JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
1, **39**, 2001

INFLUENCE OF THE SUPPORTING SPRING STIFFNESS ON THE VIBRATIONS AND STABILITY OF A GEOMETRICALLY NON-LINEAR COLUMN

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In the paper, the non-linear vibrations and stability of a compound two-rod column with different bending rigidities of its members are investigated. The support of the column is pinned and has a rotational spring the stiffness of which can be either constant or dependent on the applied load. A perturbation technique is chosen to solve the problem. As the column is loaded by a partially follower load it loses its stability via divergence or flutter. For the case of the nonconservative load an adjoint system is formulated for finding the amplitude-frequency relation. The linear part of the natural frequency for adjoint systems is the same, but the non-linear terms differ for each system because they depend on the vibration modes. The support stiffening may lower the critical load of the column.

Key words: natural vibrations, divergence, flutter

1. Introduction

The stability and vibration of elastically restrained columns and frames subjected to a conservative or nonconservative loads have been studied thoroughly by many researchers. The problem became especially important after the experimental investigation on the behaviour of a steel column base connection done and described by Piccard and Beaulieu (1985), Piccard et al. (1986). Their results showed that the compressive force significantly increases the flexural stiffness of this connection, and because of that, in the design of such systems, the rotational spring with its stiffness dependent on the external load should be considered. The effect of such support stiffening on the column buckling for two columns, one pinned at the top and the second clamped at the top was studied by Plaut (1989). The support stiffened with the compressing force according to a linear relation, and when the stiffening parameter increased the critical force increased as well. Guran and Plaut (1993) extended the investigation on the case of an elastic column under a follower load. The rotational spring supporting the column had stiffness either constant or increasing with a linear or quadratic function. It was found that the initial spring stiffness and the stiffening rate may lower the critical load and have a crucial effect on the vibration and stability of such a system.

The main purpose of this work is to show how for the stiffening of the support can affect the natural vibration and stability of a geometrically non-linear two-member elastic slender structure loaded by a partially follower force. In this way the conservative or nonconservative load can be studied dependently on the follower parameter, and as a result, the divergence or flutter instability of the system. The problem of the flutter and divergence instabilities applicable to engineering was thoroughly described by Kurnik (1997) in his book. Methods of the analysis of columns under follower forces were presented by Bogacz and Janiszewski (1986). There are two physical models of the structure: a column made of two coaxial tubes, or a tube and a bar, or a planar frame made of a strip located in the centre of the structure in which the second member is formed by two identical strips, symmetrically located at both sides of the central strip. Buckling of the prestressed frame with one clamped end was experimentally and theoretically investigated by Godley and Chilver (1970). Przybylski et al. (1996), examining theoretically and experimentally the free vibration and stability of a compound pinned frame, found that the distribution of bending rigidity of the frame members had the essential influence on the natural vibrations of the system and this effect can be diminished by introducing a prestress into the structure. The stochastic stability of a compound column was studied by Tylikowski (1991).

The problem investigated in this work is a continuation, in the sense of the same object, of the problem studied by the author (1999) where the influence of the prestress on the stability of a cantilever column loaded by a follower force was presented. It was shown in the latter work that the prestress can cause discontinuities in the critical force and that the instability of such a system can occur for small values of the external load.

While creating a mathematical model of a given system, especially when follower loads are concerned the results of experimental investigation and their comparison with the applied theory should be taken into account. Sugiyama et al. (1995) presented the experimental results, which well agreed with the theoretical flutter predictions, for cantilevered columns with an intermediate mass and subjected to a rocket thrust. Knowing that the internal structural damping may stabilise or destabilise a nonconservative system, the authors neglected it in the theoretical FEM model, which proves that it can be done for structures with very small damping. Similar investigations were done by Mullagulov (1994), who compared the theoretical and experimental results of the critical flutter forces for columns with constant and stepwise varying cross-sections. Despite the absence of the internal damping in the theoretical continuous model, the differences between the results were within the range of $4.0 \div 6.5$ per cent. The theoretical predictions done in the last two cited works and also by Gasparini et al. (1995) were made on the base of the kinetic instability criterion saying that the critical flutter load is the value of the load at which the two smallest natural frequencies approach each other until they coalesce. The divergence critical load is the value of the load at which the smallest eigenvalue becomes equal to zero. The kinetic criterion is also used in this work.

2. Solution of the problem

The scheme of deformed axes of both rods of the column subject to a partially follower load is given in Fig.1. The external force acts depending on the follower parameter η which implies that the system is conservative for $\eta = 0$ and nonconservative for $\eta > 0$. The ends of the members of the column are rigidly connected to each other in both the displacement and rotational senses. The column is pinned with a rotational spring, the stiffness of which can be constant or may increase when the spring is compressed.

The governing equations of the system are identical to those derived by Przybylski et al. (op. cit.) on the basis of the strain-displacement relations for a beam undergoing a moderately large deflection, described by von Kármán and applied by Woinowsky-Krieger (1950), and presented by Levinson (1996) Hamilton's principle. The equations have the following non-dimensional form:
— for the lateral vibration of the ith of the column $(i = 1, 2)$

$$\frac{\partial^4 w_i(\xi,\tau)}{\partial \xi^4} + k_i \frac{\partial^2 w_i(\xi,\tau)}{\partial \xi^2} + \varpi_{ni} \frac{\partial^2 w_i(\xi,\tau)}{\partial \tau^2} = 0 \tag{2.1}$$

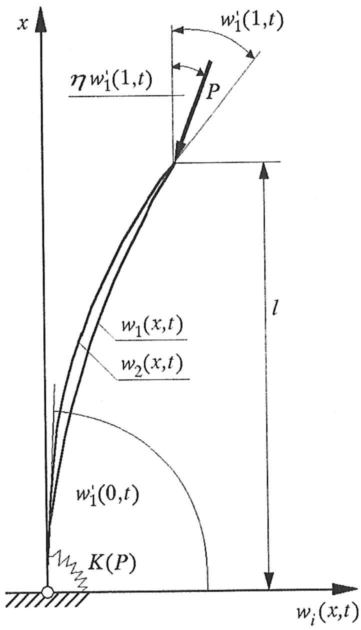


Fig. 1. Scheme of the deflected axis of rods of the two-member column subject to a partially follower force

— for the longitudinal displacement $u_i(\xi,\tau)$ of the ith rod $(i=1,2)$

$$u_i(\xi,\tau) = -\frac{k_i}{\lambda_i}\xi - \frac{1}{2}\int\limits_0^{\xi} \frac{\partial w_i(\xi,\tau)}{\partial\zeta}\,d\zeta \tag{2.2}$$

where

$$w_i = \frac{W_i(x,\tau)}{l} \qquad k_i = \frac{S_i l^2}{E_i I_i} \qquad \varpi_{ni} = \Omega_n^2 l^4 \frac{\rho_i A_i}{E_i I_i}$$

for $i=1,2$ denote the non-dimensional transverse displacements, load parameters and non-dimensional frequency parameter, respectively, and

l	–	length of the column
Ω_n	–	nth natural frequency
$E_i I_i$	–	bending stiffness of the ith rod
$\rho_i A_i$	–	mass per the unit length of the ith rod.

The other dimensionless quantities are as follows

$$\xi = \frac{x}{l} \qquad \tau = \Omega_n t$$
$$\lambda_i = \frac{A_i l^2}{I_i} \qquad u_i(\xi,\tau) = \frac{U_i(x,\tau)}{l} \tag{2.3}$$

A perturbation technique is used to study the behaviour of the system in the neighbourhood of linear solutions since a closed form of the solution to equations (2.1) and (2.2) is not possible to obtain. By making use of the perturbation method, the corresponding quantities are expanded into exponential series with respect to the small amplitude parameter ε (cf Evansen, 1968; Öz et al., 1998), $i = 1, 2$

$$w_i(\xi,\tau) = \sum_{j=1}^{N} \varepsilon^{2j-1} w_{i(2j-1)}(\xi,\tau) + O(\varepsilon^{N+1})$$

$$k_i(\tau) = k_{i0} + \sum_{j=1}^{N} \varepsilon^{2j} k_{i(2j)}(\tau) + O(\varepsilon^{N+1}) \tag{2.4}$$

$$\varpi_{ni} = \omega_{ni} + \sum_{j=1}^{N} \varepsilon^{2j} \omega_{ni}^{(2j)} + O(\varepsilon^{N+1})$$

where $\omega_{ni}^{(2j)}$ stands for the frequency correction coefficients, and for $i = 1, 2$

$$\begin{aligned}
w_{i1}(\xi,\tau) &= w_{i1}^{(1)}(\tau)\cos\tau \\
w_{i3}(\xi,\tau) &= w_{i3}^{(1)}(\tau)\cos\tau + w_{i3}^{(3)}(\tau)\cos 3\tau \\
w_{i5}(\xi,\tau) &= w_{i5}^{(1)}(\tau)\cos\tau + w_{i5}^{(3)}(\tau)\cos 3\tau + w_{i5}^{(5)}(\tau)\cos 5\tau \\
&\vdots
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
k_{i2}(\tau) &= k_{i2}^{(0)} + k_{i2}^{(2)}\cos 2\tau \\
k_{i4}(\tau) &= k_{i4}^{(0)} + k_{i4}^{(2)}\cos 2\tau + k_{i4}^{(4)}\cos 4\tau \\
&\vdots
\end{aligned} \tag{2.6}$$

Following the standard procedure, equations (2.4) are substituted into equations of motion (2.1) and axial displacements (2.2). The coefficient of each power of ε is collected and equated with zero which results in the following set of equations of motion and longitudinal displacements $(i = 1, 2)$

$$O(\varepsilon^0) \qquad u_{i0}(\xi) = -\frac{k_{i0}}{\lambda_i}\xi \tag{2.7}$$

$$O(\varepsilon^1) \qquad w_{i1}^{IV}(\xi,\tau) + k_{i0} w_{i1}^{II}(\xi,\tau) + \omega_{ni}\ddot{w}_{i1}(\xi,\tau) = 0 \tag{2.8}$$

$$O(\varepsilon^2) \qquad u_{i2}(\xi,\tau) = -\frac{k_{i2}(\tau)}{\lambda_i}\xi - \frac{1}{2}\int\limits_0^{\xi} [w_{i1}^{I}(\zeta,\tau)]^2 \, d\zeta \tag{2.9}$$

$$\begin{aligned} O(\varepsilon^3) \qquad & w_{i3}^{IV}(\xi,\tau) + k_{i0} w_{i3}^{II}(\xi,\tau) + \omega_{ni}\ddot{w}_{i3}(\xi,\tau) = \\ & -k_{i0}(\tau) w_{i1}^{II}(\xi,\tau) - \omega_{ni}^{(2)}\ddot{w}_{i1}(\xi,\tau) \end{aligned} \tag{2.10}$$

$$\vdots$$

The roman numerals and dots denote derivatives with respect to ξ and τ, respectively.

Equations (2.7) $\div$ (2.10) are associated with the following boundary conditions $(j = 1, 3, 5, ...)$

$$\begin{aligned} & w_{1j}(0,\tau) = w_{2j}(0,\tau) = 0 \\ & w_{1j}(1,\tau) = w_{2j}(1,\tau) \\ & w_{1j}^{I}(\xi,\tau)|_{\xi=0} = w_{2j}^{I}(\xi,\tau)|_{\xi=0} \\ & w_{1j}^{I}(\xi,\tau)|_{\xi=1} = w_{2j}^{I}(\xi,\tau)|_{\xi=1} \\ & w_{1j}^{II}(\xi,\tau)|_{\xi=0} + \mu w_{2j}^{II}(\xi,\tau)|_{\xi=0} - k_m(p) w_{1j}^{I}(\xi,\tau)|_{\xi=0} = 0 \\ & w_{1j}^{II}(\xi,\tau)|_{\xi=1} + \mu w_{2j}^{II}(\xi,\tau)|_{\xi=1} = 0 \\ & w_{1j}^{III}(\xi,\tau)|_{\xi=1} + \mu w_{2j}^{III}(\xi,\tau)|_{\xi=1} + p(1-\eta)(1+\mu) w_{1j}^{I}(\xi,\tau)|_{\xi=1} = 0 \end{aligned} \tag{2.11}$$

and

$$u_{1j}(0,\tau) = u_{2j}(0,\tau) = 0 \qquad u_{1j}(1,\tau) = u_{2j}(1,\tau) \qquad j = 2, 4, ... \tag{2.12}$$

$$k_{10}\mu_1 + k_{20}\mu_2 = p \tag{2.13}$$

$$k_{1j}(\tau)\mu_1 + k_{2j}(\tau)\mu_2 = 0 \qquad j = 2, 4, ... \tag{2.14}$$

where

$$p = \frac{Pl^2}{E_1I_1 + E_2I_2} \qquad \mu_1 = \frac{E_1I_1}{E_1I_1 + E_2I_2}$$
$$\mu_2 = \frac{E_2I_2}{E_1I_1 + E_2I_2} \qquad \mu = \frac{E_2I_2}{E_1I_1}$$
$$k_m(p) = K(P)\frac{l}{E_1I_1}$$

and P is the external load applied to the column, $K(P)$ is the rotational spring stiffness dependent on the load P.

In this work the nondimensional rotational spring stiffness $k_m(p)$ is a quadratic function of p as it was assumed by Guran and Plaut (op. cit.)

$$k_m(p) = k_{m1}(1 + \beta p^2) \tag{2.15}$$

where k_{m1} is the initial nondimensional spring stiffness and β is the stiffening parameter.

For an unloaded column $p = 0$, which gives $k_m(0) = k_{m1}$. While considering the spring stiffness independent of p, $\beta = 0$ should be introduced in (2.15), which gives $k_m(p) = k_{m1} = \text{const}$.

Equation (2.7) expresses the axial displacement – force relation in the ith column member. The inserting of these equations into boundary conditions (2.12) for $j = 0$ gives a linear relationship between the axial forces S_{i0} in each rod due to the external force P in the following form $(i = 1, 2)$

$$S_{i0} = \frac{E_iA_i}{E_1A_1 + E_2A_2}P \tag{2.16}$$

For the column with rods of the same axial stiffness $(E_1A_1 = E_2A_2)$ the force $S_{10} = S_{20} = P/2$.

The general solution to equation (2.8), after separation of the ξ and τ variables according to equation $(2.5)_1$ is as follows $(i = 1, 2)$

$$w_{i1}^{(1)}(\xi) = A_{i1}\cosh(\alpha_{i1}\xi) + B_{i1}\sinh(\alpha_{i1}\xi) + C_{i1}\cos(\beta_{i1}\xi) + D_{i1}\sin(\beta_{i1}\xi) \tag{2.17}$$

where

$$\alpha_{i1} = \sqrt{-\frac{1}{2}k_{i0} + \sqrt{\frac{1}{4}k_{i0}^2 + \omega_{ni}}} \qquad \beta_{i1} = \sqrt{\frac{1}{2}k_{i0} + \sqrt{\frac{1}{4}k_{i0}^2 + \omega_{ni}}} \tag{2.18}$$

By substituting equation (2.17) for $i = 1, 2$ into boundary conditions (2.11) for $j = 1$ one obtains the system of eight homogeneous equations with the unknown integration constants A_{i1}, B_{i1}, C_{i1} and D_{i1}, $(i = 1, 2)$. The determinant of matrix coefficient of the system must be equal to zero for a nontrivial solution of the problem. This implies a relation between the external load and the natural frequency, which is solved numerically afterwards. To find the vibration mode for the calculated frequency the normalisation condition was applied $w_{i1}^{(1)}(1) = 1$. The equation resulting from this condition is to replace an arbitrary one from the above-mentioned system of eight equations. Numerical solution to the new system of equations yields the values of constants A_{i1}, B_{i1}, C_{i1} and D_{i1}, $(i = 1, 2)$ in (2.17) expressing the mode shapes of both rods.

3. Amplitude-frequency relation for the conservative load (case $\eta = 0$)

The amplitude force parameter $k_{i2}(\tau)$ existing in equation (2.9) depends on the vibration amplitude. It is derived from conditions $(2.12)_2$ with equations $(2.5)_1$ and $(2.6)_1$ as well as condition (2.14) being taken into consideration. This leads to the equations

$$\begin{aligned} k_{12}^{(0)} &= k_{12}^{(2)} = \frac{1}{4}\frac{\mu\lambda_1\lambda_2}{\lambda_1 + \mu\lambda_2}\int\limits_0^1 \Big[\Big(\frac{\partial w_{21}^{(1)}(\xi)}{\partial \xi}\Big)^2 - \Big(\frac{\partial w_{11}^{(1)}(\xi)}{\partial \xi}\Big)^2\Big]\,d\xi \\ k_{22}^{(0)} &= k_{22}^{(2)} = -\frac{1}{\mu}k_{12}^{(0)} \end{aligned} \tag{3.1}$$

On the grounds of the above equations it is easy to find that the amplitude force parameters $k_{12}^{(0)}$ and $k_{22}^{(0)}$ differ from zero if the vibration modes of both rods do not overlap each other, or if the symmetric modes characterised by the same amplitude for each rod and the opposite sign of the curvature do not appear. These parameters were calculated numerically after finding the analytical solution, i.e. when equations (2.17) for $i = 1, 2$ were inserted into equation $(3.1)_1$. The internal force parameter in each rod is now equal to

$$k_i(\tau) = k_{i0} + \varepsilon^2 k_{i2}^{(0)}(1 + \cos 2\tau) \tag{3.2}$$

and is dependent on the value of the amplitude parameter ε.

The frequency correction parameter $\omega_{ni}^{(2)}$ from equation (2.10) can be find from the orthogonality condition proposed by Keller and Ting (1966). For the conservative load of the column $(\eta = 0)$ equation (2.10) for each $i = 1, 2$, is an inhomogeneous form of adequate equation (2.8) and both equations have similar boundary conditions. Equation (2.10) will have a periodic solution if and only if its right-hand side is orthogonal to all the solutions to adjoint homogeneous equation (2.8). After separation of the ξ and τ variables according to equations $(2.5)_2$ and $(2.6)_2$, the orthogonality condition is obtained by multiplying equation (2.10) by $E_iI_iw_{i1}^{(1)}(\xi)$, integrating it over the range $< 0,1 >$ and adding with respect to i $(i = 1,2)$. It is easy to find that the integral of the left-hand side vanishes when conditions (2.11) for $j = 1,3$ are taken into account. The right-hand side of equation (2.10) yields

$$\begin{gathered}\frac{3}{2}E_1I_1k_{12}^{(0)}\int\limits_0^1\Big[\Big(\frac{\partial w_{11}^{(1)}(\xi)}{\partial\xi}\Big)^2-\Big(\frac{\partial w_{21}^{(1)}(\xi)}{\partial\xi}\Big)^2\Big]\,d\xi+\\ +\omega_n^{(2)}\frac{E_1I_1+E_2I_2}{\rho_1A_1+\rho_2A_2}\Big\{\rho_1A_1\int\limits_0^1[w_{11}^{(1)}(\xi)]^2\,d\xi+\rho_2A_2\int\limits_0^1[w_{21}^{(1)}(\xi)]^2\,d\xi\Big\}=0\end{gathered}\tag{3.3}$$

The nondimensional correction frequency $\omega_n^{(2)}$

$$\omega_n^{(2)}=(\Omega_n^{(2)})^2l^4\frac{\rho_1A_1+\rho_2A_2}{E_1I_1+E_2I_2}$$

is introduced here instead of $\omega_{ni}^{(2)}$ (comp. equation $(2.2)_4$) to get the reference to the whole structure, not only to one of its members which is particulary significant when one presents the numerical results. Between those parameters the following relation holds

$$\omega_n^{(2)}=\omega_{ni}^{(2)}\frac{E_iI_i}{\rho_iA_i}\frac{\rho_1A_1+\rho_2A_2}{E_1I_1+E_2I_2}\tag{3.4}$$

By inserting $(3.1)_1$ into (3.3) one obtains

$$\begin{gathered}\omega_n^{(2)}=\frac{3}{8(1+\mu)}\frac{\mu\lambda_1\lambda_2}{\lambda_1+\mu\lambda_2}(\rho_1A_1+\rho_2A_2)\cdot\\ \cdot\frac{\Big(\int\limits_0^1\Big[\Big(\frac{\partial w_{21}^{(1)}(\xi)}{\partial\xi}\Big)^2-\Big(\frac{\partial w_{11}^{(1)}(\xi)}{\partial\xi}\Big)^2\Big]\,d\xi\Big)^2}{\rho_1A_1\int\limits_0^1[w_{11}^{(1)}(\xi)]^2\,d\xi+\rho_2A_2\int\limits_0^1[w_{21}^{(1)}(\xi)]^2\,d\xi}\end{gathered}\tag{3.5}$$

The first correction frequency parameter $\omega_n^{(2)}$ takes non-zero values when both rods of the column vibrate with different mode shapes. Having this parameter numerically calculated after solving analytically equation (3.3), the frequency according to $(2.4)_3$ is

$$\varpi_n = \omega_n + \varepsilon^2\omega_n^{(2)} + O(\varepsilon^4) \tag{3.6}$$

The effects brought about by the non-linearity in the system begin with terms of $O(\varepsilon^2)$ so the changes in the vibration amplitude controlled by ε (comp. equation $(2.4)_1$) affect the amplitude force parameter $k_{12}^{(0)}$ and the frequency correction parameter $\omega_n^{(2)}$. The amplitude-frequency relation which exists for non-linear problems can be exhibited for an arbitrary level of the load if at least the first frequency correction parameter is calculated. It is customary to restrict practical considerations to terms up to the second order in expansion $(2.4)_3$ for the problems of vibration at lower modes as it was done by Evansen (1968) and Aravamudan and Murthy (1973). The higher order terms need investigating the response at higher modes.

4. Amplitude-frequency relation for the nonconservative load (case $0 < \eta \leqslant 1$)

For the nonconservative load $(0 < \eta \leqslant 1)$ the boundary-value problem is a non-self-adjoint one, so the formulation of the adjoint system is necessary to obtain the amplitude-frequency relation. This procedure was applied by Nemat-Nasser and Herrmann (1966), Plaut (1972) and Anderson (1975). For the case of the Beck type problem, as which can be treated a column made of two identical rods and subjected at its free end to a compressive follower force, the Reut problem is the adjoint one.

This remark can be confirmed by realising the above described procedure resulting from Keller and Ting's orthogonality condition for the conservative load. The left-hand side of equations (2.10), after separation of the ξ and τ variables according to equations $(2.5)_2$ and $(2.6)_1$, multiplication by $E_iI_iw_{i1}^{(1)}(\xi)$, integration over the range $<0,1>$ and addition with respect to i $(i=1,2)$ with the use of boundary conditions (2.11) takes the following form

$$\begin{aligned}
L = E_1J_1\Big\{&(w_{13}^{(1)})^I(\xi)|_{\xi=1}\cdot\\
&\cdot\Big[(w_{11}^{(1)})^{II}(\xi)|_{\xi=1} + \mu(w_{21}^{(1)})^{II}(\xi)|_{\xi=1} + p(1+\mu)\eta w_{11}^{(1)}(1)\Big] + \\
&-w_{13}^{(1)}(1)\Big[(w_{11}^{(1)})^{III}(\xi)|_{\xi=1} + \mu(w_{21}^{(1)})^{III}(\xi)|_{\xi=1} + p(1+\mu)(w_{11}^{(1)}))^I(\xi)|_{\xi=1}\Big]\Big\}
\end{aligned} \tag{4.1}$$

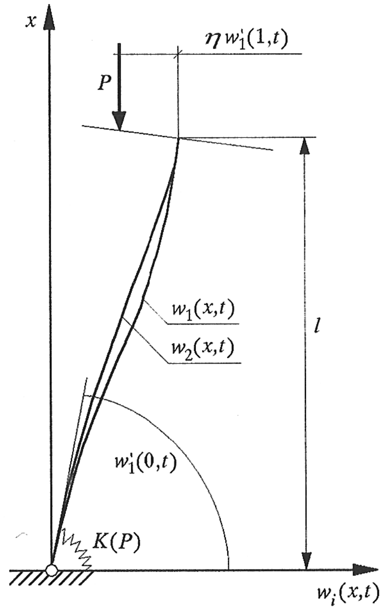


Fig. 2. Scheme of the deflected axis of rods of the two-member adjoint system

This left-hand side can be equal to zero if the terms in the square brackets are equal to zero. It imposes two boundary conditions adequate to conditions $(2.11)_{6,7}$. The system fulfilling such conditions and conditions $(2.11)_{1-5}$, (2.12) and (2.13) is depicted in Fig.2. A single rod cantilevered column under the load resulting from these conditions for $\eta = 1$ was experimentally and theoretically examined by Sugiyama (1982), a column with spring supports along its span by Qiu and Nemat-Nasser (1983).

Replacing $w_{ij}(\xi,\tau)$ by $v_{ij}(\xi,\tau)$ as well as $k_{ij}(\tau)$ by $q_{ij}(\tau)$ to denote the lateral displacement and the dimensionless longitudinal force parameter for the adjoint system, respectively, the set of its boundary conditions before separating the space and time variables is as follows ($j = 1, 3, 5, ...$)

$$
\begin{aligned}
&v_{1j}(0,\tau) = v_{2j}(0,\tau) = 0 \\
&v_{1j}(1,\tau) = v_{2j}(1,\tau) \\
&v_{1j}^{I}(\xi,\tau)|_{\xi=0} = v_{2j}^{I}(\xi,\tau)|_{\xi=0} \\
&v_{1j}^{I}(\xi,\tau)|_{\xi=1} = v_{2j}^{I}(\xi,\tau)|_{\xi=1} \\
&v_{1j}^{II}(\xi,\tau)|_{\xi=0} + \mu v_{2j}^{II}(\xi,\tau)|_{\xi=0} - k_m(p) v_{1j}^{I}(\xi,\tau)|_{\xi=0} = 0
\end{aligned}
\tag{4.2}
$$

$$v_{1j}^{II}(\xi,\tau)|_{\xi=1} + \mu v_{2j}^{II}(\xi,\tau)|_{\xi=1} + p(1+\mu)\eta v_{1j}(1,\tau) = 0$$
$$v_{1j}^{III}(\xi,\tau)|_{\xi=1} + \mu v_{2j}^{III}(\xi,\tau)|_{\xi=1} + p(1+\mu)v_{1j}^{I}(\xi,\tau)|_{\xi=1} = 0$$

and

$$u_{1j}(0,\tau) = u_{2j}(0,\tau) = 0 \qquad u_{1j}(1,\tau) = u_{2j}(1,\tau) \qquad j = 2,4,... \tag{4.3}$$

$$k_{10}\mu_1 + k_{20}\mu_2 = p \tag{4.4}$$

$$q_{1j}(\tau)\mu_1 + q_{2j}(\tau)\mu_2 = 0 \qquad j = 2,4,... \tag{4.5}$$

The governing equations for the adjoint problem are identical to equations (2.1) and (2.2), but new expansions into the amplitude parameter ε must be introduced $(i = 1,2)$

$$\begin{aligned}
v_i(\xi,\tau) &= \sum_{j=1}^{N} \varepsilon^{2j-1} v_{i(2j-1)}(\xi,\tau) + O(\varepsilon^{N+1}) \\
q_i(\tau) &= k_{i0} + \sum_{j=1}^{N} \varepsilon^{2j} q_{i(2j)}(\tau) + O(\varepsilon^{N+1}) \\
\varpi_{ni}^{R} &= \omega_{ni} + \sum_{j=1}^{N} \varepsilon^{2j} (\omega_{ni}^{(2j)})^{R} + O(\varepsilon^{N+1})
\end{aligned} \tag{4.6}$$

The form of the above expansions is determined by the properties of the adjoint system. The static axial force S_{i0} $(S_{i0} = k_{i0}E_iJ_il^{-2})$, which appears in each member of the column due to the external load is the same for each system. Further dynamic terms of the force depend on the vibration modes of a particular system. These modes are different for each of the adjoint systems so the dynamic terms $k_{i(2j)}(\tau)$ from equation $(2.4)_2$ and $q_{i(2j)}(\tau)$ from equation $(4.6)_2$, for any $j = 1,2,...,N$, must differ from each other.

The eigenfrequency ω_{ni} from expansions $(2.4)_3$ and $(4.6)_3$ assumes the same values for each system, as for the linear nonconservatively loaded adjoint systems (cf Nemat-Nasser and Herrmann, 1966; Anderson, 1975). This frequency appears in linear equation (2.8) and the relevant ones for the adjoint system, the solutions to which fulfil linear boundary conditions (2.11) for the first system, and relevant (4.2) for the second one. For the geometrically nonlinear adjoint systems the frequencies ϖ_{ni} from equation $(2.4)_3$ and ϖ_{ni}^{R} from equation $(4.6)_3$ must take different values because their further terms, those multiplyied by the amplitude parameter ε in adequate expansions, depend on the vibration amplitude.

Summarising: to obtain the frequency correction parameter $\omega_{ni}^{(2)}$ from equations (2.10) for the case of the nonconsevatively loaded column, those equations, after separating the time and space variables, are multiplied by $E_i J_i v_{i1}^{(1)}(\xi)$, then integrated over the range $< 0, 1 >$ and added with respect to i ($i = 1, 2$). The left-hand side of the resulting identity takes the form

$$
\begin{aligned}
L = E_1 J_1 \Big\{ & (w_{13}^{(1)})^I(\xi)|_{\xi=1} \cdot \\
& \cdot \Big[(v_{11}^{(1)})^{II}(\xi)|_{\xi=1} + \mu (v_{21}^{(1)})^{II}(\xi)|_{\xi=1} + p(1+\mu)\eta v_{11}^{(1)}(1) \Big] + \\
& - w_{13}^{(1)}(1) \Big[(v_{11}^{(1)})^{III}(\xi)|_{\xi=1} + \mu (v_{21}^{(1)})^{III}(\xi)|_{\xi=1} + p(1+\mu)(v_{11}^{(1)}))^I(\xi)|_{\xi=1} \Big] \Big\}
\end{aligned}
\tag{4.7}
$$

and now is equal to zero because for the adjoint system boundary conditions $(4.2)_{6,7}$ are fulfilled.

By integrating by parts the right-hand side

$$
R = \sum_{i=1}^{2} E_i J_i \int_0^1 \Big[-\frac{3}{2} k_{12}^{(0)} (w_{11}^{(1)})^{II}(\xi) + \omega_{ni}^{(2)} w_{i1}^{(1)}(\xi) \Big] v_{i1}^{(1)}(\xi) \, d\xi = 0 \tag{4.8}
$$

which is now equal to zero, and performing some algebraic transformations one obtains the following expression for the correction frequency parameter $\omega_n^{(2)}$

$$
\omega_n^{(2)} = \frac{3(\rho_1 A_1 + \rho_2 A_2)}{2(1+\mu)} k_{12}^{(0)} \frac{\int_0^1 \Big[\frac{\partial w_{21}^{(1)}(\xi)}{\partial \xi} \frac{\partial v_{21}^{(1)}(\xi)}{\partial \xi} - \frac{\partial w_{11}^{(1)}(\xi)}{\partial \xi} \frac{\partial v_{11}^{(1)}(\xi)}{\partial \xi} \Big] \, d\xi}{\rho_1 A_1 \int_0^1 w_{11}^{(1)}(\xi) v_{11}^{(1)}(\xi) \, d\xi + \rho_2 A_2 \int_0^1 w_{21}^{(1)}(\xi) v_{21}^{(1)}(\xi) \, d\xi} \tag{4.9}
$$

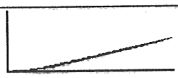
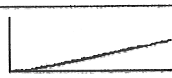
To calculate this parameter after analytical processing of equation (4.9) a general solution for $v_{11}^{(1)}(\xi)$ must be applied at first. It has an analogous form to equation (2.17) with new eight integration constants and identical values of α_{i1}, β_{i1} expressed by (2.18). The integration constants are numerically found from the obtained eight equations, when seven boundary conditions $(4.2)_{1-6}$, and the normalisation condition $v_{i1}^{(1)} = 1$ are satisfied.

5. Results of numerical analysis

For the needs of the numerical analysis certain assumptions were made to limit the number of quantities to be taken into account. A column made of two bars of identical both the mass per unit length and the axial rigidity (e.a. $\rho_1 A_1 = \rho_2 A_2$, $E_1 A_1 = E_2 A_2$, respectively) was examined. According to the results presented by Przybylski et al. (op. cit.), the distribution of these parameters between the column members is less influential on the vibration frequency than the bending rigidity distribution. The sum of the bending rigidity of both rods $(EI = E_1 I_1 + E_2 I_2)$ was taken as constant, but the relation $\mu = E_2 I_2/(E_1 I_1)$ changed its values within the range $0.1 \leqslant \mu \leqslant 1$. The compound column for $\mu = 1$ becomes a geometrically linear system because its rods always vibrate with the same absolute amplitude. As a result, the force parameter $k_{12}^{(0)}$ from equation $(3.1)_1$ and the frequency parameters $\omega_n^{(2)}$ from equations (3.3) and (4.9) are equal to zero.

The rotational spring support had stiffness either constant or increasing according to the quadratic function described by equation (2.15). The stiffening parameter was included within the range $0 \leqslant \beta \leqslant 0.3$.

Table 1. Amplitude-frequency relation for the chosen values of the amplitude parameter $(k_{m1} = 10,\ \beta = 0.0,\ \mu = 0.1)$

A	ϖ_n/ω_n							
	$p/p_{cr} = 0.3$				$p/p_{cr} = 0.6$			
	$\eta = 0.0$		$\eta = 1.0$		$\eta = 0.0$		$\eta = 1.0$	
	First mode	Second mode	First mode	Second mode	First mode	Second mode	First mode	Second mode
1	1	1.0001	1	1.0001	1	1.0001	1	1.0001
2	1	1.0007	1	1.0006	1	1.0008	1	1.0008
3	1	1.0108	1	1.0102	1	1.0191	1	1.0106
4	1	1.0725	1	1.0692	1	1.0836	1	1.0737
5	1.0001	1.0803	1.0002	1.0736	1.0002	1.0975	1.0001	1.0819
Two-rod Beck's column					Two-rod Reut's column			
First mode			Second mode		First mode		Second mode	
								

where $A = \varepsilon w_{11}^{(1)}(1)l/r$,

To find the effect of the nonlinearity during vibration five values of the small parameter related to the nondimensional radius of gyration, see Table 1, were taken into account $(r = \sqrt{I/A})$. The first modes inrespectively of the way of the column loading applications, and whether they result from boundary conditions (2.11) $\div$ (2.14) for Beck's type column, or boundary conditions (4.2) $\div$ (4.5) for Reut's one, have similar shapes, as shown in the table. As the deflections of each rod are almost identical, even though they were obtained for $\mu = 0.1$, the quotient ϖ_n/ω_n for different amplitudes of the first modes is equal to one (except for the last value of ε). The effect of the nonlinearity is visible for the second and further modes.

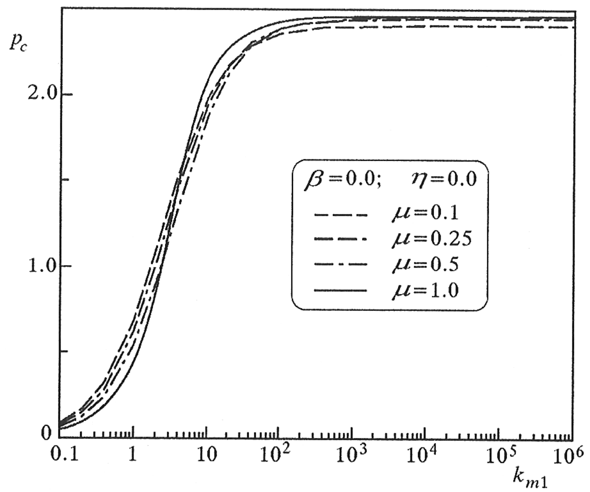


Fig. 3. Divergence critical force versus initial spring stiffness $(\beta = 0)$ for the columns with different bending rigidity ratios and loaded by the conservative force $(\eta = 0)$

All the results presented below were obtained for $\varepsilon w_{11}^{(1)}(1)l/r = 4$. Fig.3 illustrates the effect of the spring stiffness independent of the external load $(\beta = 0)$ on the divergence critical force parameter for the columns with different relation between the bending rigidities of their rods, when they are loaded by a conservative load $(\eta = 0)$. When the stiffness increases up to $k_{m1} = 100$, the critical load also increases; for $k_{m1} > 100$ the critical load is independent of the stiffness. It is worth noticing that for small values of the support stiffness, greater asymmetry in the bending rigidity of the rods corresponds to greater values of the critical force. Then the pattern reverses and the critical force assumes the smallest values for the column of the greatest rigidity ratio $\mu = 0.1$. The same phenomenon is observed for the column which is loaded by the partially follower force with the parameter $\eta = 0.5$, see Fig.4. The differences in the divergence critical load for the support stiffness $k_{m1} > 10$ are more significant in this case, and for $k_{m1} = 10^6$ the critical load for the linear column $(\mu = 1)$ is 1.76 of that for the column with $\mu = 0.1$.

In Fig.5 corresponding to the case of the follower load with $\eta = 1$ one can notice different courses of the critical force curves. For this type of load the system loses its stability via flutter and the critical load decreases when

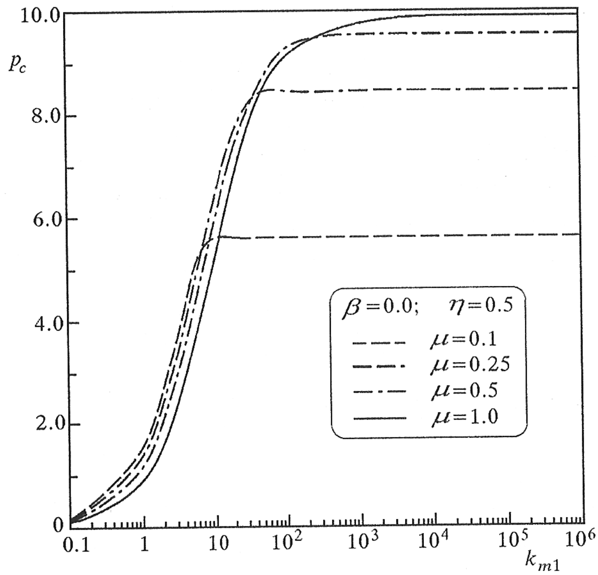


Fig. 4. Divergence critical force versus initial spring stiffness $(\beta = 0)$ for the columns with different bending rigidity ratios and loaded by the conservative force $(\eta = 0.5)$

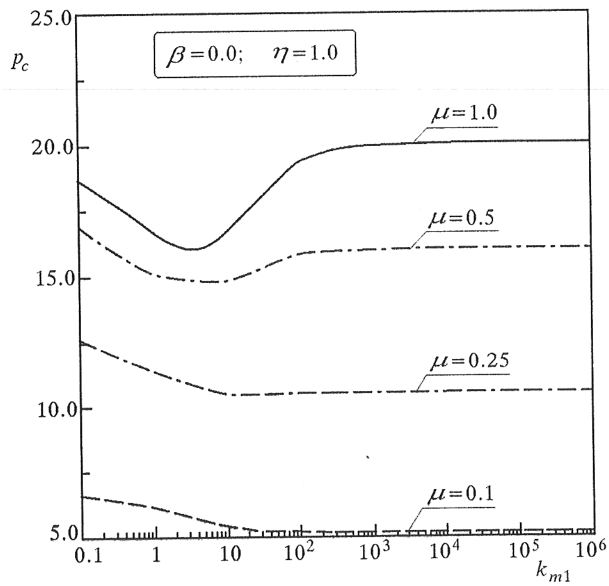


Fig. 5. Flutter critical force versus initial spring stiffness $(\beta = 0)$ for the columns with different bending rigidity ratios and loaded by the conservative force $(\eta = 1)$

the support stiffness increases up to $k_{m1} = 10$. For the linear column and for the column with $\mu = 0.5$, further increasing of the support stiffness gives an increase in the critical load. The maximum critical load parameter regarding the linear column made of identical rods and is equal to 20.0509, which is the value obtained by Beck (1953) for a single rod column. Greater asymmetry in the bending rigidity ratio of the rods gives a smaller critical load. Thus the best reinforcement of the single rod column would be obtained, if necessary, by adding to it the second member of identical bending stiffness.

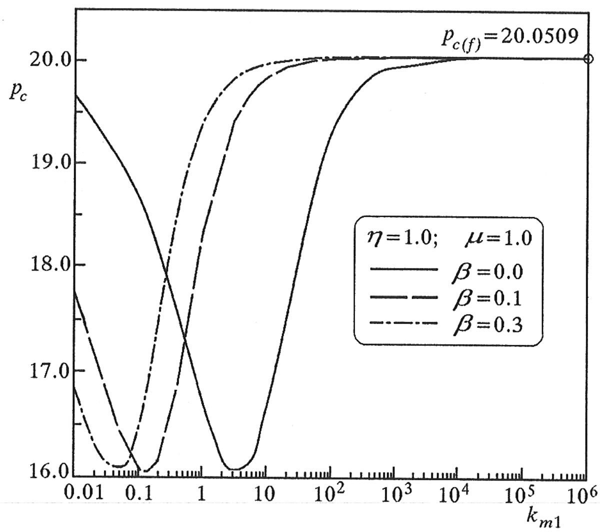


Fig. 6. Flutter critical force versus spring stiffness when $k_m(p) = k_{m1}(1 + \beta p^2)$ for the linear column $(\mu = 1.0)$ loaded by the follower force $(\eta = 1)$

The influence of the stiffening rate expressed by the stiffening rate constant β on the critical flutter force for the column made of identical rods, when the stiffness of the rotational supporting spring is the quadratic function of the load $k_m(p) = k_{m1}(1 + \beta p^2)$, is presented in Fig.6. The results correspond to those obtained by Guran and Plaut (1993) for a tangentially loaded single rod column. The stiffening rate may either decrease or increase the critical load; for $k_{m1} > 5 \cdot 10^3$ it has no effect on the critical load. For the column with considerable asymmetry in the bending rigidity of its rods $\mu = 0.1$, as shown in Fig.7, an increase in the stiffening rate gives a decrease in the divergence critical load if $k_{m1} < 2 \cdot 10^2$; if $k_{m1} > 2 \cdot 10^2$ the value of the critical load stabilises on the level of $p_c = 5.155$.

To illustrate less significant effects of the supporting spring stiffening on

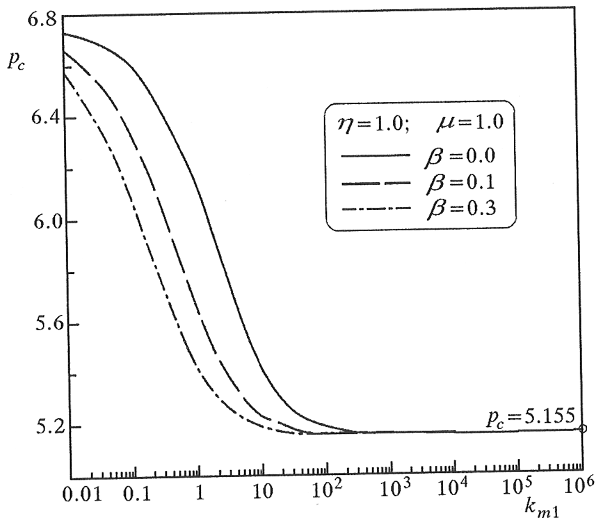


Fig. 7. Flutter critical force versus spring stiffness when $k_m(p) = k_{m1}(1 + \beta p^2)$ for the non-linear column $(\mu = 0.1)$ loaded by the follower force $(\eta = 1)$

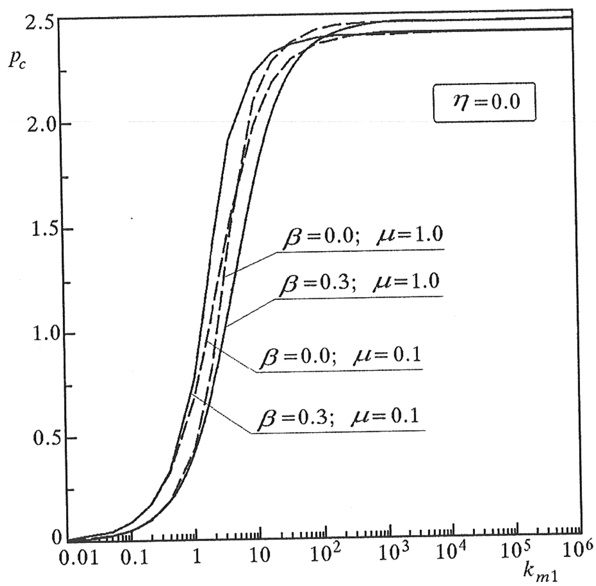


Fig. 8. Divergence critical force versus spring stiffness when $k_m(p) = k_{m1}(1 + \beta p^2)$ for the linear $(\mu = 0.1)$ and non-linear column $(\mu = 0.1)$ loaded by the conservative force $(\eta = 1)$

the divergence critical force for the conservatively loaded column ($\eta = 0.0$), two values of the stiffening rate constant were taken into consideration – one for the column made of identical rods, and the other for the column with rods having the rigidity ratio $\mu = 0.1$ – these results are grouped in Fig.8.

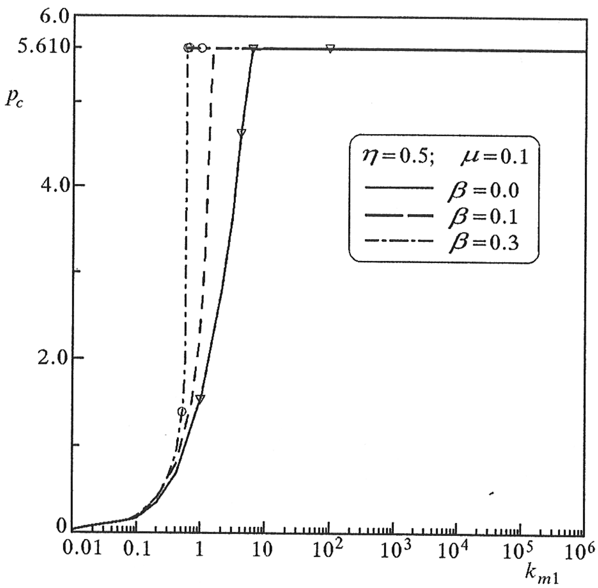


Fig. 9. Divergence critical force versus spring stiffness when $k_m(p) = k_{m1}(1 + \beta p^2)$ for the linear ($\mu = 0.1$) and non-linear column ($\mu = 0.1$) loaded by the partially follower force ($\eta = 0.5$)

Some of the obtained results of the critical force require considering together with the natural frequency curves like in the case of the column loaded by a partially follower force with the parameter $\eta = 0.5$, for which the bending rigidity ratio was $\mu = 0.1$, see Fig.9.

The triangles and circles are the points for which the frequency curves are plotted in successive Fig.10 and Fig.11, respectively. It can be seen in Fig.9 that an initial common gradual growth in the critical load for the increasing value of the support stiffness for different β, changes to a sudden increase, becoming even a jump in the critical load for the stiffening rate $\beta = 0.3$. Such an unusual course of the curve for $\beta = 0.0$ results from the character of the natural frequency curves drawn in Fig.10 with bolder lines for the first frequency curves. The triangles on the y-axis (external load axis) are the points where a particular frequency curve crosses this axis, when $k_{m1} = 1.1$, 4.4 and 100, or is tangent to this axis which happens for $k_{m1} = 6.6$. At

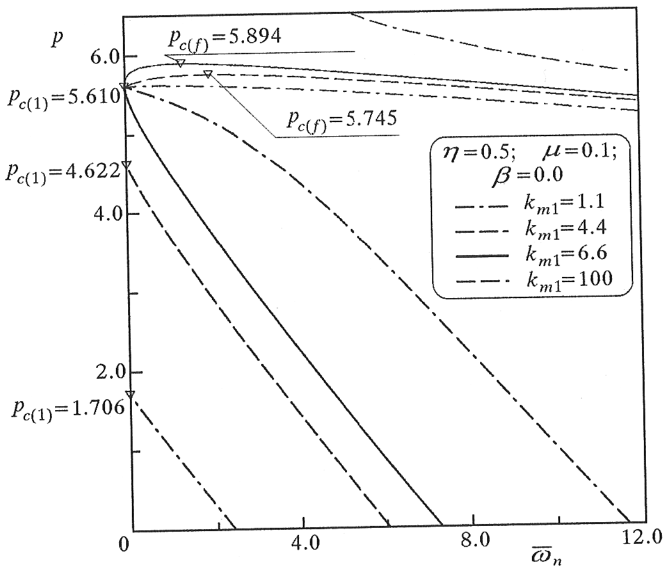


Fig. 10. Natural vibration curves for the non-linear column $(\mu = 0.1)$ loaded by the partially follower force $(\eta = 0.5)$ for different values of the supporting spring stiffness independent of the load $(\beta = 0)$

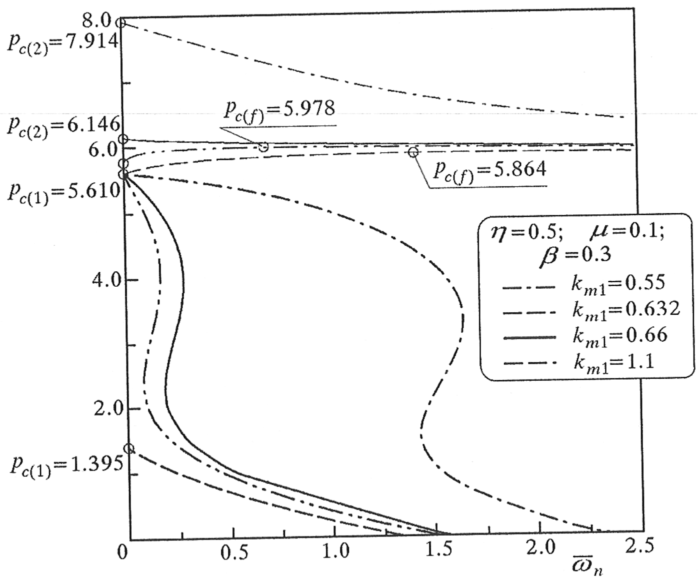


Fig. 11. Natural vibration curves for the non-linear column $(\mu = 0.1)$ loaded by the partially follower force $(\eta = 0.5)$ for different values of the supporting spring stiffness independent of the load $(\beta = 0.3)$

these points the divergence instability occurs. The triangles marked on the branches of the second frequency curves for $k_{m1} = 4.4$, 6.6 are placed at their convexity, so these are the points corresponding to the flutter instability. It is worth noticing that the points of the flutter instability lie above the points of the divergence instability for each value of the support stiffness. For $k_{m1} \in < 0.01, 6.6 >$ the curves of the first frequency intersect the y-axis at the points of their first critical force, whereas all the curves of the second frequency cross that axis at one point $p_c = 5.610$ of their second critical force. If $k_{m1} = 6.6$ the curves of the first and second frequency coincide at the point of the divergence instability existing also for $p_c = 5.610$. For $k_{m1} \in (6.6, 10^6 >$ all the first frequency curves go through the y-axis on the level of their instability existing for the same critical force $p_c = 5.610$.

The natural frequency curves presented in Fig.11 and obtained for the supporting stiffness dependent on the external load with the stiffening constant $\beta = 0.3$ are qualitatively similar to those shown in Fig.10. The only difference consisted in the fact that for smaller values of k_{m1} the first frequency curves, which change their curvature with the external load, intersect the y-axis at the ordinate $p_c = 5.610$.

6. Conclusions

On the basis of the perturbation method, the natural vibration and stability of the geomerically non-linear two-rod column subject to a partially follower force, and supported by the spring with stiffness dependent on the external load have been studied.

The adjoint system to solve the problem for the case of a nonconservative load has been found. For the geometrically non-linear adjoint systems the natural frequency assumes different values because the non-linear terms of the frequency depend on the vibration modes, which are different for each system.

For the first mode of vibration the amplitude-frequency relation is almost constant like in the linear problems due to similar deflections of the vibrating rods. The effect of the nonlinearity becomes noticeable in the second and higher modes.

The critical instability force of the system changes with the supporting stiffness and the bending rigidity distribution between the column members. This distribution has the smallest effect on the divergence critical load when the column is under the conservative force.

Independently of the bending rigidity relation, the supporting spring stiffening considerably changes the values of the critical forces. For the follower loads there are ranges of the spring stiffness within which an increase in the rate of the spring stiffening causes a decrease in the critical load.

The spring stiffening affects also the courses of the natural vibration curves. For a partially follower load all the vibration curves corresponding to the majority of the supporting spring stiffness values, intersect the load axis at one point of the divergence instability.

Acknowledgement

This research has been supported by the State Committee for Scientific Research, Warszawa, Poland, under the grant no. 7T07C03218.

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Wpływ sztywności sprężyny podpierającej geometrycznie nieliniową kolumnę na jej drgania i stateczność

Streszczenie

W pracy analizowane są drgania i stateczność kolumny złożonej z dwóch prętów i podpartej na sprężynie rotacyjnej, której sztywność jest stała lub zależna od przyłożonego obciążenia zewnętrznego. Ponieważ siła zewnętrzna jest częściowo śledząca, to kolumna może tracić stateczność przez dywergencję lub flatter. Celem znalezienia relacji amplituda-częstość dla obciążenia niezachowawczego znaleziono układ sprzężony. Wykazano, że liniowy składnik rozwinięcia częstości drgań obu układów jest taki sam, natomiast składowe nielinowe są różne. Przeprowadzono badania numeryczne, na podstawie których stwierdzono m.in., że wzrost sztywności sprężyny podpierającej kolumnę może obniżać wartość jej siły krytycznej.

Manuscript received May 15, 2000; accepted for print June 21, 2000