-1}\Big(\rho_2 \int\limits_{-l/2}^{-a} h_2(\xi)\, d\xi + \rho_1 \int\limits_{-a}^{a} h_1(\xi)\, d\xi + \rho_2 \int\limits_{a}^{l/2} h_2(\xi)\, d\xi\Big) = 0 \tag{5.12}$$

Taking into account the symmetry of the segment geometry and the material constants we can present the solution to the eigenvalue problem as a sum of the odd function $h^*(\cdot)$ and the even function $h^{**}(\cdot)$.

For the odd function averaged condition (5.12) is trivially satisfied. Additionally, the following conditions must be satisfied

$$\begin{aligned} &h_2^*(l/2) = h_2^*(-l/2) = 0 \\ &(h_2^*)''(l/2) = (h_2^*)''(-l/2) = 0 \end{aligned} \tag{5.13}$$

We look for the odd mode shape function in the following form

$$\begin{aligned} &h_1^*(\xi) = a_1^* \sin \lambda_1^* \xi + b_1^* \sinh \lambda_1^* \xi && \text{for} \quad \xi \in (-a, a) \\ &h_2^*(\xi) = a_2^* \sin \lambda_2^*(\xi - l/2) + b_2^* \sinh \lambda_2^*(\xi - l/2) + \\ &\qquad + c_2^* \cos \lambda_2^*(\xi - l/2) + d_2^* \cosh \lambda_2^*(\xi - l/2) && \text{for} \quad \xi \in (a, l/2) \\ &h_2^*(\xi) = -h_2^*(-\xi) && \text{for} \quad \xi \in (-l/2, -a) \end{aligned} \tag{5.14}$$

From conditions (5.13) we obtain $c_2^* = 0$, $d_2^* = 0$. By the aforementioned choice of the arguments in (5.14) the function $h_2^*(\cdot)$ satisfies the periodic boundary conditions of mode shape functions (5.10). On the junctions of the adjacent segments the conditions of equality of the functions $h_2^*(\cdot)$ and all its derivatives hold.

The nontrivial solution to eigenvalue problem (5.1) exists in form (5.14), provided that the system of the first four equations given by (5.11) for the four unknowns $a_1^*$, $a_2^*$, $b_1^*$, $b_2^*$ has the determinant equal to zero. In this way we find numerically the successive values of $\lambda_1^{*\alpha}$, $\alpha = I, II, III, \ldots$.

Taking the following data

$$a = \frac{l}{4} \qquad \frac{B_1}{B_2} = 8 \qquad \frac{\rho_1}{\rho_2} = 2 \tag{5.15}$$

related to the rod with a constant width and the height $2H$ on the segment $< -a, a >$ and $H$ on the $< -l/2, -a > \cup < a, l/2 >$, we obtain the following solutions for the first three eigenvalues $\lambda_1^{*\alpha}$, see Table 1.

**Table 1**

| | | | |
|---|---|---|---|
| $\lambda_1^*$ | $4.7207l^{-1}$ | $10.7696l^{-1}$ | $15.3998l^{-1}$ |
| $a_1^*$ | $l^2$ | $l^2$ | $l^2$ |
| $b_1^*$ | $0.266451l^2$ | $-0.009631l^2$ | $-0.017767l^2$ |
| $a_2^*$ | $-1.73096l^2$ | $1.92886l^2$ | $-1.3455l^2$ |
| $b_2^*$ | $0.15834l^2$ | $0.036796l^2$ | $0.000581l^2$ |

As an arbitrary constant $a_1^*$ we choose $l^2$.

Fig.2 presents plots of the functions $g^{*\alpha}(\eta) = h^{*\alpha}(\xi)l^{-2}$ of the dimensionless variable $\eta = \xi l^{-1}$, $\eta \in< 0, 0.5 >$ for the successive values of $\lambda_1^{*\alpha}$. In the remaining part of the line segment, for $\eta \in< -0.5, 0 >$, the plots of the functions $g^{*\alpha}(\eta)$ are antisymmetric.

For the even shape function averaged condition (5.12) is not trivially satisfied. We must use equations (5.9) substituting $h_\beta = B_\beta h_\beta^{IV}/(\rho_\beta\omega^2)$, $\beta = 1, 2$, to the left-hand side of (5.12) obtaining

$$\begin{aligned} & l^{-1}\Big(\rho_2 \int\limits_{-l/2}^{-a} h_2(\xi)\, d\xi + \rho_1 \int\limits_{-a}^{a} h_1(\xi)\, d\xi + \rho_2 \int\limits_{a}^{l/2} h_2(\xi)\, d\xi\Big) = \\ & = \frac{1}{l\omega^2}\Big[B_2 h_2'''(\xi)\Big|_{-l/2}^{-a} + B_1 h_1'''(\xi)\Big|_{-a}^{a} + B_2 h_2'''(\xi)\Big|_{a}^{l/2}\Big] = \\ & = \frac{1}{l\omega^2}\Big[B_2 h_2'''(-a) - B_2 h_2'''(-l/2) + B_1 h_1'''(a) - B_1 h_1'''(-a) + \\ & + B_2 h_2'''(l/2) - B_2 h_2'''(a)\Big] \equiv 0 \end{aligned} \tag{5.16}$$

For the even shape function the following extra conditions must be satisfied

$$\begin{aligned} (h_2^*)'(l/2) &= (h_2^*)'(-l/2) = 0 \\ (h_2^*)'''(l/2) &= (h_2^*)'''(-l/2) = 0 \end{aligned} \tag{5.17}$$

We look for the even mode shape function in the following form

$$\begin{aligned} & h_1^{**}(\xi) = a_1^{**} \cos \lambda_1^{**}\xi + b_1^{**} \cosh \lambda_1^{**}\xi & & \text{for} \quad \xi \in (-a, a) \\ & h_2^{**}(\xi) = a_2^{**} \cos \lambda_2^{**}(\xi - l/2) + b_2^{**} \cosh \lambda_2^{**}(\xi - l/2) + \\ & \quad + c_2^{**} \sin \lambda_2^{**}(\xi - l/2) + d_2^{**} \sinh \lambda_2^{**}(\xi - l/2) & & \text{for} \quad \xi \in (a, l/2) \\ & h_2^{**}(\xi) = -h_2^{**}(-\xi) & & \text{for} \quad \xi \in (-l/2, -a) \end{aligned} \tag{5.18}$$

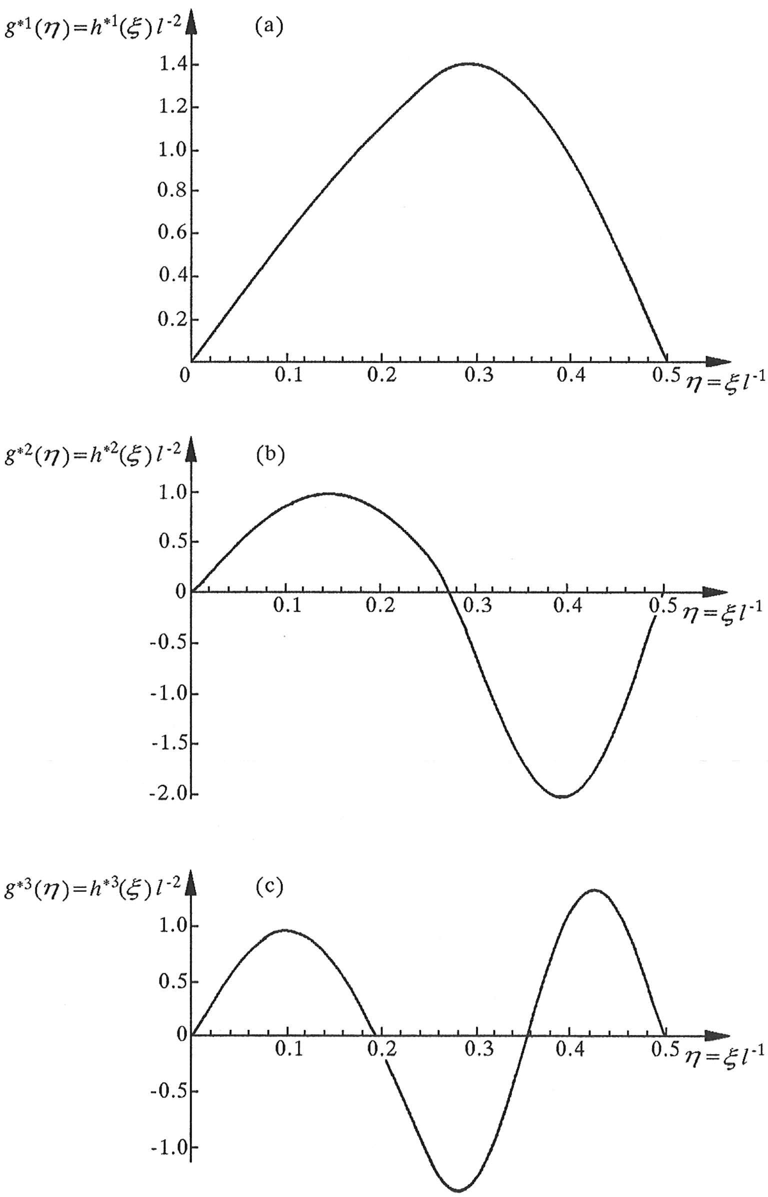


Fig. 2. The plot of the function $g^{*\alpha}(\eta) = h^{*\alpha}(\xi)l^{-2}$, (a) $\alpha = 1$, (b) $\alpha = 2$, (c) $\alpha = 3$, for $\eta \in < 0, 0.5 >$

From conditions (5.17) we obtain $c_2^* = 0$, $d_2^* = 0$. By making use of the choice of the arguments introduced into (5.18), $h_2^*(\cdot)$ satisfies the periodic boundary conditions of mode shape functions (5.10).

The nontrivial solution to eigenvalue problem (5.10) in form (5.18) exist provided that the system of the first four equations in (5.11) for unknowns $a_1^{**}$, $a_2^{**}$, $b_1^{**}$, $b_2^{**}$ has the determinant equal to zero. In this way we find numerically the successive values of $\lambda_1^{**\alpha}$, $\alpha = I, II, III, \ldots$.

For data (5.15) we obtain the following solutions for the first three eigenvalues $\lambda_1^{**\alpha}$, see Table 2.

**Table 2**

| | | | |
|---|---|---|---|
| $\lambda_1^{**}$ | $5.64768l^{-1}$ | $10.0391l^{-1}$ | $15.815l^{-1}$ |
| $a_1^{**}$ | $l^2$ | $l^2$ | $l^2$ |
| $b_1^{**}$ | $-0.094638l^2$ | $-0.069884l^2$ | $-0.002356l^2$ |
| $a_2^{**}$ | $-1.70403l^2$ | $1.49056l^2$ | $-2.11347l^2$ |
| $b_2^{**}$ | $-0.20041l^2$ | $0.007391l^2$ | $0.006552l^2$ |

As an arbitrary constant $a_1^{**}$ we take $l^2$.

Fig.3 presents plots of the functions $g^{**\alpha}(\eta) = h^{**\alpha}(\xi)l^{-2}$ of the dimensionless variable $\eta = \xi l^{-1}$, $\eta \in <0, 0.5>$ for the successive values of $\lambda_1^{**\alpha}$. In the remaining part of the line segment, for $\eta \in <-0.5, 0>$, the plots of the functions $g^{**\alpha}(\eta)$ are symmetric.

## 6. Conclusions

By applying the tolerance averaging approach, one is able to formulate equations of structured beams in the form of a system of averaged differential equations with slowly varying functional or constant coefficients. This approximation describes the effect of segment length of the beam on its global dynamic behaviour (this effect disappears in the classical homogenization solutions).

The obtained averaged equations of the structured beam are applicable to investigations of special problems provided that the mode shapes $h_x^A(\cdot)$, $A = 1, 2, \ldots, n$, can be established. In these cases the series $\sum h_x^A \psi_A(x)$ are convergent. Moreover, in most problems higher modes $h_x^A(\cdot)$ do not contribute significantly to the solution and that is why these series can be truncated and replaced by th sums $h_x^A \psi_A(x)$. In practice however the analytical exact

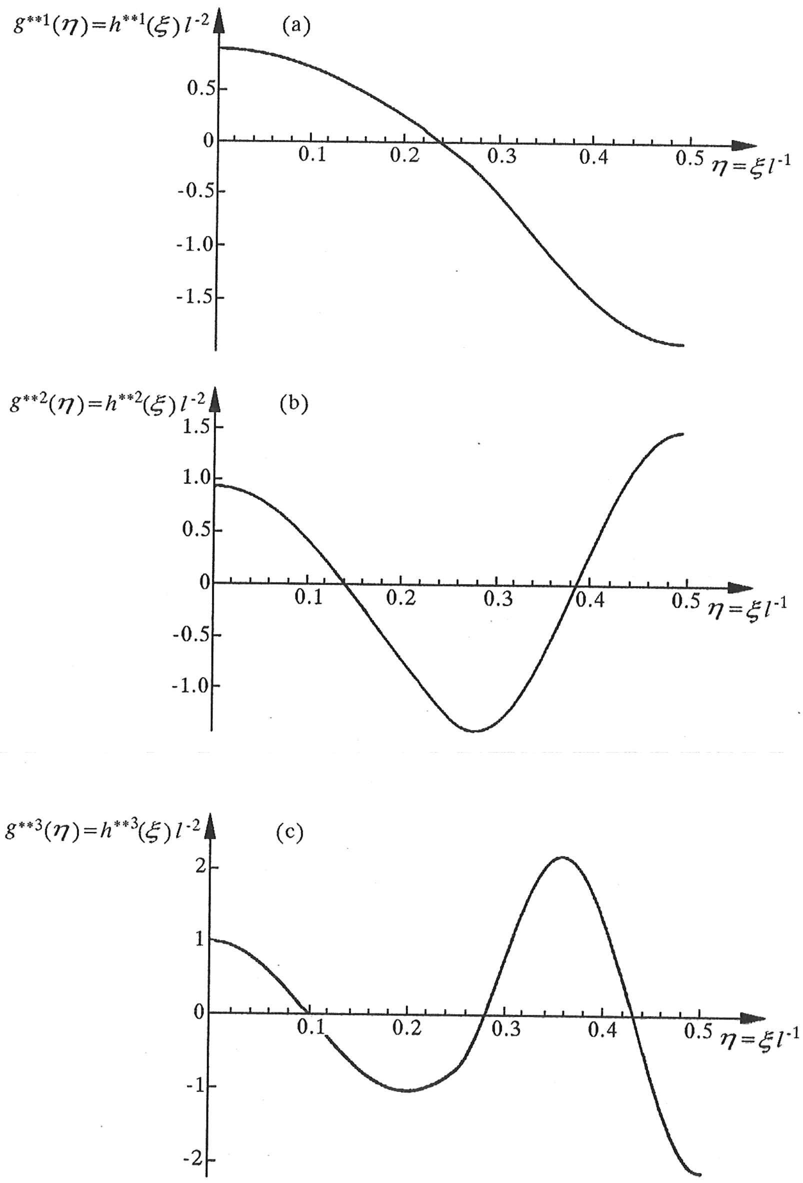


Fig. 3. The plot of the function $g^{**\alpha}(\eta) = h^{**\alpha}(\xi)l^{-2}$, (a) $\alpha = 1$, (b) $\alpha = 2$, (c) $\alpha = 3$, for $\eta \in < 0, 0.5 >$

solutions $h_x^A(\xi)$, $\xi \in \overline{\Delta}(x)$, to the eigenvalue problems of equation (4.9) can only be obtained for rather simple structured beams.

In most cases, instead of exact solutions to eigenvalue problems, we have to look for an approximate form of the vibration mode shapes $h_x^A(\cdot)$. To do this we take into account the mode shapes related to the systems similar to that described by (4.9). We also restrict considerations to a small number $n$ of mode shapes. These approximations may be significant if we calculate with sufficient accuracy bending moments and shear forces in a cross section of structured beam. On the other hand, the overall behavior of the beam can be analyzed using a properly chosen approximate form of the mode shapes $h_x^A(\cdot)$. Because in the obtained equations we only deal with the mean values of the mode shapes in the pertinent segments the exact expressions for $h_x^A(\cdot)$ are not necessary.

It is easy to see that for periodic systems all coefficients in averaged equations are constant. For periodic-like systems, which are not periodic, the aforementioned coefficients are smooth slowly varying functions. In this case their values should be calculated at some points of the interval $\Omega =< 0, L >$ and then extrapolated on $\Omega$.

The averaged equations also describe the simple beam with $B$, $\rho$ and $k$ as a slowly varying functions. In this special case equations (4.1) separate into well known beam equation (2.3) and the homogeneous system of equations for $\psi_A(\cdot)$; the latter under the homogeneous initial conditions $\psi_A(x,t_0) = 0$, $\dot{\psi}_A(x,t_0) = 0$ leads to the trivial solution $\psi_A(x,t) \equiv 0$.

Applications of the results obtained in this contribution (i.e. the averaged model equations) will be investigated in the forthcoming papers.

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## Zastosowanie modelu kinematycznych wewnętrznych zmiennych w dynamice belek o prawie periodycznej strukturze

Streszczenie

Celem pracy jest zaproponowanie nowego uśrednionego modelu dynamicznej odpowiedzi prawie periodycznej prostej belki o zmiennym przekroju. Rozważana belka współpracuje z podłożem winklerowskim o prawie okresowo zmiennych właściwościach. Zagadnienie rozpatruje się w ramach teorii Bernoulliego-Eulera, z uwzględnieniem wpływu sił osiowych na ugięcie belki. Zastosowanie metody uśredniania tolerancyjnego, Woźniak (1999), umożliwia sformułowanie uśrednionych równań belki o prawie periodycznej strukturze, które opisują efekt skali.

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