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CONTROL IN OBSTACLE THREE-LAYERED PLATE PROBLEM

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The problem to find an optimal thickness of a three-layered plate (ignoring shears in the middle plate) in a set of bounded Lipschitz continuous functions is considered. The variable thickness of the exterior layer of the plate is to be optimized to reach the minimal weight under some constraints for maximal stresses. The cost functionals represent: 1) weight of the three-layered plate, 2) positive distribution (a non-negative Radon measure). The state problem is represented by a variational inequality and the design variables influence both the coefficients and the set of admissible functions. The existence of the optimal thickness is proved and some convergence analysis for an approximate penalized optimal control problem is presented. We prove the existence of a solution to the weight minimization problem or minimization the work of interaction forces on the basis of a general theorem on the control of variational inequalities.

Key words: three-layered plate, thickness, unilateral plate bending, penalty method

1. Introduction

Plates and shells are main elements of many advanced structures. One of the most important characteristics of a construction is its weight, which determines the consumption of a material needed for production of the construction as well as some operating features of the latter.

We shall deal with an optimization problem for the unilateral contact between an elastic three-layered plate and inner obstacle. The model of the three-layered plate ignores shears in the middle layer. We assume that the homogeneous and orthotropic plate occupying the domain $\Omega \times (-[H_0+e], [H_0+e])$ of the space $\mathbb{R}^3$ is loaded by the transversal distributed force $\mathcal{O}(x, y)$ perpendicular to the plane XY. The orientation of the load is positive down along the Z-axis. The plate is supported unilaterally by an inner rigid obstacle (punch). The role of the control variable is played by the thickness of the exterior layer (appearing also in the right-hand side). The inner obstacle and the variable thickness (the exterior layer) imply that the convex set of admissible states depends on the control parameters. The cost functional represents a weight of the three-layered plate. Here, for the weight minimization problem, we introduce constraints, which express bounds for some mean values of the intensity of the stress field. The state problem is modelled by a variational inequality (fourth order elliptic variational inequality), where the control variable influences both the coefficients of the linear monotone operator and the set of admissible state functions. On the basis of the general existence theorem for a class of optimization problems with variational inequalities, we prove the existence of at least one solution to the weight minimization (is treated via a penalty method). We deduce a continuous dependence of the deflection on the control variable (thickness of the plate). Further, introduce the cost functional which represents a positive distribution on Ω (non-negative Radon measure). This measure describes the work of interaction forces between the plate and the obstacle. Next, we define a finite element discretization of the penalized optimal control problem and prove its solvability. Here, any sequence of approximate solutions, with the mesh size decreasing to zero, contains a subsequence, converging to the solution of the penalized control problem. From here, taking into account a sequence of the solutions with the penalization parameter tending to zero, any limit point is proved to coincide with the solution of the original weight minimization problem.

2. Basic relations

A three-layered plate consists of two exterior layers, which are made of a strong material (the so-called carrier layers), and of a comparatively light, non-strong middle layer (the so-called filler). The latter ensures the joint work of the exterior layers. Consider the three-layered plate whose middle layer is of the thickness $H_0(x, y)$ and two exterior layers are of the thickness $e(x, y)$. We suppose that e is much less than $H_0(e \ll H_0(x, y))$ and that the material of the middle layer is much more flexible than the material of the exterior layers. In this case, the shearing stresses perceive mainly the middle layer and the bending stresses perceive mainly the exterior ones.

Suppose also that, in the transversal direction, the elasticity modulus of the material of the middle layer is infinitely large. The material of the middle layer is usually light, so that the mass of the plate is concentrated in the exterior layers. This is why, in solving optimization problems of three-layered plates, the control is usually realised via the function $e(x, y)$ determining the thickness of the carrier layers. In what follows, we assume that the equality: $H_0 + e =$ const determining the parallelism of the midplanes of the carrier layers holds. The Kirchhoff hypotheses are supposed to be fulfilled for the three-layered plate as a whole. Then, the strain components ε_{xx}, ε_{yy}, ε_{xy}) are expressed by the formulas

$$\varepsilon_{xx}(x,y,z) = \frac{\partial \xi(x,y,z)}{\partial x} = -z\frac{\partial^2 v(x,y)}{\partial x^2}$$

$$\varepsilon_{yy}(x,y,z) = \frac{\partial \eta(x,y,z)}{\partial y} = -z\frac{\partial^2 v(x,y)}{\partial y^2}$$

$$\varepsilon_{xy}(x,y,z) = \frac{\partial \xi(x,y,z)}{\partial y} + \frac{\partial \eta(x,y,z)}{\partial x} = -2z\frac{\partial^2 v(x,y)}{\partial x \partial y}$$

where under Kirchhoff hypothesses, the components $\xi(x, y, z)$ and $\eta(x, y, z)$ of the vector of displacements of points of the plate in directions of the X and Y axes have the form

$$\xi(x,y,z) = -z\frac{\partial v(x,y)}{\partial x} \qquad \eta(x,y,z) = -z\frac{\partial v(x,y)}{\partial y}$$

where $v(x, y)$ denotes the displacements of points of the midplane along the Z axis.

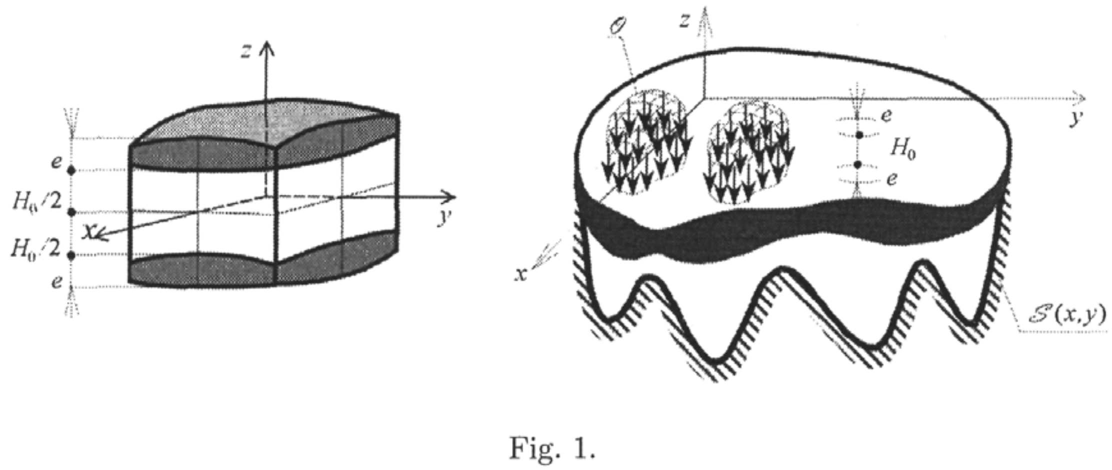

Fig. 1.

For an orthotropic plate, the stress components σ_{xx}, σ_{yy}, σ_{xy} are determined by the relations

$$\begin{aligned}
\sigma_{xx}(x,y,z) &= E_{11}\varepsilon_{xx}(x,y,z) + E_{12}\varepsilon_{yy}(x,y,z) = \\
&= -E_{11}z\frac{\partial^2 v(x,y)}{\partial x^2} - E_{12}z\frac{\partial^2 v(x,y)}{\partial y^2} \\
\sigma_{yy}(x,y,z) &= E_{12}\varepsilon_{xx}(x,y,z) + E_{22}\varepsilon_{yy}(x,y,z) = \\
&= -E_{21}z\frac{\partial^2 v(x,y)}{\partial x^2} - E_{22}z\frac{\partial^2 v(x,y)}{\partial y^2} \\
\sigma_{xy}(x,y,z) &= G\varepsilon_{xy}(x,y,z) = -2Gz\frac{\partial^2 v(x,y)}{\partial x \partial y}
\end{aligned} \tag{2.1}$$

where

$$E_{11} = \frac{E_1}{1-\mu_1\mu_2} \qquad E_{22} = \frac{E_2}{1-\mu_1\mu_2} \qquad E_{12} = E_{21} = \mu_2 E_{11} = \mu_1 E_{22} \tag{2.2}$$

E_1, E_2, G, μ_1, μ_2 being the elasticity characteristics of the material.

Suppose that

$$\begin{aligned}
&E_1,\ E_2,\ G \text{ are positive numbers} \\
&\mu_1 \text{ and } \mu_2 \text{ are constants},\quad 0 \leqslant \mu_i < 1\,,\quad i = 1,2 \\
&H_0 \geqslant \text{const}_A > 0 \\
&H_0 + e = \text{const}_B
\end{aligned} \tag{2.3}$$

where const_A and const_B are positive numbers.

For the bending moments and torque, we have in view of (2.1) the following relations

$$\begin{aligned}
M_{xx}(x,y) &= \int\limits_{-(H_0/2+e)}^{H_0/2+e} z\sigma_{xx}(x,y,z)\,dz \approx -\frac{1}{2}(H_0+e)e\sigma_{xx}\Big(x,y,-\frac{1}{2}(H_0+e)\Big) + \\
&+\frac{1}{2}(H_0+e)e\sigma_{xx}\Big(x,y,\frac{1}{2}(H_0+e)\Big) = D_{11}(e)\frac{\partial^2 v(x,y)}{\partial x^2} + D_{12}(e)\frac{\partial^2 v(x,y)}{\partial y^2}
\end{aligned} \tag{2.4}$$

where

$$D_{11}(e) = \frac{E_{11}(H_0+e)^2 e}{2} \qquad D_{12}(e) = \frac{E_{12}(H_0+e)^2 e)}{2}$$

and E_{11}, E_{12} are the elasticity characteristics of the exterior layers for which (2.2) holds true.

Similarly, we have

$$M_{yy}(x,y) = \int_{-(H_0/2+e)}^{(H_0/2)+e} z\sigma_{yy}(x,y,z)\,dz \approx D_{21}(e)\frac{\partial^2 v(x,y)}{\partial x^2} + D_{22}(e)\frac{\partial^2 v(x,y)}{\partial y^2}$$

$$M_{xy}(x,y) = \int_{-(H_0/2+e)}^{(H_0/2)+e} z\sigma_{xy}(x,y,z)\,dz \approx D_{33}(e)\frac{\partial^2 v(x,y)}{\partial x \partial y} \tag{2.5}$$

Here one has

$$D_{21}(e) = \frac{E_{21}(H_0+e)^2 e}{2} = D_{12}(e) \qquad D_{22}(e) = \frac{E_{22}(H_0+e)^2 e}{2}$$

$$D_{33}(e) = G(H_0+e)^2 e \tag{2.6}$$

and E_{21}, E_{22}, G are the elasticity characteristics of the exterior layers.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$ and let $\mathcal{S}(x,y)$ be a smooth function in $\overline{\Omega}$ (closure of Ω). $C^m(\overline{\Omega})$ is a space of m-times continuously differentiable real-valued functions for which all the derivatives up to order m are continuous in $\overline{\Omega}$, $\operatorname{supp}(v)$ is closure of the set $\{[x,y] \in \Omega : \ v(x,y) \neq\}$, $C_0^m(\Omega) = \{v \in C^m(\overline{\Omega}),\ v$ has a compact support in $\Omega\}$, $\mathcal{D}(\Omega) = C_0^m(\Omega)$, $v|_{\partial\Omega}$ is trace of v on $\partial\Omega$. By the symbol $C^{0,1}(\overline{\Omega})$, we denote the set of all functions u satisfying the Lipschitz conditions in $\overline{\Omega}$ ($|u(x) - u(y)| \leqslant M|x-y|$). $V^*(\Omega)$ is the dual of $V(\Omega)$ and $\mathcal{D}^*(\Omega)$ is the space of generalized functions (or distributions). We denote the standard Sobolev function spaces by $H^k(\Omega) \equiv W_2^k(\Omega)$, $k = 1, 2$. In the following $L_2(\Omega)$ and $L_\infty(\Omega)$ denote the space of Lebesgue-square integrable functions on Ω and the space of essentially bounded functions on Ω with the standard norms $\|\cdot\|_{L_2(\Omega)}$ and $\|\cdot\|_{L_\infty(\Omega)}$, respectively. The inner product in $L_2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle_{L_2(\Omega)}$, $\langle\cdot,\cdot\rangle_{\mathcal{D}(\Omega)}$ denotes the duality product between $\mathcal{D}(\Omega)$ and $\mathcal{D}^*(\Omega)$. The function $\langle\cdot,\cdot\rangle_{V(\Omega)}$ defined on $V^*(\Omega) \times V(\Omega)$, is called the scalar product of $V^*(\Omega)$ and $V(\Omega)$.

The following energy space is considered

$$V(\Omega) = \left\{ v \in H^2(\Omega) : v = 0, \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \text{ in the sense of traces} \right\} = H_0^2(\Omega)$$

(Obviously, for the clamped plate homogeneous kinematic conditions are prescribed) or

$$V(\Omega) = \Big\{ v \in H^2(\Omega) : \; v = 0, \; \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega_{disp} \text{ in the sense of traces} \Big\}$$

$$V(\Omega) \subset H^2(\Omega)$$

where $\partial\Omega_{disp} = \partial\Omega_{\text{Displacement}}$.

Here we have $\partial\Omega = \partial\Omega_{disp} \cup \partial\Omega_{cont}$ (where $\partial\Omega_{cont} = \partial\Omega_{\text{Contact}}$), meas $\partial\Omega_{disp} > 0$, meas $\partial\Omega_{cont} > 0$. The plate at the part $\partial\Omega_{cont}$ is unilaterally supported. The transversal displacements (deflections) v belong to the energy space $V(\Omega)$.

Let us recall some connections between continuous functionals and the Radon measures. We denote by $C_{comp}(\Omega) = C_{\text{Comppact}}(\Omega)$ the space of all continuous functions with compact support in Ω. A sequence $\{\theta_n\}_{n\in N}$, $\theta_n \in C_{comp}(\Omega)$ converges to $\theta \in C_{comp}(\Omega)$, if the supports of the functions θ_n belong to a compact subset of Ω and $\{\theta_n\}_{n\in N}$ converges to θ uniformly on Ω. Due to the representation theorem every continuous linear functional $\mathcal{V}$ over $C_{comp}(\Omega)$ can be represented by the integral

$$\langle \mathcal{V}, \theta \rangle_{C_{comp}(\Omega)} = \int_\Omega \theta \, d\mu \qquad \forall \theta \in C_{comp}(\Omega) \tag{2.7}$$

where μ belongs to the set $\mathcal{M}(\Omega)$ (the set of all measures defined on Ω). A linear continuous functional $\mathcal{V}$ on $C_{comp}(\Omega)$ is said to be positive, if $\mathcal{V} \geqslant 0$ for all $\theta \in C_{comp}(\Omega)$, $\theta(x, y) \geqslant 0$. Positive functionals on $C_{comp}(\Omega)$ possess an important property: the linear and positive functional $\mathcal{V}$ on $C_{comp}(\Omega)$ is continuous and can be represented in form (2.7) with a non-negative measure μ.

Taking into consideration relations (2.1) to (2.6), we get the following expression for the strain energy of the orthotropic plate as a functional of $v(x, y)$, say

$$\begin{aligned} \mathcal{E}(e, v) = \frac{1}{2} \int_\Omega \Big[& D_{11}(e) \Big(\frac{\partial^2 v}{\partial x^2} \Big)^2 + 2D_{12}(e) \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \\ & + D_{22}(e) \Big(\frac{\partial^2 v}{\partial y^2} \Big)^2 + 2D_{33}(e) \Big(\frac{\partial^2 v}{\partial x \partial y} \Big)^2 \Big] \, d\Omega \end{aligned} \tag{2.8}$$

The convex function $\mathcal{E}(e, v)$ is weakly or Gâteaux differentiable on $V(\Omega)$, the corresponding element from $\mathcal{D}^*(\Omega)$ will be denoted by $\operatorname{grad}_v\mathcal{E}(e,v)$

$$\langle \operatorname{grad}_v\mathcal{E}(e,v),\widehat{o}\rangle_{\mathcal{D}(\Omega)} = \lim_{\lambda\to 0}\frac{\mathcal{E}(e,v+\lambda\widehat{o})-\mathcal{E}(e,v)}{2} =$$

$$= \int_\Omega \Big[\frac{\partial^2}{\partial x^2}\Big(D_{11}(e)\frac{\partial^2 v}{\partial x^2}+D_{12}(e)\frac{\partial^2 v}{\partial y^2}\Big)+ \tag{2.9}$$

$$+\frac{\partial^2}{\partial y^2}\Big(D_{22}(e)\frac{\partial^2 v}{\partial y^2}+D_{12}(e)\frac{\partial^2 v}{\partial x^2}\Big)+2\frac{\partial^2}{\partial x\partial y}\Big(D_{33}(e)\frac{\partial^2 v}{\partial x\partial y}\Big)\Big]\widehat{o}\,d\Omega$$

for any $\widehat{o}\in\mathcal{D}(\Omega)$.

Hence, we introduce the potential operator:

$$\mathcal{A}(e)(=\operatorname{grad}_v\mathcal{E}(e,v)):\ V(\Omega)\to\mathcal{D}^*(\Omega)$$

such that

$$\mathcal{A}(e)v = \frac{\partial^2}{\partial x^2}\Big(D_{11}(e)\frac{\partial^2 v}{\partial x^2}+D_{12}(e)\frac{\partial^2 v}{\partial y^2}\Big)+ \tag{2.10}$$

$$+\frac{\partial^2}{\partial y^2}\Big(D_{22}(e)\frac{\partial^2 v}{\partial y^2}+D_{12}(e)\frac{\partial^2 v}{\partial x^2}\Big)+2\frac{\partial^2}{\partial x\partial y}\Big(D_{33}(e)\frac{\partial^2 v}{\partial x\partial y}\Big)$$

The thickness e will be sought in the following set of admissible functions

$$U_{ad}(\Omega) := \Big\{e\in C^{(0),1}(\overline{\Omega}):\ e_{\min}\leqslant e(x,y)\leqslant e_{\max},\ \Big|\frac{\partial e}{\partial x}\Big|\leqslant C_1,\ \Big|\frac{\partial e}{\partial y}\Big|\leqslant C_2$$

$$\Big(\text{or }\ \Big|\frac{\partial e}{\partial \xi}\Big|\leqslant C_{1^*},\ \Big|\frac{\partial e}{\partial \eta}\Big|\leqslant C_{2^*}\Big)\Big\}$$

where $C^{(0),1}(\overline{\Omega})$ denotes the set of Lipschitz functions $e_{\min}$, $e_{\max}$ and C_1, C_2, C_{1^*}, C_{2^*} are given positive parameters, ξ, η some skew coordinates. Due to Arzela theorem (Haslinger and Neittaanmäki, 1988), $U_{ad}(\Omega)$ is a compact subset of $U(\Omega)(=C(\overline{\Omega}))$.

The loading of the homogeneous, orthotropic plate (e.g. a concrete plate reinforced by welded ribs) is given by:

1° The surface forces $\mathcal{O}(x,y)$

2° The body forces (within the plate) $\omega_1 H_0+2\omega_2 e$, where ω_i $(i=1,2)$ is the specific weight of the material. Here, ω_i are positive constants.

Further, we introduce the functional space: $\mathcal{H}(\Omega)=\overset{\circ}{W}{}^1_p(\Omega)$ with some $p\in(2,\infty)$. Here, the loading functional $(\mathcal{O}+\omega_1 H_0+2\omega_e)\in\mathcal{H}^*(\Omega)$ and the function $\mathcal{S}\in C(\overline{\Omega})$ be given describing a lower unilateral obstacle. Next the obstacle function $\mathcal{S}:\ \overline{\Omega}\to\mathbb{R}$ fulfils the condition:

(H0) $$\max_{x,y\in\partial\Omega} \mathcal{S}(x,y) + e_{\max} + \frac{1}{2}H_0 < 0$$

Let us use the virtual displacement principle to establish a variational formulation of the problem. To this end we introduce the set

$$\mathcal{K}(e,\Omega) = \Big\{v \in V(\Omega) : \ \ \mathcal{F}v \geqslant \mathcal{S} + e + \frac{1}{2}H_0\Big\} \tag{2.11}$$

or

$$\mathcal{K}(\Omega) = \{v \in V(\Omega) : \mathcal{F}v \geqslant 0 \text{ on } \Omega_* \text{ and } v \geqslant 0 \text{ on } \partial\Omega_{cont} \text{ in the sense of traces}\} \tag{2.12}$$

where $\mathcal{F}$ is the embedding of $V(\Omega)$ into $L_\infty(\Omega)$, $\overline{\Omega}_* \subset \Omega$. It is important to note that $\mathcal{K}(\Omega)$ is a cone with the vertex at zero.

According to (2.8) and (2.9), the following bilinear form on $V(\Omega) \times V(\Omega)$ corresponds to the strain energy of the orthotropic plate

$$a(e,v,z) = \int_\Omega \Big[D_{11}(e)\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 z}{\partial x^2} + D_{22}(e)\frac{\partial^2 v}{\partial y^2}\frac{\partial^2 z}{\partial y^2} +$$

$$+D_{12}(e)\Big(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\frac{\partial^2 z}{\partial x^2}\Big) + 2D_{33}(e)\frac{\partial^2 v}{\partial x \partial y}\frac{\partial^2 z}{\partial x \partial y}\Big]\, d\Omega$$

where $D_{11}(e)$, $D_{22}(e)$, $D_{12}(e)$ and $D_{33}(e)$ are defined by (2.6).

Now, define the mapping $\mathcal{A}(e) : V(\Omega) \to V^*(\Omega)$ by the formula

$$\langle \mathcal{A}(e)v, z\rangle_{V(\Omega)} = a(e,v,z) \qquad \forall v, z \in V(\Omega) \tag{2.13}$$

On the basis of the virtual displacement principle, we introduce the following **state problem**:

Given any $e \in U_{ad}(\Omega)$, find $u(e) \in \mathcal{K}(e,\Omega)$ such that

$$\langle \mathcal{A}(e)u(e), v - u(e)\rangle_{V(\Omega)} \geqslant \langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e, \widetilde{O}v - \widetilde{O}u(e)\rangle_{\mathcal{H}(\Omega)} \tag{2.14}$$

holds for all $v \in \mathcal{K}(e,\Omega)$.

Here, $\widetilde{O}$ is the embedding of $V(\Omega)$ into $\mathcal{H}(\Omega)$. Later on, we shall prove that the variational inequality has a unique solution for any $e \in U_{ad}(\Omega)$.

This is a mathematical model of an elastic orthotropic plate in the state of a static equilibrium, interacting with the obstacle $\mathcal{S}(x,y)$.

On the other hand the linear continuous operator $\widetilde{\mathcal{A}}(e) : V(\Omega) \to \mathcal{D}^*(\Omega)$ is determined by the formula

$$\langle \widetilde{\mathcal{A}}(e)v, \theta\rangle_{\mathcal{D}(\Omega)} = a(e,v,\theta) \qquad \begin{matrix} v \in V(\Omega) \\ \theta \in \mathcal{D}(\Omega)\end{matrix} \tag{2.15}$$

Then, in view of (2.15) from (2.14) (inserting $v = \theta$) it follows that

$$\mu([e,v],\Omega) = \widetilde{\mathcal{A}}(e)v - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e) \tag{2.16}$$

is a positive distribution on Ω, and consequently, a non-negative Radon measure in Ω. This measure describes the work of the interaction forces between the plate and the obstacle.

The weight of the homogeneous, orthotropic (three-layered) plate is determined by

$$\mathcal{L}_{wg}(e) = \int_{\Omega} (\omega_1 H_0 + 2\omega_2 e)\, d\Omega$$

where $\mathcal{L}_{wg}(e) = \mathcal{L}_{\text{Weight}}(e)$.

Moreover, the following constraints will be considered (the Norris strength criterion)

(A1) $\quad \mathcal{S}_i(e, \mathbf{M}(e)) \leqslant 0 \qquad i = 1, 2, ..., N_i \qquad N_i < +\infty$ where

$$\mathcal{S}_i(e,\mathbf{M}(e)) = \frac{9}{4\,\mathrm{meas}\Omega_i^*} \int_{\Omega_i^*} \frac{1}{(H_0/2+e)^4} \cdot$$

$$\cdot \Big[M_{xx}^2(e) + M_{yy}^2(e) + \Big(\frac{\sigma_R}{\tau_R}\Big)^2 M_{xy}^2(e)\Big]\, d\Omega - \sigma_R^2$$

$\Omega_i^* \subset \overline{\Omega}$ are given subdomains, σ_R, τ_R are given positive constants and $\mathbf{M}(e)$ is the vector of the bending moment and torque, derived by relations (2.4) and (2.5) from solution $u(e)$ of (2.14).

Let us introduce the set of statically admissible control variables

$$\mathcal{G}_{ad}(\Omega) = \Big\{e \in U_{ad}(\Omega) : \sum_{i=1}^{N_i} [\mathcal{S}_i(e,\mathbf{M}(e))]^+ = 0\Big\}$$

where $a^+ = \max\{0, a\}$ denotes the positive part of a.

Here, we assume

$$\mathcal{G}_{ad}(\Omega) \neq \emptyset \tag{2.17}$$

Now, our main task is to solve the Optimal Control Problem $(\mathcal{P})$:

$(\mathcal{P})$ $\qquad e_* = \underset{e \in \mathcal{G}_{ad}(\Omega)}{\mathrm{ArgMin}}\, \mathcal{L}_{wg}(e)$

In the following, we remove constraints (A1) by means of a penalty method. To this end we introduce a penalized cost functional

$$\mathcal{L}_{\varepsilon,wg}(e,\mathbf{M}(e)) = \mathcal{L}_{wg}(e) + \frac{1}{\varepsilon}\sum_{i=1}^{N_i}[\mathcal{S}_i(e,\mathbf{M}(e))]^+ \qquad \varepsilon > 0$$

and a penalized optimal control problem

$$(\mathcal{P}_\varepsilon) \qquad e_\varepsilon = \operatorname*{ArgMin}_{e\in U_{ad}(\Omega)} \mathcal{L}_{\varepsilon,wg}(e,\mathbf{M}(e))$$

3. Existence of a solution to the optimal control problem

We shall consider a class of abstract optimal control problems and prove their solvability. Then, we shall apply the general result to our optimal control problem $(\mathcal{P})$.

Let $U(\Omega)$ be a Banach space of controls, $U_{ad}(\Omega)$ is a subset of admissible controls. We assume that $U_{ad}(\Omega)$ is compact in $U(\Omega)$. Let the reflexive Banach space $V(\Omega)$ be endowed with a norm $\|\cdot\|_{V(\Omega)}$ and let $V^*(\Omega)$ be its dual with a norm $\|\cdot\|_{V^*(\Omega)}$, the duality pairing between $V(\Omega)$ and $V^*(\Omega)$ being denoted by $\langle\cdot,\cdot\rangle_{V(\Omega)}$.

Definition 1. We say that a sequence $\{K_n(\Omega)\}_{n\in N}$ of convex subsets of $V(\Omega)$ converges to a set $K(\Omega)$, i.e. $K(\Omega) = \lim_{n\to\infty} K_n(\Omega)$ (convergence in the sense of Mosco) if the following two conditions are satisfied:

1° For any $v \in K(\Omega)$ a sequence $\{v_n\}_{n\in N}$ exists, such that $v_n \in K_n(\Omega)$ and $\lim_{n\to\infty} v_n = v \in V(\Omega)$

2° If $v_n \in K_n(\Omega)$ and $v_n \to v$ weakly in $V(\Omega)$, then $v \in K(\Omega)$.

Let us consider a system $\{\mathcal{K}(e_n,\Omega)\}_{n\in N}$, $e_n \in U_{ad}(\Omega)$, of closed convex subsets, $\mathcal{K}(e_n,\Omega) \subset V(\Omega)$ and a family $\{A(e_n)\}_{n\in N}$ of operators $A(e_n) : V(\Omega) \to V^*(\Omega)$, satisfying the following assumptions:

$$
\text{(H1)}\begin{cases}
1^\circ \quad \bigcap_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega)\neq\emptyset\\
2^\circ \quad n \to e \text{ strongly in } U(\Omega),\ e_n \in U_{ad}(\Omega) \Rightarrow \mathcal{K}(e,\Omega) = \lim_{n\to\infty}\mathcal{K}(e_n,\Omega)\\
3^\circ \quad \text{There exist constants: } 0<\alpha_A<M_A \text{ independent of } e\in U_{ad}(\Omega) \\
\qquad \text{and such that } \alpha_A\|v-z\|^2_{V(\Omega)} \leqslant \langle A(e)v - A(e)z, v-z\rangle_{V(\Omega)},\\
\qquad \|A(e)v-A(e)z\|_{V^*(\Omega)} \leqslant M_A\|v-z\|_{V(\Omega)}\\
4^\circ \quad e_n \to e \text{ strongly in } U(\Omega),\ e_n\in U_{ad}(\Omega) \Rightarrow A(e_n)v \to A(e)v\\
\qquad \text{strongly in } V^*(\Omega) \text{ holds for all } v\in V(\Omega)
\end{cases}
$$

Finally, let a functional $f \in V^*(\Omega)$ and a continuous mapping $B:\ U(\Omega) \in V^*(\Omega)$ be given.

For any $e \in U_{ad}(\Omega)$ let us consider the following variational inequality:

Find $u(e) \in \mathcal{K}(e,\Omega)$ such that

$$\langle A(e)u(e), v-u(e)\rangle_{V(\Omega)} \geqslant \langle f+Be, v-u(e)\rangle_{V(\Omega)} \qquad \forall v\in\mathcal{K}(e,\Omega) \tag{3.1}$$

Here, we note that there exists a unique solution $u(e) \in \mathcal{K}(e,\Omega)$ for any $e \in U_{ad}(\Omega)$. In fact, we may employ the general theory of variational inequalities (see Barbu, 1987; Khludnev and Sokolowski, 1997; Lions, 1969; Panagiotopoulos, 1985).

Next, let a functional $\mathcal{L}:\ U(\Omega)\times V(\Omega)\to\mathbb{R}$ be given such that

$$\left.\begin{array}{l} e_n \to e \text{ strongly in } U(\Omega)\\ e_n\in U_{ad}(\Omega)\\ v_n\to v \text{ weakly in } V(\Omega)\end{array}\right\} \Rightarrow \liminf_{n\to\infty}\mathcal{L}(e_n,v_n)\geqslant\mathcal{L}(e,v) \tag{3.2}$$

Let us introduce a functional $J:\ U_{ad}(\Omega)\to\mathbb{R}$ by the formula $J(e) = \mathcal{L}(e,u(e))$ is the solution to the state problem (3.1). Here we shall solve the optimization problem $(\mathcal{B})$

$$(\mathcal{B}) \qquad e_* = \operatorname*{ArgMin}_{e\in U_{ad}(\Omega)} J(e).$$

Theorem 1. Let the data of state problem (2.1) satisfy assumptions (H1). Let $e_n\in U_{ad}(\Omega)$, $e_n\to e_*$ strongly in $U(\Omega)$. Then, one has: $u(e_n)\to u(e_*)$ strongly in $V(\Omega)$.

Proof. Let us consider inequality (3.1) for any e_n, $n = 1, 2, \ldots$. We take an arbitrary $v_o \in \mathcal{K}(e_o, \Omega)$ and by $(\mathrm{H1})_{2^\circ}$ there exists a sequence $\{a_n\}_{n\in N} \in \prod_{n\in N} \mathcal{K}(e_n, \Omega)$ such that $a_n \to v_o$ strongly in $V(\Omega)$. Further, we set $v = a_n$ in (3.1), adding the term $\langle A(e_n)a_n, u(e_n) - a_n\rangle_{V(\Omega)}$ to both sides, we derive the inequality

$$\begin{aligned} &\langle A(e_n)u(e_n) - A(e_n)a_n, u(e_n) - a_n\rangle_{V(\Omega)} \leqslant \\ &\leqslant \langle f + Be_n, u(e_n) - a_n\rangle_{V(\Omega)} + \langle A(e_n)a_n, a_n - u(e_n)\rangle_{V(\Omega)} \end{aligned} \tag{3.3}$$

Hence in view of $(\mathrm{H1})_{3^\circ,4^\circ}$ and the continuity of B, we deduce: $\|u(e_n)\|_{V(\Omega)} \leqslant$ const for all n.

Thus, there exists a subsequence $\{u(e_{n_k})\}_{k\in N} \subset \{u(e_n)\}_{n\in N}$ and element $u_* \in V(\Omega)$, such that

$$u(e_{n_k}) \to u_* \qquad \text{weakly in } V(\Omega) \tag{3.4}$$

Assumption $(\mathrm{H1})_{2^\circ}$ implies that: $u_* \in \mathcal{K}(e_*, \Omega)$ and we can find a sequence $\{\theta_k\}_{k\in N}$, such that $\theta_k \in \mathcal{K}(e_k, \Omega)$

$$\theta_k \to u_* \qquad \text{strongly in } V(\Omega) \tag{3.5}$$

Here, we consider again inequality (3.1) for $e = e_{n_k}$ insert $v := \theta_k$, and add the term

$$\langle A(e_{n_k})\theta_k, u(e_{n_k}) - \theta_k\rangle_{V(\Omega)}$$

to both sides. We obtain

$$\begin{aligned} &\limsup_{k\to\infty} \langle A(e_{n_k})u(e_{n_k}) - A(e_{n_k})\theta_k, u(e_{n_k}) - \theta_k\rangle_{V(\Omega)} \leqslant \\ &\leqslant \limsup_{k\to\infty} \langle A(e_{n_k})\theta_k, \theta_k - u(e_{n_k})\rangle_{V(\Omega)} + \\ &+ \limsup_{k\to\infty} \langle f + Be_{n_k}, u(e_{n_k}) - \theta_k\rangle_{V(\Omega)} = 0 \end{aligned} \tag{3.6}$$

The last inequality follows from weak convergence (3.4) of $\{u(e_{n_k})\}_{k\in N}$, (3.5), the continuity of B and the following assertion

$$\left.\begin{aligned} &e_k \to e \text{ strongly in } U(\Omega) \\ &e_k \in U_{ad}(\Omega) \\ &v_k \to v \text{ strongly in } V(\Omega) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\|A(e_k)v_k - A(e)v\|_{V^*(\Omega)} \leqslant \\ &\leqslant M_A\|v_k - v\|_{V(\Omega)} + \\ &+ \|A(e_k)v - A(e)v\|_{V^*(\Omega)} \to 0 \end{aligned}\right. \tag{3.7}$$

which is a consequence of $(\mathrm{H1})_{3^\circ,4^\circ}$.

Moreover, due to the uniform monotonicity of $A(e_{n_k})$ $(\mathrm{H1})_{3^\circ}$ and due to (3.6), we can write

$$\lim_{k\to\infty} \|u(e_{n_k}) - \theta_k\|_{V(\Omega)} = 0 \tag{3.8}$$

Thus, by virtue of (3.8) and (3.5), we arrive at

$$u(e_{n_k}) \to u_* \qquad \text{strongly in } V(\Omega) \tag{3.9}$$

Then, relations (3.7) and (3.9) give

$$A(e_{n_k})u(e_{n_k}) \to A(e_*)u_* \qquad \text{strongly in } V^*(\Omega) \tag{3.10}$$

Hence, passing to the $\limsup_{k\to\infty}$ on both sides the inequality

$$\langle A(e_{n_k})u(e_{n_k}), u(e_{n_k}) - \theta_k\rangle_{V(\Omega)} \leqslant \langle f + Be_{n_k}, u(e_{n_k}) - \theta_k\rangle_{V(\Omega)}$$

we arrive at (by virtue of (3.9), (3.10))

$$\langle A(e_*)u_*, u_* - v\rangle_{V(\Omega)} \leqslant \langle f + Be_*, u_* - v\rangle_{V(\Omega)}$$

Then, from the uniqueness of $u(e_*)$, we deduce $u_* = u(e_*)$. Hence, the whole sequence $\{u(e_n)\}_{n\in N}$ converges to $u(e_*)$ in $V(\Omega)$.

Theorem 2. Let the data of state problem (2.1) satisfy assumptions (H1). Let the functional $\mathcal{L}$ satisfy condition (3.2). Then there exists at least one solution to the **optimal control problem** $(\mathcal{B})$.

Proof. Since the set $U_{ad}(\Omega)$ is compact in $U(\Omega)$, there exists a sequence $\{e_n\}_{n\in N}$, such that $e_n \in U_{ad}(\Omega)$, $e_n \to e_*$ strongly in $U(\Omega)$, $e_* \in U_{ad}(\Omega)$, $J(e_n) \to \inf_{e\in U_{ad}(\Omega)} J(e)$.

Then, (3.2) and Theorem 1 imply that

$$\mathcal{L}(e_*, u(e_*)) \leqslant \liminf_{n\to\infty} \mathcal{L}(e_n, u(e_n)) = \inf_{e\in U_{ad}(\Omega)} \mathcal{L}(e, u(e))$$

As a consequence, e_* is a solution to the problem $(\mathcal{B})$.

We now consider a family of the optimization problems $(\mathcal{P}_{\varepsilon_n})$, which depend on $\varepsilon_n > 0$. Here we apply a penalty method for the existence of the optimal solution $(\mathcal{P})$.

Lemma 1. For any $e \in U_{ad}(\Omega)$ the set $\mathcal{K}(e,\Omega)$ defined in (2.11), is a non-empty closed and convex subset of $V(\Omega)$ and $e_n \in U_{ad}(\Omega)$, $e_n \to e$ strongly in $U(\Omega) \Rightarrow \mathcal{K}(e,\Omega) = \lim_{n\to\infty} \mathcal{K}(e_n,\Omega)$.

Proof. For any $v \in \mathcal{K}(e,\Omega)$ there exists a sequence

$$\begin{gathered}\{v_n\}_{n\in N},\ \text{such that: } v_n \in V(\Omega),\ v_n \in K(e_n,\Omega) \text{ for } n \\ \text{sufficiently great and } v_n \to v \text{ strongly in } V(\Omega),\ \text{as } n\to\infty\end{gathered} \tag{3.11}$$

Indeed, let us define: $\widehat{o} = v - (\mathcal{S} + e + H_0/2)$ so that $\widehat{o} \in C(\overline{\Omega})$, $\widehat{o} \geqslant 0$ in $\overline{\Omega}$ and

$$\vartheta_n = (e_n - e) - \widehat{o} = e_n - v + \mathcal{S} + \frac{1}{2}H_0 \in C(\overline{\Omega})$$

$$\widetilde{O}_n = \Big\{[x,y] \in \Omega:\ \vartheta_n(x,y) \geqslant \frac{1}{2}\mathcal{C}\Big\}$$

where

$$\mathcal{C} = \max_{[x,y]\in\partial\Omega} \mathcal{S}(x,y) + e_{\max} + \frac{1}{2}H_0 < 0$$

due to assumption (H0).

Next, there exists an open set $\widetilde{O} \subset \overline{\widetilde{O}} \subset \Omega$ such that

$$\widetilde{O}_n \subset \widetilde{O} \qquad \forall n \tag{3.12}$$

To see this, we realise that: $\vartheta_n = \mathcal{S} + e_n + H_0/2 \leqslant \mathcal{C}$ on the boundary $\partial\Omega$. Hence, the continuity of $\vartheta_n(x,y)$ and the constraints $|\partial e_n/\partial x| \leqslant \text{const}_1$ and $|\partial e_n/\partial y| \leqslant \text{const}_2$ imply that $\bigcup_{n=1}^{\infty} \widetilde{O}_n \Subset \Omega$ and (3.12) follows. Obviously, there exists a function $\mathcal{N} \in C^\infty(\overline{\Omega})$ such that $\mathcal{N}(x,y) = 1$ for $[x,y] \in \widetilde{O}$ and $\mathcal{N}(x,y) = 0$, $\partial\mathcal{N}/\partial n = 0$ for $[x,y] \in \partial\Omega$, $0 < \mathcal{N}(x,y) \leqslant 1$ for $[x,y] \in \Omega$. Let us set: $v_n = v + \|e_n - e\|_{L_\infty(\Omega)}\mathcal{N}$. Then $v_n \in V(\Omega)$ and

$$\|v - v_n\|_{V(\Omega)} = \|e - e_n\|_{L_\infty(\Omega)}\|\mathcal{N}\|_{V(\Omega)} \to 0 \qquad \text{as } n\to\infty$$

On the other hand, we can show that there exists $n_* > 0$ such that

$$n > n_* \Rightarrow v_n \geqslant \mathcal{S} + e_n + \frac{1}{2}H_0 \quad \text{ in } \ \overline{\Omega} \Rightarrow v_n \in \mathcal{K}(e_n,\Omega) \tag{3.13}$$

Indeed, let

1° $[x,y] \in \widetilde{O}$. Then, one has

$$v_n = v + \|e_n - e\|_{L_\infty(\Omega)} \geqslant v + e_n - e \geqslant \mathcal{S} + e_n + \frac{1}{2}H_0 \tag{3.14}$$

2° Let $[x,y] \in \overline{\Omega}\backslash\widetilde{O}$. Then, we have

$$v_n \geqslant \mathcal{S} + e + \frac{1}{2}H_0 + \widehat{o} + |e_n - e|\mathcal{N} \tag{3.15}$$

Taking into account that $[x,y] \notin \widetilde{O}$, $[x,y] \notin \widetilde{O}_n$ for any n and $\vartheta_n \leqslant \mathcal{C}$ one has $e_n - e - \widehat{o} \leqslant \mathcal{C}$, $-\mathcal{C}\mathcal{N} + (1-\mathcal{N})\widehat{o} \leqslant \widehat{o} + |e_n - e|\mathcal{N}$.

Hence (inserting into (3.15)), we obtain

$$v_n \geqslant \mathcal{S} + e + \frac{1}{2}H_0 + \widehat{\mathcal{O}}$$

where $\widehat{\mathcal{O}} = -\mathcal{N}\mathcal{C} + (1-\mathcal{N})\widehat{o}$.

The function $\widehat{\mathcal{O}}$ is continuous and attains a positive minimum in the compact set $\overline{\Omega}\backslash\widetilde{O}$: $\mathcal{M} = \widehat{\mathcal{O}}([x_*,y_*]) = \min_{\overline{\Omega}\backslash\widetilde{O}} \widehat{\mathcal{O}} > 0$. Notice that if $\mathcal{N}(x_*,y_*) = 0$, then $[x_*,y_*] \in \partial\Omega$ and we have

$$\widehat{\mathcal{O}}([x_*,y_*]) = \widehat{o}(x_*,y_*) = -\Big[\mathcal{S}(x_*,y_*) + e(x_*,y_*) + \frac{1}{2}H_0\Big] \geqslant -\mathcal{C} > 0$$

Next, taking into account: $\mathcal{N}(x_*,y_*) > 0$ then one has $\widehat{\mathcal{O}}([x_*,y_*]) \geqslant -\mathcal{C}\mathcal{N}(x_*,y_*) > 0$. On the other hand there exists $n_*(\mathcal{M})$ such that: $n \geqslant n(\mathcal{M}) \Rightarrow \|e_n - e\|_{L_\infty(\Omega)} \leqslant \mathcal{M}$. Hence, then we have: $\widehat{\mathcal{O}}([x,y]) \geqslant \widehat{\mathcal{O}}([x_*,y_*]) \geqslant \|e_n - e\|_{L_\infty(\Omega)} \geqslant e_n(x,y) - e(x,y)$, so that: $v_n(x,y) \geqslant \mathcal{S}(x,y) + e_n(x,y) + H_0/2$ for $n > n_*(\mathcal{M}) \Rightarrow v_n \in \mathcal{K}(e_n,\Omega)$. This means that condition 1° in Definition 1 is verified. Next, we verify condition 2°. As $v_n \in \mathcal{K}(e_n,\Omega)$, $e_n \to e$ strongly in $U(\Omega)$ and $v_n \to v$ weakly in $V(\Omega)$, then $v_n \to v$ and $e_n \to e$ strongly in $C(\overline{\Omega})$ and the inequality for the limit remains valid.

The form of $\mathcal{K}(e,\Omega)$ follows directly from its definition. Since: $\mathcal{S} + e_{\max} + H/2 \leqslant 0$ on Ω and due to assumption (H0), the zero function belongs to $\mathcal{K}(e,\Omega)$ for any $e \in U_{ad}(\Omega)$. As a consequence, $(\mathrm{H1})_{1^\circ,2^\circ}$ are satisfied.

The subspace $\mathcal{R}(\Omega) := \{v \in V(\Omega) : \langle \mathcal{A}(e)v, v\rangle_{V(\Omega)} = 0\}$ is the set of rigid body motion of the plate.

Let $P_V(\Omega)$ be the subspace of all possible (virtual) rigid body displacements of the middle plane, i.e.

$$P_V(\Omega) := \Big\{v \in V(\Omega) : \ \Big(\frac{\partial^2 v}{\partial x^2}\Big)^2 = 0, \ \ \Big(\frac{\partial^2 v}{\partial y^2}\Big)^2 = 0, \ \ \Big(\frac{\partial^2 v}{\partial x \partial y}\Big)^2 = 0\Big\}$$

Lemma 2. Let $v \in H^2(\Omega)$ and $(\partial^2 v/\partial x^2)^2 = 0$, $(\partial^2 v/\partial y^2)^2 = 0$, $(\partial^2 v/\partial x \partial y)^2 = 0$. Then $P_V(\Omega) = \{0\}$, i.e. $P_V(\Omega)$ reduces to the zero element.

Proof. The regularization of the displacement v gives an element for which

$$\frac{\partial^2 v^h}{\partial x^2} = \Big[\frac{\partial^2 v}{\partial x^2}\Big]^h = 0 \qquad \frac{\partial^2 v^h}{\partial y^2} = \Big[\frac{\partial^2 v}{\partial y^2}\Big]^h = 0 \qquad \frac{\partial^2 v^h}{\partial x \partial y} = \Big[\frac{\partial^2 v}{\partial x \partial y}\Big]^h = 0 \tag{3.16}$$

holds for every domain Ω_* such that $\overline{\Omega}_* \subset \Omega$, provided h is sufficiently small $h < \mathrm{dis}(\overline{\Omega}_*, \partial\Omega)$. Then, from conditions (3.10) we conclude that v^h is a linear polynomial. Since v^{h_n} converges to v in $L_2(\Omega)$ as $h_n \to 0$ and finite-dimensional subspaces are closed in $L_2(\Omega)$, conclude that v^{h_n} is a linear polynomial in every interior subdomain Ω, $\overline{\Omega}_* \subset \Omega$ and, thus, throughout Ω. The homogeneous boundary condition of $\partial\Omega_{disp}$ yields however $v = 0$. On the other hand the definition of $\mathcal{R}(\Omega)$, inequality (3.18) and Lemma 2 imply that $\mathcal{R}(\Omega) = \{0\}$.

Lemma 3. The family of the operators $\{\mathcal{A}(e_n)\}_{n\in N}$, $e_n \in U_{ad}(\Omega)$, defined by (2.13), satisfies assumption $((\mathrm{H1})_{3^\circ,4^\circ}$.

Proof. It is readily seen that

$$\langle \mathcal{A}(e)v, v\rangle_{V(\Omega)} \geqslant \alpha_A \|v\|^2_{V(\Omega)} \tag{3.17}$$

for any $e \in U_{ad}(\Omega)$, for any $v \in V(\Omega)$, with the constant α_A, independent of $[e, v]$.

Indeed, due to the Sylvester criterion and assumption (2.3) we deduce the quadratic form

$$\frac{E_1}{1-\mu_1\mu_2}\xi_1^2 + \frac{2\mu_1 E_2}{1-\mu_1\mu_2}\xi_1\xi_2 + \frac{E_2}{1-\mu_1\mu_2}\xi_2^2 \qquad \forall \xi_1, \xi_2 \in \mathbb{R}$$

to be positive definite. Hence, we have

$$a(e, v, v) \geqslant C \int_\Omega \Big[\Big(\frac{\partial^2 v}{\partial x^2}\Big)^2 + \Big(\frac{\partial^2 v}{\partial y^2}\Big)^2 + \Big(\frac{\partial^2 v}{\partial x \partial y}\Big)^2\Big]\, d\Omega \tag{3.18}$$

where $v \in V(\Omega)$, $e \in U_{ad}(\Omega)$, $C = \mathrm{const} > 0$.

Then, by corollary 1.6.1 (Litvinov, 2000) the formula

$$\sqrt{\int_\Omega \Big[\Big(\frac{\partial^2 v}{\partial x^2}\Big)^2 + \Big(\frac{\partial^2 v}{\partial y^2}\Big)^2 + \Big(\frac{\partial^2 v}{\partial x \partial y}\Big)^2\Big]\, d\Omega}$$

defines a norm in $V(\Omega)$, which is equivalent to the original one, i.e., to the norm of $H^2(\Omega)$.

Further, we can write (in view of (2.2) to (2.6))

$$
\begin{aligned}
&|\langle\mathcal{A}(e)v,w\rangle_{V(\Omega)}-\langle\mathcal{A}(e)z,w\rangle_{V(\Omega)}|=\Big|\int_\Omega\Big[D_{11}(e)\frac{\partial^2(v-z)}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+\\
&+D_{22}(e)\frac{\partial^2(v-z)}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+D_{12}(e)\Big(\frac{\partial^2(v-z)}{\partial^2(v-z)}\frac{\partial^2 w}{\partial y^2}+\\
&+\frac{\partial^2(v-z)}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\Big)+2D_{33}(e)\frac{\partial^2(v-z)}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big]\,d\Omega\Big|\leqslant\\
&\leqslant D_{11}(e_{\max})\int_\Omega\Big|\frac{\partial^2(v-z)}{\partial x^2}\frac{\partial^2 w}{\partial x^2}\Big|\,d\Omega+D_{22}(e_{\max})\int_\Omega\Big|\frac{\partial^2(v-z)}{\partial y^2}\frac{\partial^2 w}{\partial y^2}\Big|\,d\Omega+\\
&+D_{12}(e_{\max})\int_\Omega\Big(\Big|\frac{\partial^2(v-z)}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Big|+\Big|\frac{\partial^2(v-z)}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\Big|\Big)\,d\Omega+\\
&+2D_{33}(e_{\max})\int_\Omega\Big|\frac{\partial^2(v-z)}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big|\,d\Omega\leqslant\\
&\leqslant\max[D_{11}(e_{\max}),D_{22}(e_{\max}),2D_{12}(e_{\max}),2D_{33}(e_{\max})]\|v-z\|_{V(\Omega)}\|w\|_{V(\Omega)}\leqslant\\
&\leqslant C[e_{\max}+e_{\max}^2+e_{\max}^3]\|v-z\|_{V(\Omega)}\|w\|_{V(\Omega)}
\end{aligned}
\tag{3.19}
$$

where $C=$ const. As a consequence, assumption $(\mathrm{H1})_{3^\circ}$)is satisfied.

To verify $(\mathrm{H1})_{4^\circ}$, we write

$$
\begin{aligned}
&|\langle\mathcal{A}(e_n)v-\mathcal{A}(e)v,w\rangle_{V(\Omega)}|=\Big|\int_\Omega\Big\{[(D_{11}(e_n)-D_{11}(e)]\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+\\
&+[D_{22}(e_n)-D_{22}(e)]\frac{\partial^2 v}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+[D_{12}(e_n)-D_{12}(e)]\Big(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\\
&+\frac{\partial^2 v}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\Big)+2[D_{33}(e_n)-D_{33}(e)]\frac{\partial^2 v}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big\}\,d\Omega\Big|\leqslant\\
&\leqslant C\Big(\|e_n-e\|_{L_\infty(\Omega)}+\|e_n^2-e^2\|_{L_\infty(\Omega)}+\|e_n^3-e^3\|_{L_\infty(\Omega)}\Big)\|v\|_{V(\Omega)}\|w\|_{V(\Omega)}
\end{aligned}
$$

Then, one has

$$
\begin{aligned}
&\|\mathcal{A}(e_n)v-\mathcal{A}(e)v\|_{V^*(\Omega)}\leqslant\\
&\leqslant C\Big(\|e_n-e\|_{L_\infty(\Omega)}+\|e_n^2-e^2\|_{L_\infty(\Omega)}+\|e_n^3-e^3\|_{L_\infty(\Omega)}\Big)\|v\|_{V(\Omega)}\to 0
\end{aligned}
$$

as $e_n \to e$ strongly in $U(\Omega)$.

Next, we introduce state variational inequality (2.14) for $u(e_n) \in \mathcal{K}(e_n, \Omega)$

$$\langle \mathcal{A}(e_n)u(e_n), v - u(e_n)\rangle_{V(\Omega)} \geqslant \langle \widehat{O} + \omega_1 H_0 + 2\omega_2 e_n, \widetilde{O}v - \widetilde{O}u(e_n)\rangle_{\mathcal{H}(\Omega)}$$

for all $v \in \mathcal{K}(e_n, \Omega)$.

Consequently, due to Lemma 1, we can write (inserting the sequence $\{v_n\}_{n\in N}$ from (3.11) into the above variational inequality for $e_n \in U_{ad}(\Omega)$)

$$\langle \mathcal{A}(e_n)u(e_n), v_n - u(e_n)\rangle_{V(\Omega)} \geqslant \langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e, \widetilde{O}v_n - \widetilde{O}u(e_n)\rangle_{\mathcal{H}(\Omega)} \quad (3.20)$$

for $n > n_*$.

Hence, in view of (3.17), we have

$$\begin{aligned} &\alpha_A \|u(e_n)\|^2_{V(\Omega)} \leqslant \langle \mathcal{A}(e_n)u(e_n), v\rangle_{V(\Omega)} + \\ &+\langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e_n, \widetilde{O}v_n - u(e_n)\rangle_{\mathcal{H}(\Omega)} \leqslant C\Big[\|u(e_n)\|_{V(\Omega)}\|v\|_{V(\Omega)} + \\ &+\Big(\|\mathcal{O}\|_{\mathcal{H}^*(\Omega)} + \omega_1 H_0 + 2\omega_2\|e\|_{L_\infty(\Omega)}\Big)\cdot \\ &\cdot\Big(\|v_n\|_{V(\Omega)} + \|u(e_n)\|_{V(\Omega)}\Big)\Big] \leqslant C(\|u(e_n)\|_{V(\Omega)} + 1) \end{aligned}$$

and $\|u(e_n)\|_{V(\Omega)} \leqslant C$ for any n.

This means that there exists $u \in V(\Omega)$ and a subsequence $\{u(e_{n_k})\}_{k\in N} \subset \{u(e_n)\}_{n\in N}$, such that

$$u(e_{n_k}) \to u \qquad \text{weakly in } V(\Omega) \quad (3.21)$$

The functional $v \to \langle \mathcal{A}(\widetilde{o})v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous on $V(\Omega)$ for any $\widetilde{o} \in U_{ad}(\Omega)$. Consequently

$$\liminf_{n\to\infty}\langle \mathcal{A}(e)u(e_{n_k}), u(e_{n_k})\rangle_{V(\Omega)} \geqslant \langle \mathcal{A}(e)u, u\rangle_{V(\Omega)} \qquad \text{since } e \in U_{ad}(\Omega)$$

Moreover, we have

$$\begin{aligned} &|\langle \mathcal{A}(e_{n_k})u(e_{n_k}), u(e_{n_k})\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u(e_{n_k}), u(e_{n_k})\rangle_{V(\Omega)}| \leqslant \\ &\leqslant C\Big(\|e_{n_k} - e\|_{L_\infty(\Omega)} + \|e^2_{n_k} - e^2\|_{L_\infty(\Omega)} + \|e^3_{n_k} - e^3\|_{L_\infty(\Omega)}\Big)\|u(e_{n_k})\|^2_{V(\Omega)} \to 0 \end{aligned} \quad (3.22)$$

as $e_{n_k} \to e$ strongly in $U(\Omega)$.

From the above argument, we conclude that

$$\liminf_{k\to\infty}\langle\mathcal{A}(e_{n_k})u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}=\liminf_{k\to\infty}\Big(\langle\mathcal{A}(e)u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}+$$
$$+\langle\mathcal{A}(e_{n_k})u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}-\langle\mathcal{A}(e)u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}\Big)\geqslant \tag{3.23}$$
$$\geqslant\liminf_{k\to\infty}\langle\mathcal{A}(e)u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}\geqslant\langle\mathcal{A}(e)u,u\rangle_{V(\Omega)}$$

Further, we can write (using the decomposition

$$\langle\mathcal{A}(e_{n_k})u(e_{n_k}),\tilde{o}\rangle_{V(\Omega)}-\langle\mathcal{A}(e)u,\tilde{o}\rangle_{V(\Omega)}=$$
$$+\Big[\langle\mathcal{A}(e_{n_k})u(e_{n_k}),\tilde{o}\rangle_{V(\Omega)}-\langle\mathcal{A}(e)u(e_{n_k}),\tilde{o}\rangle_{V(\Omega)}\Big]+\langle\mathcal{A}(e)(u(e_{n_k})-u),\tilde{o}\rangle_{V(\Omega)}$$

and the weak convergence of $\{u(e_{n_k})\}_{k\in N}$)

$$\lim_{k\to\infty}\langle\mathcal{A}(e_{n_k})u(e_{n_k}),\tilde{o}\rangle_{V(\Omega)}=\langle\mathcal{A}(e)u,\tilde{o}\rangle_{V(\Omega)}\qquad\forall\tilde{o}\in V(\Omega) \tag{3.24}$$

Taking into account (3.11), we obtain as $k\to\infty$

$$|\langle\mathcal{A}(e_{n_k})u(e_{n_k}),v_k-v\rangle_{V(\Omega)}|\leqslant C\|u(e_{n_k})\|_{V(\Omega)}\|v_k-v\|_{V(\Omega)}\to 0 \tag{3.25}$$

Then, due to (3.24) and (3.25), we arrive at

$$|\langle\mathcal{A}(e_{n_k})u(e_{n_k}),v_k\rangle_{V(\Omega)}-\langle\mathcal{A}(e)u,v\rangle_{V(\Omega)}|\leqslant|\langle\mathcal{A}(e_{n_k})u(e_{n_k}),v_k-v\rangle_{V(\Omega)}|+ \tag{3.26}$$
$$+|\langle\mathcal{A}(e_{n_k})u(e_{n_k}),v\rangle_{V(\Omega)}-\langle\mathcal{A}(e)u,v\rangle_{V(\Omega)}|\to 0$$

Furthermore, the weak convergence of $\{u(e_{n_k})\}_{k\in N}$ and (3.11) yield that

$$\langle\mathcal{O}+\omega_1H_0+2\omega_2e_{n_k},\tilde{O}v_n-\tilde{O}u(e_{n_k})\rangle_{\mathcal{H}(\Omega)}\to\langle\mathcal{O}+\omega_1H_0+2\omega_2e,\tilde{O}v-\tilde{O}u\rangle_{\mathcal{H}(\Omega)} \tag{3.27}$$

when $e_{n_k}\to e$ strongly in $U(\Omega)$.

On the other hand, from inequality (3.20), we deduce that

$$\langle\mathcal{A}(e_{n_k})u(e_{n_k}),u(e_{n_k})\rangle_{V(\Omega)}+\langle\mathcal{O}+\omega_1H_0+2\omega_2e,\tilde{O}v_k-\tilde{O}u(e_{n_k})\rangle_{\mathcal{H}(\Omega)}\leqslant \tag{3.28}$$
$$\leqslant\langle\mathcal{A}(e_{n_k})u(e_{n_k}),v\rangle_{V(\Omega)}$$

Passing here (in (3.28)) to the limit inferior on both sides with $k\to\infty$, using (3.23), (3.26) and (3.27), we obtain

$$\langle\mathcal{A}(e)u,u\rangle_{V(\Omega)}+\langle\mathcal{O}+\omega_1H_0+2\omega_2e,\tilde{O}v-\tilde{O}u\rangle_{(\Omega)}\leqslant\langle\mathcal{A}(e)u,v\rangle_{V(\Omega)}$$

Consequently, u satisfies inequality (2.14). Since the solution $u(e)$ to (2.14) is unique, $u = u(e)$ follows and the whole sequence $\{u(e_n)\}_{n\in N}$ converges to $u(e)$ weakly in $V(\Omega)$.

Finally, it remains to verify the strong convergence. By virtue of (3.28), (3.26) and (3.27), we can write

$$\limsup_{n\to\infty}\langle \mathcal{A}(e_n)u(e_n), u(e_n)\rangle_{V(\Omega)} \leqslant \\ \leqslant \langle \mathcal{A}(e)u(e), v\rangle_{V(\Omega)} + \langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e, \widetilde{O}u(e) - \widetilde{O}v\rangle_{\mathcal{H}(\Omega)} \tag{3.29}$$

for any $v \in \mathcal{K}(e,\Omega)$.

Hence (we put $v := u(e)$ in (3.29)) due to (3.23), we get

$$\langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)} \leqslant \liminf_{n\to\infty}\langle \mathcal{A}(e_n)u(e_n), u(e_n)\rangle_{V(\Omega)} \leqslant \\ \leqslant \limsup_{n\to\infty}\langle \mathcal{A}(e_n)u(e_n), u(e_n)\rangle_{V(\Omega)} \leqslant \langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)}$$

This means that

$$\lim_{n\to\infty}\langle \mathcal{A}(e_n)u(e_n), u(e_n)\rangle_{V(\Omega)} = \langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)} \tag{3.30}$$

Taking into account (3.22) and (3.30), we arrive at

$$\lim_{n\to\infty}\langle \mathcal{A}(e)u(e_n), u(e_n)\rangle_{V(\Omega)} = \langle \mathcal{A}(e)u(e), u(e)\rangle_{V(\Omega)} \tag{3.31}$$

Further, if $V(\Omega)$ is equipped with the scalar product $\langle \mathcal{A}(e)u(e), v\rangle_{V(\Omega)} = (u(e), v)_A$, then (3.31) implies that the associated norms $\|u(e_n)\|_A$ tend to $\|u(e)\|_A$. Since the norms $\|\cdot\|_A$ and $\|\cdot\|_{V(\Omega)}$ are equivalent, we are led to the strong convergence

$$\|u(e_n) - u(e)\|_{V(\Omega)} \to 0 \tag{3.32}$$

Lemma 4. Let $e_n \to e$ strongly in $U(\Omega)$ as $n\to\infty$, $e_n \in U_{ad}(\Omega)$. Then, one has

$$\mathbf{M}(e_n) \to \mathbf{M}(e) \qquad \text{strongly in } [L_2(\Omega)]^4$$

Proof. We can write

$$\|M_{xx}(e_n) - M_{xx}(e)\|_{L_2(\Omega)} \leqslant \\ \leqslant \Big\|[D_{11}(e_n) - D_{11}(e)]\frac{\partial^2 u(e_n)}{\partial x^2} + [D_{12}(e_n) - D_{12}(e)]\frac{\partial^2 u(e_n)}{\partial y^2}\Big\|_{L_2(\Omega)} + \\ + \Big\|D_{11}(e)\Big(\frac{\partial^2 u(e_n)}{\partial x^2} - \frac{\partial^2 u(e)}{\partial x^2}\Big) + D_{12}(e)\Big(\frac{\partial^2 u(e_n)}{\partial y^2} - \frac{\partial^2 u(e)}{\partial y^2}\Big)\Big\|_{L_2(\Omega)} = \\ = S_{1n} + S_{2n}$$

Next, in view of (3.32) we have: $S_{1n} \to 0$, $S_{2n} \to 0$ as $n \to \infty$. Equally, we obtain

$$\|M_{yy}(e_n) - M_{yy}(e)\|_{L_2(\Omega)} \to 0$$
$$\|M_{xy}(e_n) - M_{xy}(e)\|_{L_2(\Omega)} \to 0$$

Lemma 5. Let $e_n \to e$ strongly in $U(\Omega)$ as $n \to \infty$, $e_n \in U_{ad}(\Omega)$. Then, for any $i = 1, 2, ..., N_i$

$$[\mathcal{S}_i(e_n, \mathbf{M}(e_n))]^+ \to [\mathcal{S}_i(e, \mathbf{M}(e))]^+$$

Proof. Due to the estimate: $|a^+ - b^+| \leqslant |a - b|$, we can write

$$|[\mathcal{S}_i(e_n, \mathbf{M}(e_n))]^+ - [\mathcal{S}_i(e, \mathbf{M}(e))]^+| \leqslant |\mathcal{S}_i(e_n, \mathbf{M}(e_n)) - \mathcal{S}_i(e, \mathbf{M}(e))| \leqslant$$

$$\leqslant \frac{36}{\text{meas}\Omega_i^*} \int\limits_{\Omega_i^*} \Big| \frac{1}{(H_{02} + e_n)^4} \Big[M_{xx}^2(e_n) + M_{yy}^2(e_n) + \Big(\frac{\sigma_0}{\tau_0}\Big)^2 M_{xy}^2(e_n)\Big] -$$

$$- \frac{1}{(H_{02} + e_n)^4} \Big[M_{xx}^2(e) + M_{yy}^2(e) + \Big(\frac{\sigma_0}{\tau_0}\Big)^2 M_{xy}^2(e)\Big] \Big| \, d\Omega \leqslant$$

$$\leqslant C \int\limits_{\Omega_i^*} \Big| \frac{1}{(H_{02} + e_n)^4} \Big\{ [M_{xx}^2(e_n) - M_{xx}^2(e)] + [M_{yy}^2(e_n) - M_{yy}^2(e)] +$$

$$+ \Big(\frac{\sigma_0}{\tau_0}\Big)^2 [M_{xy}^2(e_n) - M_{xy}^2(e)] \Big\} + \Big(\frac{1}{[2(H_{02} + e_n)]^4} -$$

$$- \frac{1}{(H_{02} + e)^4} [M_{xx}^2(e) + M_{yy}^2(e)] + \Big(\frac{\sigma_0}{\tau_0}\Big)^2 M_{xy}^2(e) \Big) \Big| \, d\Omega \leqslant$$

$$\leqslant C \frac{1}{[2(H_{02} + e_{\min})]^4} \Big\{ \int\limits_{\Omega_i^*} \Big(|M_{xx}(e_n) + M_{xx}(e)| \, |M_{xx}(e_n) - M_{xx}(e)| +$$

$$+ |M_{yy}(e_n) + M_{yy}(e)| \, |M_{yy}(e_n) - M_{yy}(e)| +$$

$$+ \Big(\frac{\sigma_0}{\tau_0}\Big)^2 |M_{xy}(e_n) + M_{xy}(e)| \, |M_{xy}(e_n) - M_{xy}(e)| \Big) \, d\Omega \Big\} +$$

$$+ C(e) \Big\| \frac{1}{[2(H_{02} + e_n)]^4} - \frac{1}{[2(H_{02} + e)]^4} \Big\|_{L_\infty(\Omega)} \to 0$$

as $n \to \infty$ and $H_{02} = H_0/2$, due to Lemma 4.

Lemma 6. The penalized optimal control problem $(\mathcal{P}_{\varepsilon_n})$ has a solution for any $\varepsilon_n > 0$.

Proof. Note that the functionals $\mathcal{L}(e)$ and $[\mathcal{S}_i(e, \mathbf{M}(e))]^+$ are continuous in $U_{ad}(\Omega)$ and $U_{ad}(\Omega)$ is compact in $U(\Omega)$. Hence, there exists a minimizer e_{ε_n} of $\mathcal{L}_{\varepsilon_n}(e, \mathbf{M}(e))$.

Theorem 3. Let condition (2.17) be satisfied. Let $\{\varepsilon_n\}_{n\in N}$, $\varepsilon \to 0^+$ be a sequence and $\{e_{\varepsilon_n}\}_{n\in N}$ a sequence of solutions to the penalized optimal control problems $(\mathcal{P}_{\varepsilon_n})$, $\{\mathbf{M}(e_{\varepsilon_n})\}_{n\in N}$ the sequence of corresponding moment fields.

Then, there exists a subsequence $\{\varepsilon_{n_k}\}_{k\in N} \subset \{\varepsilon_n\}_{n\in N}$ and an element $e_* \in \mathcal{G}_{ad}(\Omega)$ such that

$$\begin{cases} e_{\varepsilon_{n_k}} \to e_* & \text{strongly in } U(\Omega) \\ \mathbf{M}(e_{\varepsilon_{n_k}}) \to \mathbf{M}(e_*) & \text{strongly in } [L_2(\Omega)]^4 \end{cases} \tag{3.33}$$

where e_* is the solution to the optimal control problem $(\mathcal{P})$.

Proof. There exists a subsequence $\{\varepsilon_{n_k}\}_{k\in N} \subset \{\varepsilon_n\}_{n\in N}$ (here, $U_{ad}(\Omega)$ is compact in $U(\Omega)$) such that $(3.33)_1$ holds with $e_* \in U_{ad}(\Omega)$. In view of Lemma 4, we obtain $(2.32)_2$.

Further, the definition yields

$$\mathcal{L}_{wg}(e_{\varepsilon_{n_k}}) + \frac{1}{\varepsilon_{n_k}} \sum_{i=1}^{N_i} [\mathcal{S}_i(e_{\varepsilon_{n_k}}, \mathbf{M}(e_{\varepsilon_{n_k}}))]^+ \leqslant \mathcal{L}_{wg}(e) + \frac{1}{\varepsilon_{n_k}} \sum_{i=1}^{N_i} [\mathcal{S}_i(e, \mathbf{M}(e))]^+ \tag{3.34}$$

holds for any $e \in U_{ad}(\Omega)$.

On the other hand, for an arbitrary element e from $\mathcal{G}_{ad}(\Omega)$, we have

$$\varepsilon_{n_k} \mathcal{L}_{wg}(e_{\varepsilon_{n_k}}) + \sum_{i=1}^{N_i} [\mathcal{S}_i(e_{\varepsilon_{n_k}}, \mathbf{M}(e_{\varepsilon_{n_k}}))]^+ \leqslant \varepsilon_{n_k} \mathcal{L}_{wg}(e)$$

$$0 \leqslant \sum_{i=1}^{N_i} [\mathcal{S}_i(e_{\varepsilon_{n_k}}, \mathbf{M}(e_{\varepsilon_{n_k}}))]^+ \leqslant \varepsilon_{n_k} \mathcal{L}_{wg}(e)$$

Hence, passing to the limit with $\varepsilon_{n_k} \to 0$ and by virtue of Lemma 5, we arrive at

$$\sum_{i=1}^{N_i} [\mathcal{S}_i(e_*, \mathbf{M}(e_*))]^+ = 0$$

This means that the element $e_* \in \mathcal{G}_{ad}(\Omega)$.

Then, on account of (3.34), we obtain

$$\mathcal{L}_{wg}(e_{\varepsilon_{n_k}}) \leqslant \mathcal{L}_{wg}(e_{\varepsilon_{n_k}}) + \frac{1}{\varepsilon_{n_k}} \sum_{i=1}^{N_i} [\mathcal{S}_i(e_{\varepsilon_{n_k}}, \mathbf{M}(e_{\varepsilon_{n_k}}))]^+ \leqslant \mathcal{L}_{wg}(e) \tag{3.35}$$

for any $e \in \mathcal{G}_{ad}(\Omega)$.

We deduce from (3.35) the estimate (passing to the limit with $\varepsilon_{n_k} \to 0$ and we use $(3.33)_1$)

$$\mathcal{L}_{wg}(e_*) \leqslant \mathcal{L}_{wg}(e) \qquad \forall e \in \mathcal{G}_{ad}(\Omega)$$

Lemma 7. For a non-empty set $\mathcal{G}_{ad}(\Omega)$ there exists at least one solution to the optimal control problem $(\mathcal{P})$.

Proof. The proof follows immediately from Lemma 5 and Theorem 3.

Cost functional with measures

Consider the following optimal control problem $(\mathcal{B})$: Find $e_* \in U_{ad}(\Omega)$ such that

$$J(e_*) \leqslant J(e) \qquad \forall e \in U_{ad}(\Omega) \tag{3.36}$$

where

$$\begin{aligned} &J(e) = \mu([e, u(e)], \Omega) \\ &\mathcal{U}_{ad}(\Omega) = \{e \in U_{ad}(\Omega) \cap H^2(\Omega) : \ \|e\|_{H^2(\Omega)} \leqslant C\} \end{aligned}$$

The non-negative Radon measure μ in Ω is given by the relation (2.16).

For a given $e \in \mathcal{U}_{ad}(\Omega)$, the state function $u(e) \in \mathcal{K}(e, \Omega)$ is the solution to variational inequality (2.14).

Lemma 8. The optimization problem $(\mathcal{B})$ has a solution.

Proof. Let $\{e_n\}_{n \in N}$ be a minimizing sequence. The sequence is bounded, and hence one has $e_{n_k} \to e_*$ weakly in $H^2(\Omega)$ and uniformly in $\overline{\Omega}$ as $k \to \infty$. By the same arguments as the proof of Theorem 1, the sequence $\{u(e_{n_k})\}_{k \in N}$ of the solutions to variational inequality (3.20) is bounded in $V(\Omega)$. Hence, we conclude that $u(e_{n_k}) \to u(e_*)$ weakly in $V(\Omega)$ as $k \to \infty$. Moreover, $u(e)$ is a solution to variational inequality (2.14).

In the following, we show weak-star convergence of the sequence of measures $\{\mu([e_{n_k}, u(e_{n_k})], \Omega)\}_{k \in N}$. To this end, it is enough to prove that for every compact subset $\mathcal{Q} \subset \Omega$ the values $\mu([e_{n_k}, u(e_{n_k})], \mathcal{Q})$ are bounded uniformly with respect to $k = 1, 2, \ldots$. For any $\widetilde{o} \in C_0^\infty(\Omega)$, $\widetilde{o} \equiv 1$ on $\mathcal{Q}$, $\widetilde{o} \geqslant 0$ (taking into account (2.15) and (3.19)), we may write (independent of $k = 1, 2, \ldots$)

$$\mu([e_{n_k}, u(e_{n_k})], \mathcal{Q}) \leqslant \int_\Omega \widetilde{o}\, d\mu([e_{n_k}, (e_{n_k})], \Omega) =$$

$$= \langle [\widetilde{\mathcal{A}}(e_{n_k})(e_{n_k})u(e_{n_k}) - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e_{n_k})], \widetilde{o}\rangle_{L_2(\Omega)} \leqslant C$$

Hence, we deduce that

$$\mu([e_{n_k}, u(e_{n_k})], \mathcal{Q}) \equiv \widetilde{\mathcal{A}}(e_{n_k})u(e_{n_k}) - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e_{n_k}) \rightarrow$$

$$\rightarrow \mu([e_*, u(e_*)], \Omega) \equiv \widetilde{\mathcal{A}}(e_*)u(e_*) - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e_*)$$

weakly star.

The measure $\mu([e_*, u(e_*)], \Omega)$ is obtained as the limit, since the sequence $\{u(e_{n_k})\}_{k \in N}$ is bounded in $V(\Omega)$. Thus, we have proved the following

$$\widetilde{\mathcal{A}}(e_{n_k})u(e_{n_k}) - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e_{n_k}) \rightarrow$$
$$\rightarrow \widetilde{\mathcal{A}}(e_*)u(e_*) - (\mathcal{O} + \omega_1 H_0 + 2\omega_2 e_*)$$

weakly in $V^*(\Omega)$.

Therefore, from the weakly star convergence of the sequence $\{\mu([e_{n_k}, u(e_{n_k})], \Omega)\}_{k \in N}$, we obtain

$$\liminf_{k \to \infty} \mu([e_{n_k}, u(e_{n_k})], \Omega) \geqslant \mu([e_*, u(e_*)], \Omega)$$

Then, the following relation holds

$$\widetilde{\mathcal{E}} = \liminf_{k \to \infty} J(e_{n_k}) \geqslant J(e_*) \geqslant \widetilde{\mathcal{E}}$$

where $\widetilde{\mathcal{E}} = \inf\limits_{e \in U_{ad}(\Omega)} J(e)$.

Consequently, e_* is a solution to the problem $(\mathcal{B})$.

4. Approximate optimal control. The numerical solution by the finite element method

We shall propose approximate solutions to the optimization problem for an elastic three-layered plate by the finite element method. We restrict ourselves to particular domains, namely we suppose that Ω is parallelogram. We consider only the convex set given by (2.12).

Let $\mathcal{T}_{h_n}$ denote a uniform partition of Ω into a finite number of small (open) parallelograms H_i by means of two systems of equidistant straight lines parallel with the sides of Ω. Then, we can write $\overline{\Omega} = \bigcup_{i=1}^{N(h)} \overline{\mathcal{H}_i}$, $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ for $i \neq j$ and denote $h = \operatorname{diam} \mathcal{H}_i$.

Assume that $\mathcal{T}_h$ is consistent with the partition of the boundary $\partial\Omega = \partial\Omega_{cont} \cup \partial\Omega_{disp}$ i.e. the number of points $\partial\overline{\Omega}_{disp} \cap \partial\overline{\Omega}_{cont}$ is finite and every point of this kind coincides with the node of $\mathcal{T}_h$. Thus, we can write: $\partial\Omega_{cont} = \sum_{i=1}^{\mathcal{N}(h)} \overline{A^*_{(i-1)_h} A^*_{i_h}}$.

We introduce the spaces $Q_k(\mathcal{H})$ of bilinear $(k = 1)$ or bicubic $(k = 3)$ polynomials defined on the parallelogram $\mathcal{H}$. If $\mathcal{H}$ is not rectangular, the spaces $Q_k(\mathcal{H})$ are defined via the affine mapping

$$\boldsymbol{x} = \mathcal{V}(\boldsymbol{y}): \quad x_1 = y_1 + y_2 \cos\alpha, \qquad x_2 = y_2 \sin\alpha \tag{4.1}$$

which maps a rectangle $\mathcal{H}_*$ onto $\mathcal{H}$. We set

$$v \in Q_k(\mathcal{H}) \Leftrightarrow v \circ \mathcal{V} = \widehat{v} \in Q_k(\mathcal{H}_*)$$

Let Σ_h be the set of all vertices $A_{\mathcal{Q}}$, $1 \leqslant \mathcal{Q} \leqslant M(h)$ (nodes of $\mathcal{T}_h$) of the parallelograms. Let $V_h(\Omega)$ be a finite-dimensional subspace of $V(\Omega)$ defined by $(N(h)$ is the set of all parallelograms)

$$V_h(\Omega) = \{v \in V(\Omega): \ v\big|_{\mathcal{H}_{\hat{o}}} \in Q_3(\mathcal{H}_{\hat{o}}), \ 1 \leqslant \hat{o} \leqslant N(h)\}$$

i.e. $V_h(\Omega)$ contains those functions which are continuous and continuously differentiable in $\overline{\Omega}$ and piecewise bicubic in each $\mathcal{H}_{\hat{o}}$. Then, the set $\mathcal{K}_h(\Omega)$ is defined in the following way

$$\mathcal{K}_h(\Omega) = \Big\{v \in V_h(\Omega): \ 0 \leqslant v(\mathring{A}_{ih}), \text{ where } \mathring{A}_{ih} \text{ are nodes of } \mathcal{T}_h \text{ such that } \mathring{A}_{ih} \in \Omega_* \text{ or all nodes } \mathring{A}_{ih} \in \Sigma^*_h - \text{ the set of all vertices of the rectangles } \mathcal{H}_i \in \mathcal{T}_h, \ \mathcal{H}_i \in \overline{\Omega}_*) \text{ and } 0 \leqslant v(A^*_{ih}), \ 1 \leqslant i \leqslant n(h)\Big\}$$

Next, let

$$U_{ad_h}(\Omega) = \Big\{e \in U_{ad}(\Omega): \ e\big|_{\mathcal{H}_{\hat{o}}} \in Q_1(\mathcal{H}_{\hat{o}}), \ 1 \leqslant \hat{o} \leqslant N(h)\Big\}$$

Convergence of Ritz approximations

In what follows, we shall consider any families $\{\mathcal{T}_{h_n}\}_{n\in N}$, $h_n \to 0^+$ of partitions, which refine the (original) partition $\mathcal{T}_{h_*}$. We say that the family $\{\mathcal{T}_{h_n}\}_{n\in N}$ is regular if there exists a positive constant such that

$$\begin{cases} \dfrac{h_n}{\rho} \leqslant C & \text{for any } \mathcal{H}_i \in \bigcup\limits_{h_n} \mathcal{T}_{h_n} \\ \Sigma_{h_1} \subset \Sigma_{h_2} & \text{if } h_1 > h_2 \end{cases} \tag{4.2}$$

where ρ denotes the diameter of the maximal circle contained $\mathcal{H}_i$. Now, we can define the following **approximate state problem:** given any $e_h \in U_{ad_h}(\Omega)$, find $u_h(e_h) \in \mathcal{K}_h(\Omega)$ such that

$$\begin{cases} \langle \mathcal{A}(e_h)u_h(e_h), v_h - u_h(e_h)\rangle_{V(\Omega)} \geqslant \langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e_h, \widetilde{O}v_h - \widetilde{O}u_h(e_h)\rangle_{\mathcal{H}(\Omega)} \\ \text{holds } \ \forall\, v_h \in \mathcal{K}_h(\Omega) \end{cases} \tag{4.3}$$

Finally, let us define the penalized cost functional

$$\mathcal{L}_{\varepsilon\, wg_h} = \int\limits_{\Omega} (\omega_1 H_0 + 2\omega_2 e_h)\, d\Omega + \frac{1}{\varepsilon}\sum_{i=1}^{N_i} [\mathcal{S}_i(e_h, \mathbf{M}_h(e_h))]^+ = 0$$

Here, the approximate optimal control problem consists in finding a function $e_{\varepsilon_h, wg}$ of the approximate optimal control problem such that

$$(\mathcal{P}_{\varepsilon_h}) \qquad e_{\varepsilon_h, wg} = \operatorname*{ArgMin}_{e_h \in U_{ad_h}(\Omega)} \Big\{ \mathcal{L}_{wg}(e_h) + \frac{1}{\varepsilon}\sum_{i=1}^{N_i} [\mathcal{S}_i(e_h, \mathbf{M}_h(e_h))]^+ \Big\}$$

Further, we shall prove solvability of the problem $(\mathcal{P}_{\varepsilon_h})$. To this end we first establish the following lemmas.

Lemma 9. The set $\mathcal{K}_h(\Omega)$ is a closed and convex subset of $V_h(\Omega)$ and $\mathcal{K}(\Omega) = \lim\limits_{n\to\infty} \mathcal{K}_{h_n}(\Omega)$ (convergence in the sense of Glowinski).

Proof. Let $\{v_{h_n}\}_{n\in N}, v_{h_n} \in \mathcal{K}_{h_n}(\Omega)$ being the sequence such that $v_{h_n} \to v$ weakly in $V(\Omega)$ for $n \to \infty$. In view of the Dirac function, concentrated at $[x,y] \in \overline{\Omega}_*$ (since $\delta(x,y) \in V^*(\Omega)$), we have: $v_{h_n}(x,y) \to v(x,y)$ for all $[x,y] \in \overline{\Omega}_*$. Let us suppose that there exists $[x_{\tilde{o}}, y_{\tilde{o}}] \in \overline{\Omega}_*$ such that $v(x_{\tilde{o}}, y_{\tilde{o}}) < 0$ and that $v < 0$ in an interval $\widehat{S} \subset \partial\Omega_{cont}$, respectively. Here, $v \in C(\overline{\Omega})$ (the embedding of $H^2(\Omega)$ is compact). Hence, we conclude that

$v(x_{\tilde{o}}, y_{\tilde{o}}) < 0$ holds in some neighbourhood: $\mathcal{U}([x_{\tilde{o}}, y_{\tilde{o}}], \varepsilon) \cap \overline{\Omega}_*$, $\varepsilon > 0$, where $\mathcal{U}([x_{\tilde{o}}, y_{\tilde{o}}], \varepsilon) = \{[x, y] \in \mathbb{R}_2 : \ \rho([x, y], [x_{\tilde{o}}, y_{\tilde{o}}]) \leqslant \varepsilon\}$.

Furthermore, $\varepsilon > 0$ and a subinterval $\widehat{S}_0 \subset \widehat{S}$ exist such that $v(x_{\tilde{o}}, y_{\tilde{o}}) < -\varepsilon$ for all $[x_{\tilde{o}}, y_{\tilde{o}}] \in \widehat{S}_0$. Now, taking into account that $\operatorname{diam}(\mathcal{H}_i) \leqslant h_n$ for any $\mathcal{H}_i \in \mathcal{T}_{h_n}$ and $h_n \to 0^+$. Thus, there exists $\mathring{A}_{ih_{\tilde{o}}} \in \Sigma^*_{h_{\tilde{o}}}$ such that $\mathring{A}_{ih_{\tilde{o}}} \in \mathcal{U}([x_{\tilde{o}}, y_{\tilde{o}}], \varepsilon) \cap \overline{\Omega}_*$ for any $h_n \leqslant h_{\tilde{o}}$. Further, $v_{h_n}(\mathring{A}_{ih_{\tilde{o}}}) \geqslant 0$ for any $h_n \leqslant h_{\tilde{o}}$. Hence, one has: $v(\mathring{A}_{ih_{\tilde{o}}}) = \lim_{h_n \to 0^+} v_{h_n}(\mathring{A}_{ih_{\tilde{o}}}) \geqslant 0$, which is a contradiction to the previous considerations. On the other hand, for sufficiently small h_n there exists always a node $A^*_{ih_n} \in \widehat{S}_0$, $A^*_{ih_n} \in \Sigma^*_h$ and we can write

$$|v_{h_n}(A^*_{ih_n}) - v(A^*_{ih_n}| = v_{h_n}(A^*_{ih_n}) - v(A^*_{ih_n}) \geqslant -v(A^*_{ih_n}) \geqslant \varepsilon$$

which contradicts to: $\lim_{h_n \to 0^+} \|v_{h_n} - v\|_{C(\overline{\Omega})} = 0$. Consequently, $v \geqslant 0$ i.e. in Ω_* and $v \geqslant 0$ on $\partial\Omega_{cont}$ in the sense of traces. This means: $v \in \mathcal{K}(\Omega)$.

Next, we verify that for any $v \in \mathcal{K}(\Omega)$ there exists a subset $\widehat{O}(\Omega)$ dense in $\mathcal{K}(\Omega)$ and mapping $\mathcal{R}_{h_n} : \ \widehat{O}(\Omega) \to V(\Omega)$ such that $\mathcal{R}_{h_n}(\widehat{O}(\Omega)) \subset \mathcal{K}_{h_n}(\Omega)$ and $\mathcal{R}_{h_n} v \to v$ strongly in $V(\Omega)$ as $h_n \to 0^+$ for all $v \in \mathcal{K}(\Omega)$.

Let us assume: $v \in \mathcal{K}(\Omega) \cap H^4(\Omega)$. Now, define the set $\widehat{O}(\Omega)$ by the following relation:

$$\widehat{O}(\Omega) = \Big\{ v \in C^\infty(\overline{\Omega}) : \ \ v = 0, \ \ \tfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega_{disp} \text{ and } v \geqslant 0 \text{ on } \partial\Omega_{cont} \\ \text{in sense of traces and } v \geqslant 0 \text{ on } \Omega_* \Big\}$$

Then, the functions $v_{\hat{o}_n} \in \widehat{O}(\Omega)$ exist such that: $v_{\hat{o}_n} \to v$ strongly in $V(\Omega)$ as $n \to \infty$. We introduce a mapping $\mathcal{R}_{h_n}$ by the relation: $\theta_{h_n} = \mathcal{R}_{h_n} v_{\hat{o}_n}$ where θ_{h_n} is the $V_{h_n}(\Omega)$ – interpolate of $v_{\hat{o}_n}$ over the partition $\mathcal{T}_{h_n}$. Then, $\theta_{h_n} \in \mathcal{K}_{h_n}(\Omega)$ holds, since the modal parameters involve all the values $v_{\hat{o}_n}(A^*_{ih_n})$ and $v_{\hat{o}_n}(\mathring{A}_{ih_n})$. Furthermore, the estimate

$$|\mathcal{R}_{h_n} v_{\hat{o}_n} - v_{\hat{o}_n}\|_{V(\Omega)} \leqslant C h_n^2 \|v_{\hat{o}_n}\|_{H^4(\Omega)}$$

holds for any regular family $\{\mathcal{T}_{h_n}\}$.

Hence, one has

$$\theta_{h_n} \to v \qquad \text{strongly in } \ V(\Omega) \text{ as } n \to \infty \tag{4.4}$$

which concludes the proof.

Lemma 10. The approximate problem $\mathcal{P}(_{\varepsilon_h})$ has at least one solution for any fixed rectangulation $\mathcal{T}_{h_{\bar{o}}}$ and any $\varepsilon \to 0$.

Proof. For fixed $\mathcal{T}_{h_{\bar{o}}}$ and for $e_{h[n]} \to e_h$ strongly in $U(\Omega)$, $e_{h[n]} \in U_{ad[n]}(\Omega)$, $n = 1, 2, \ldots$, we can prove (parallelly with Lemma 4) that

$$\mathbf{M}_{h[n]}(e_{h[n]}) \to \mathbf{M}_h(e_h) \qquad \text{strongly in } [L_2(\Omega)]^4$$

Then, taking into account this relation, we prove that the functions $[\mathcal{S}_i(e_{h[n]}, \mathbf{M}(e_{h[n]}))]^+$ are continuous in $U_{ad[n]}(\Omega)$ (the proof of analogous Lemma 5). Thus, we have proved the following: the cost functional in $(\mathcal{P}_{\varepsilon_h})$ is continuous, as well. Obviously, one has

$$e_h \in U_{ad_h}(\Omega) \Leftrightarrow \left\{e_h(A_{qh})\right\}_{q=1}^{M(h)} \in \mathcal{J}_h \subset \mathbb{R}^{M(h)}$$

where A_{qh} are the vertices of $\mathcal{T}_h$. But here the set $\mathcal{J}_h$ is compact in $\mathbb{R}^{M(h)}$, being bounded and closed. Hence the cost functional attains its minimum in $U_{ad_h}(\Omega)$.

Convergence results

In the following, we will study the convergence of finite element approximations when the mesh size tends to zero. To this end, we establish the crucial

Lemma 11. Let $e_{h_n} \in U_{ad_{h_n}}(\Omega)$, $e_{h_n} \to e$ strongly in $U(\Omega)$, as $h_n \to 0^+$. Then, one has

$$u_{h_n}(e_{h_n}) \to u(e) \qquad \text{strongly in } U(\Omega), \text{ as } h_n \to 0^+$$

holds for any regular family of partitions $\{\mathcal{T}_{h_n}\}_{n\in N}$, which refine $\mathcal{T}_{h_o}$.

Proof. We deduce that

$$\langle \mathcal{A}(e_h)v, v\rangle_{V(\Omega)} \geqslant C\|v\|^2_{V(\Omega)} \qquad \forall\, v \in V(\Omega),\ e_h \in U_{ad_h}(\Omega) \tag{4.5}$$

Hence, the functional

$$\mathcal{E}(v) = \frac{1}{2}\langle \mathcal{A}(e_h)v, v\rangle_{V(\Omega)} - \langle \mathcal{O} + \omega_1 H_0 + 2\omega_2 e_h, \widetilde{O}v\rangle_{\mathcal{H}(\Omega)}$$

(in view of (3.17)) is quadratic, strictly convex on the space $V(\Omega)$. Further $\mathcal{K}_h(\Omega)$ is convex and closed in $V(\Omega)$. Consequently, there exists a unique solution to the approximate state problem $(\mathcal{P}_{e_h})$. Obviously, $\mathcal{K}_h(\Omega)$ is a convex cone with the vertex at the origin. Hence, we can insert $v_h = 0$ and $v_h = 2u_h(e_h)$ into (4.3) to obtain

$$\langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = \langle \mathcal{O}+\omega_1 H_0+2\omega_2 e_{h_n}, \widetilde{O}u_{h_n}(e_{h_n})\rangle_{\mathcal{H}(\Omega)} \quad (4.6)$$

Making use of this equality and estimate (4.5), we deduce that

$$\alpha_A\|u_{h_n}(e_{h_n})\|^2_{V(\Omega)} \leqslant \langle \mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \leqslant C\|u_{h_n}(e_{h_n})\|_{V(\Omega)}$$

Hence, we conclude

$$\|u_{h_n}(e_{h_n})\|_{V(\Omega)} \leqslant C \qquad \text{for} \quad h_n \to 0^+ \quad (4.7)$$

Thus, a subsequence $\{u_{h_{n_k}}(e_{h_{n_k}})\}_{k\in N}$ exists such that

$$u_{h_{n_k}}(e_{h_{n_k}}) \to u_* \qquad \text{weakly in} \;\; V(\Omega) \quad (4.8)$$

By virtue of Lemma 9, we have: $u_* \in \mathcal{K}(\Omega)$. On the other hand, we substitute: $v_{h_k} = \theta_{h_k}$ into (4.3). Then, one has

$$\begin{aligned}&\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}),\, \theta_{h_k} - u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \geqslant \\ &\geqslant \langle \mathcal{O}+\omega_1 H_0 + 2\omega_2 e_{h_{n_k}},\, \widetilde{O}\theta_{h_k} - \widetilde{O}u_{h_{n_k}}(e_{h_{n_k}})\rangle_{\mathcal{H}(\Omega)}\end{aligned} \quad (4.9)$$

Notice, that the form $\langle \mathcal{A}(e)z, z\rangle_{V(\Omega)}$ being positive definite in $V(\Omega)$ for $e \in U_{ad}(\Omega)$. Hence, the functional $\mathcal{G}(v) = \langle \mathcal{A}(e)v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous, so that

$$\liminf_{k\to\infty}\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}),\; u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} \geqslant \langle \mathcal{A}(e)u_*, u_*\rangle_{V(\Omega)} \quad (4.10)$$

Taking into account the relation

$$\begin{aligned}&\Big|\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}),\; u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}),\; u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)}\Big| \leqslant \\ &\leqslant C\Big[\|e_{h_{n_k}} - e\|_{C(\overline{\Omega})} + \|e^2_{h_{n_k}} - e^2\|_{C(\overline{\Omega})} + \|e^3_{h_{n_k}} - e^3\|_{C(\overline{\Omega})}\Big]\cdot \quad (4.11) \\ &\cdot\|u_{h_{n_k}}(e_{h_{n_k}})\|_{V(\Omega)} \to 0\end{aligned}$$

we deduce that

$$\begin{aligned}&\liminf_{k\to\infty}\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} = \\ &= \liminf_{k\to\infty}\Big\{\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} + \\ &+\Big[\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} - \\ &-\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)}\Big]\Big\} \geqslant \langle \mathcal{A}(e)u_*, u_*\rangle_{V(\Omega)}\end{aligned} \quad (4.12)$$

Next in view of (4.9) and (4.12), we can write

$$-\langle \mathcal{A}(e)u_*, u_*\rangle_{V(\Omega)} \geqslant \limsup_{k\to\infty} \langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), -u_{h_{n_k}}(e_{h_{n_k}})\rangle_{V(\Omega)} \geqslant$$
$$\geqslant \limsup_{k\to\infty} \Big\{ -\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), \theta_{h_k}\rangle_{V(\Omega)} + \tag{4.13}$$
$$+\langle \mathcal{O} + \omega_0 H_0 + 2\omega_1 e_{h_{n_k}}, \widetilde{O}\theta_{h_k} - \widetilde{O}u_{h_{n_k}}(e_{h_{n_k}})\rangle_{\mathcal{H}(\Omega)} \Big\}$$

By virtue of (4.7) and (4.8), we obtain (for $h_{n_k} \to 0^+$)

$$|\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), o\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_*, o\rangle_{V(\Omega)}| \leqslant$$
$$\leqslant |\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), o\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), o\rangle_{V(\Omega)}| +$$
$$+|\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), o\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_*, o\rangle_{V(\Omega)}| \leqslant$$
$$\leqslant \int_\Omega \Big[\Big| (D_{11}(e_{h_{n_k}}) - D_{11}(e)) \frac{\partial^2 u_{h_{n_k}}(e_{h_{n_k}})}{\partial x^2} \frac{\partial^2 o}{\partial x^2} \Big| +$$
$$+ \Big| (D_{22}(e_{h_{n_k}}) - D_{22}(e)) \frac{\partial^2 u_{h_{n_k}}(e_{h_{n_k}})}{\partial y^2} \frac{\partial^2 o}{\partial y^2} \Big| +$$
$$+2 \Big| (D_{33}(e_{h_{n_k}}) - D_{33}(e)) \frac{\partial^2 u_{h_{n_k}}(e_{h_{n_k}})}{\partial x \partial y} \frac{\partial^2 o}{\partial x \partial y} \Big| +$$
$$+ \Big| (D_{12}(e_{h_{n_k}}) - D_{12}(e)) \Big(\frac{\partial^2 u_{h_{n_k}}(e_{h_{n_k}})}{\partial x^2} \frac{\partial^2 o}{\partial y^2} + \frac{\partial^2 u_{h_{n_k}}(e_{h_{n_k}})}{\partial y^2} \frac{\partial^2 o}{\partial x^2} \Big) \Big| \Big] \, d\Omega +$$
$$+ \Big| \int_\Omega \Big[D_{11}(e) \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial x^2} \frac{\partial^2 o}{\partial x^2} + D_{22}(e) \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial y^2} \frac{\partial^2 o}{\partial y^2} +$$
$$+ D_{12}(e) \Big(\frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial x^2} \frac{\partial^2 o}{\partial y^2} + \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial y^2} \frac{\partial^2 o}{\partial x^2} \Big) +$$
$$+2 D_{33}(e) \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial x \partial y} \frac{\partial^2 o}{\partial x \partial y} \Big] \, d\Omega \Big| \leqslant$$
$$\leqslant C \mathcal{N}_{n_k} \|u_{h_{n_k}}(e_{h_{n_k}})\|_{V(\Omega)} \|o\|_{V(\Omega)} + \max[D_{11}(e), D_{22}(e), D_{12}(e), D_{33}(e)] \cdot$$
$$\cdot \Big| \int_\Omega \Big(\frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial x^2} \frac{\partial^2 o}{\partial x^2} + \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial y^2} \frac{\partial^2 o}{\partial y^2} +$$
$$+ \frac{\partial^2 [u_{h_{n_k}}(e_{h_{n_k}}) - u_*]}{\partial x \partial y} \frac{\partial^2 o}{\partial x \partial y} \Big) \, d\Omega \Big| \to 0$$

Indeed, here the constant being positive and

$$\mathcal{N}_{n_k} = \max \left\{ \sup_{i,j} \sup_{x,y\in\overline{\Omega}} |D_{ij}(e_{h_{n_k}}) - D_{ij}(e)|, \quad i,j = 1,2, \right. \\ \left. \sup_{x,y\in\overline{\Omega}} |D_{33}(e_{h_{n_k}}) - D_{33}(e)| \right\} \tag{4.14}$$

Here, we have $D_{11}, D_{22}, D_{12}, D_{33} \in C([e_{\min}, e_{\max}])$, then the Schwartz Theorem (Litvinov, 2000) imply: $t \to [D_{ij}(t), D_{33}(t)]$ to be a uniformly continuous mapping of the interval $[e_{\min}, e_{\max}]$ into $\mathbb{R}$. Hence, by (4.14) (as $e_{n_k} \to e$ strongly in $U(\Omega)$) we obtain $\lim_{k\to\infty} \mathcal{N}_{n_k} = 0$.

The weak convergence of $\{u_{h_{n_k}}(e_{h_{n_k}})\}_{k\in N}$ yield that

$$\langle \mathcal{A}(e)u_{h_{n_k}}(e_{h_{n_k}}), o\rangle_{V(\Omega)} \to \langle \mathcal{A}(e)u_*, o\rangle_{V(\Omega)} \qquad \forall o \in V(\Omega) \quad \text{as } h_{n_k} \to 0^+ \tag{4.15}$$

Making use of (4.7) and (4.4), we obtain $(h_{n_k} \to 0^+)$

$$|\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), \theta_{h_k} - v\rangle_{V(\Omega)} \leqslant C\|u_{h_{n_k}}(e_{h_{n_k}})\|_{V(\Omega)}\|\theta_{h_k} - v\|_{V(\Omega)} \to 0 \tag{4.16}$$

Further, due to (4.16) and (4.15), we derive

$$|\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), \theta_{h_k}\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_*, v\rangle_{V(\Omega)}| \leqslant \\ \leqslant |\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), \theta_{h_k} - v\rangle_{V(\Omega)}| + \\ + |\langle \mathcal{A}(e_{h_{n_k}})u_{h_{n_k}}(e_{h_{n_k}}), v\rangle_{V(\Omega)} - \langle \mathcal{A}(e)u_*, v\rangle_{V(\Omega)}| \to 0 \tag{4.17}$$

By virtue of (4.4) and (4.8), we may write

$$\langle \mathcal{O} + \omega_1 H_0, \widetilde{O}\theta_{h_k} - \widetilde{O}u_{h_{n_k}}(e_{h_{n_k}})\rangle_{(\Omega)} \to \langle \mathcal{O} + \omega_1 H_0, \widetilde{O}v - \widetilde{O}u_*\rangle_{(\Omega)}$$

Now, taking into account (4.13), (4.17) and (4.8), we arrive at (coming back to variational inequality (4.9) and passing to limes inferior or limes superior with $h_{n_k} \to 0^+$)

$$-\langle \mathcal{A}(e)u_*, u_*\rangle_{V(\Omega)} \geqslant -\langle \mathcal{A}(e)u_*, v\rangle_{V(\Omega)} + \langle \mathcal{O} + \omega_1 H_0 + \omega_2 e, \widetilde{O}v - \widetilde{O}u_*\rangle_{\mathcal{H}(\Omega)}$$

Thus, u_* is a solution to state inequality (2.14) for any $v \in \mathcal{K}(e, \Omega)$. Here, from the uniqueness of $u(e)$ (by virtue of (3.17)), we conclude that $u_* = u(e)$ and the whole sequence $\{u_{h_n}(e_{h_n})\}_{n\in N}$ tends to $u(e)$ weakly in $V(\Omega)$, as $h_n \to 0^+$.

It remains to prove the strong convergence. Here from (4.6) and (4.8), we conclude that

$$\lim_{n\to\infty}\langle\mathcal{A}(e_{h_n})u_{h_n}(e_{h_n}),u_{h_n}(e_{h_n})\rangle_{V(\Omega)}=\langle\mathcal{O}+\omega_1H_0+\omega_2e,\tilde{O}u(e)\rangle_{\mathcal{H}(\Omega)}=$$
$$=\langle\mathcal{A}(e)u(e),u(e)\rangle_{V(\Omega)}$$

On the other hand by virtue of (4.11), we can write

$$\lim_{n\to\infty}\langle\mathcal{A}(e)u_{h_n}(e_{h_n}),u_{h_n}(e_{h_n})\rangle_{V(\Omega)}=\langle\mathcal{A}(e)u(e),u(e)\rangle_{V(\Omega)} \tag{4.18}$$

Here, the bilinear form $\langle\mathcal{A}(e)\cdot,\cdot\rangle_{V(\Omega)}$ can be taken for a scalar product in $V(\Omega)$. Thus, due to (4.18) and the weak convergence of $\{u_{h_n}(e_{h_n})\}_{n\in N}$, we deduce that

$$\lim_{n\to\infty}\langle\mathcal{A}(e)[u_{h_n}(e_{h_n})-u(e)],u_{h_n}(e_{h_n})-u(e)\rangle_{V(\Omega)}=0$$

which, in turn, implies that: $u_{h_n}(e_{h_n})\to u(e)$ strongly in $V(\Omega)$.

Lemma 12. Let $\{e_{h_n}\}_{n\in N}$, $h_n\to 0^+$ be a sequence of $e_{h_n}\in U_{ad_h}(\Omega)$ such that $e_{h_n}\to e$ strongly in $U(\Omega)$, as $h_n\to 0^+$.

Then, one has

$$\mathbf{M}_{h_n}(e_{h_n})\to\mathbf{M}(e)\qquad\text{strongly in } [L_2(\Omega)]^4 \tag{4.19}$$

Proof. By using the inclusion: $U_{ad_{h_n}}(\Omega)\subset U_{ad}(\Omega)$ and Lemma 4, we conclude that

$$\|\mathbf{M}(e_{h_n})-\mathbf{M}(e)\|_{[L_2(\Omega)]^4}\to 0$$

Further, relations (4.2) and Lemma 11 yield

$$\|\mathbf{M}_{h_n}(e_{h_n})-\mathbf{M}(e_{h_n})\|_{[L_2(\Omega)]^4}\to 0$$

Then, in view of the triangle inequality, we obtain

$$\|\mathbf{M}_{h_n}(e_{h_n})-\mathbf{M}(e)\|_{[L_2(\Omega)]^4}\leqslant\|\mathbf{M}_{h_n}(e_{h_n})-\mathbf{M}(e_{h_n})\|_{[L_2(\Omega)]^4}+$$
$$+\|\mathbf{M}(e_{h_n})-\mathbf{M}(e)\|_{[L_2(\Omega)]^4}\to 0$$

as $h_n\to 0^+$.

Lemma 13. We have

$$\mathcal{L}_{\varepsilon,wg}(e_{h_n},\mathbf{M}_{h_n}(e_{h_n}))\to\mathcal{L}_{\varepsilon,wg}(e,\mathbf{M}(e))\qquad\text{as}\quad h_n\to 0^+$$

Proof. The proof is analogous to that of Lemma 5, being based on Lemma 12.

Lemma 14. For any $e \in U_{ad}(\Omega)$ there exists a sequence $\{o_{h_n}\}_{n\in N}$, $h_n \to 0^+$, such that $o_{h_n} \in U_{ad_{h_n}}(\Omega)$ and $o_{h_n} \to e$ strongly in $U(\Omega)$, as $h_n \to 0^+$.

Proof. Here, we introduce the parallelogram Ω and use the skew coordinates $([\xi,\eta])$ via mapping (4.1). Let $\Omega = \mathcal{F}(\Omega_0)$, $\Omega_0 = (0, L_A) \times (0, L_B)$, $h_1 = L_A/m$, $h_2 = L_B/n$. Further, denote by $\mathcal{H}_{ij}$ the grid points with the coordinates: $\xi = ih_1$, $\eta = jh_2$, $i, j = 0, 1, , ..., m$

$$\widetilde{O}^0_{ij} = [(i-1)h_1, ih_1] \times [(j-1)h_2, jh_2] \qquad \widetilde{O}_{ij} = \mathcal{F}(\widetilde{O}^0_{ij})$$

$$\mathcal{O}^0_{ij} = \Big[\Big(i-\frac{1}{2}\Big)h_1, \Big(i+\frac{1}{2}\Big)h_1\Big] \times \Big[\Big(j-\frac{1}{2}\Big)h_2, \Big(j+\frac{1}{2}\Big)h_2\Big] \cap \Omega_0$$

$$\mathcal{O}_{ij} = \mathcal{F}(\mathcal{O}^0_{ij})$$

From this, we have: $\mathcal{O}_{ij}$ is a neighbourhood of the point $\mathcal{F}(\mathcal{H}_{ij})$. Here, we set

$$o_h(\mathcal{F}(\mathcal{H}_{ij})) = \frac{1}{\operatorname{mes} \mathcal{O}_{ij}} \int\limits_{\mathcal{O}_{ij}} e(x,y)\, d\Omega \qquad \begin{matrix} 0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n \end{matrix} \tag{4.20}$$

Next, we interpolate nodal values (4.20) by functions from $Q_1(\widetilde{O}_{ij})$. Hence, we obtain $o_h \in U_{ad_h}(\Omega)$. We can write

$$\int\limits_{\mathcal{O}_{ij}} o_h\, d\Omega = \frac{1}{4} \operatorname{mes} \widetilde{O}_{ij} \sum_{k=1}^{4} o_h(\mathcal{H}^k_{ij})$$

where $\mathcal{H}^k_{ij}$ are the vertices of the parallelogram $\widetilde{O}_{ij}$. Introduce the notation: $\mathcal{S}_{ij}$ (denote the union of all parallelograms $\widetilde{O}_{ij}$ which are adjacent to the node $\mathcal{F}(\mathcal{H}_{ij})$), then we have

$$\int\limits_{\Omega} o_h\, d\Omega = \sum_{i=1}^{m}\sum_{j=1}^{n} \int\limits_{\widetilde{O}_{ij}} o_h\, d\Omega = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{4} \operatorname{mes} \widetilde{O}_{ij} \sum_{k=1}^{4} o_h(\mathcal{F}(\mathcal{H}^k_{ij})) =$$

$$= \sum_{i=0}^{m}\sum_{j=1}^{n} o_h(\mathcal{F}(\mathcal{H}_{ij})) \frac{1}{4} \operatorname{mes} \mathcal{S}_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{\operatorname{mes} \mathcal{S}_{ij}}{4 \operatorname{mes} \mathcal{O}_{ij}} \int\limits_{\mathcal{O}_{ij}} e\, d\Omega = \int\limits_{\Omega} e\, d\Omega$$

since $\operatorname{mes} \mathcal{S}_{ij} = 4 \operatorname{mes} \mathcal{O}_{ij}$, $\bigcup_{i,j} \overline{\mathcal{O}_{ij}} = \Omega$.

Further, we introduce the functions $\widetilde{o} = e \circ \mathcal{F}$, $\widetilde{o}_h = e_h \circ \mathcal{F}$. Then, we can transform (4.20) into the formula

$$\widetilde{o}_h(\mathcal{H}_{ij}) = \frac{1}{\operatorname{mes} \mathcal{O}^0_{ij}} \int\limits_{\mathcal{O}^0_{ij}} \widetilde{o} \, d\xi d\eta \tag{4.21}$$

Next, take the system $[\xi, \eta]$ as a skew system, parallel with the edges of Ω. From this identification it follows

$$\frac{\partial e}{\partial \xi} = \frac{\partial \widetilde{o}}{\partial \xi} \qquad \frac{\partial e}{\partial \eta} = \frac{\partial \widetilde{o}}{\partial \eta} \qquad \frac{\partial o_h}{\partial \xi} = \frac{\partial \widetilde{o}_h}{\partial \xi} \qquad \frac{\partial o_h}{\partial \eta} = \frac{\partial \widetilde{o}_h}{\partial \eta}$$

for the corresponding points.

Let us extend $\widetilde{o}$ onto a rectangle $(-h_1/2, L_x + h_1/2) \times (-h_2/2, L_y + h_2/2)$, so that the extension $e_0 = \widetilde{o}$ in Ω_0 and e_0 is symmetric with respect to the sides, namely

$$e_0(L_x + s, \eta) = e_0(L_x - s, \eta) \qquad \forall \eta \in \Big(-\frac{h_2}{2}, L_y + \frac{h_2}{2}\Big) \quad s \in \Big(0, \frac{h_1}{2}\Big)$$

and similarly along the other sides of $\partial\Omega_0$. Taking that into account, we can write instead of (4.21)

$$\widetilde{o}_h(\mathcal{H}_{ij}) = \frac{1}{h_1 h_2} \int\limits_{S^0_{ij}} e_0 \, d\xi d\eta \qquad \begin{array}{l} 0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n \end{array} \tag{4.22}$$

where S^0_{ij} denotes the (complete) rectangle with the center $\mathcal{H}_{ij}$ and the lengths of sides h_1, h_2.

Further, we have

$$\begin{aligned} \frac{1}{h_1} |\widetilde{o}_h(\mathcal{H}_{i+1,j}) - \widetilde{o}_h(\mathcal{H}_{i,j})| &= \frac{1}{h_1^2 h_2} \Big| \int\limits_{S^0_{i+1,j}} e_0 \, d\xi d\eta - \int\limits_{S^0_{ij}} e_0 \, d\xi d\eta \Big| = \\ = \frac{1}{h_1 h_2} \Big| \int\limits_{S^0_{ij}} \frac{1}{h_1} [e_0(\xi + h_1, \eta) - e_0(\xi, \eta)] \, d\xi d\eta \Big| &\leqslant \frac{1}{h_1 h_2} C_\xi \operatorname{mes} S^0_{ij} = C_\xi \end{aligned} \tag{4.23}$$

here we use the fact that $|\partial e_0 / \partial \xi| \leqslant C_\xi$ holds almost everywhere. It follows from $\widetilde{o}_h \in Q_1(\widetilde{O}^0_{ij})$ in $\widetilde{O}^0_{ij}$, the derivative $\partial \widetilde{o}_h / \partial \xi$ attains its maximum at the boundary $\partial \widetilde{O}^0_{ij}$. Then, in view of (4.23), we get the estimate: $|\partial \widetilde{o}_h / \partial \xi| \leqslant C_\xi$ for any $[\xi, \eta] \in \Omega$.

Moreover, the upper bound C_η can be derived in a parallel way. Here we note that the maximum of $\widetilde{o}_h$ in $\widetilde{O}^0_{ij}$ is attained at some vertex of $\widetilde{O}^0_{ij}$. Then, in view of (4.21), we easily verify that: $e_{\min} \leqslant o_h(x,y) \leqslant e_{\max}$ for any $[x,y] \in \overline{\Omega}$. Hence, we have proven that $o_h \in U_{ad_h}(\Omega)$.

Notice, that in order to get a convergence of $\{o_{h_n}\}_{n\in N}$ we consider an arbitrary point $[x,y] \in \overline{\Omega}$ and we can write (for $[\xi,\eta] = \mathcal{F}^{-1}(x,y) \in \widetilde{O}^0_{ij}$)

$$|o_{h_n}(x,y) - e(x,y)| = \Big|\sum_{k=1}^{4} \widetilde{o}_{h_n}(\mathcal{H}^k_{ij})^* \omega_k(\xi,\eta) - \sum_{k=1}^{4} e(\xi,\eta)\omega_k(\xi,\eta)\Big|$$

where ω_k are the shape functions of $Q_1(\widetilde{O}^0_{ij})$ (here one has $\omega_k(\mathcal{H}^m_{ij}) = \delta_{km}$ at the vertices). Hence, in view of (4.21), we obtain

$$\begin{aligned} |o_{h_n}(x,y) - e(x,y)| &\leqslant \sum_{k=1}^{4} |\widetilde{o}_{h_n}(\mathcal{H}^k_{ij}) - \widetilde{o}(\xi,\eta)|\omega_k(\xi,\eta) \\ &= \sum_{k-1}^{4} \Big|\frac{1}{h_1h_2}\int\limits_{S^{0k}_{ij}} e_0(s_1,s_2)\,ds_1ds_2 - \frac{1}{h_1h_2}\int\limits_{S^{0k}_{ij}} \widetilde{o}(\xi,\eta)\,ds_1ds_2\Big|\omega_k(\xi,\eta) \leqslant \\ &\leqslant \sum_{k-1}^{4} \frac{1}{h_1h_2}\int\limits_{S^{0k}_{ij}} |e_0(s_1,s_2) - \widetilde{o}(\xi,\eta)|\,ds_1ds_2 \end{aligned} \tag{4.24}$$

where S^{0k}_{ij} denotes the rectangle with the center at $\mathcal{H}^k_{ij}$ and $\text{mes}S^{0k}_{ij} = (h_1h_2)$.

On the other hand, we get

$$\begin{aligned} &|e_0(s_1,s_2) - \widetilde{o}(\xi,\eta)| = |e_0(s_1,s_2) - e_0(\xi,\eta)| \leqslant \\ &\leqslant |e_0(s_1,s_2) - e_0(\xi,s_2)| + |e_0(\xi,s_2) - e_0(\xi,\eta)| \leqslant \frac{3}{2}(h_1C_\xi + h_2C_\eta) \end{aligned} \tag{4.25}$$

Then, due to (4.25) and (4.24), we have

$$|o_{h_n}(x,y) - e(x,y)| \leqslant 12h\max(C_\xi, C_\eta)$$

which completes the proof.

Theorem 4. Let $\{e_{\varepsilon_{h_n},wg}\}_{n\in N}$, $h_n \to 0^+$ be a sequence of solutions to the approximate optimal control problem $(\mathcal{P}_{\varepsilon_{h_n}})$. Then, there exists a subsequence $\{e_{\varepsilon_{h_{n_k}}}\}_{k\in N} \subset \{e_{\varepsilon_{h_n}}\}_{n\in N}$ and an element $e_{\varepsilon^*} \in U_{ad}(\Omega)$ such that

$$\begin{cases} e_{\varepsilon_{h_{n_k}}} \to e_{\varepsilon^*} & \text{strongly in } U(\Omega) \\ \mathbf{M}_{h_{n_k}}(e_{\varepsilon_{h_{n_k}}}) \to \mathbf{M}(e_\varepsilon) & \text{strongly in } [L_2(\Omega)]^4 \end{cases} \tag{4.26}$$

and e_{ε^*} is the solution to the penalized optimal control problem $(\mathcal{P}_\varepsilon)$. Each uniformly convergent subsequence $\{e_{\varepsilon_{h_n}}\}_{n\in N}$ tends to the solution of $(\mathcal{P}_\varepsilon)$, and $(4.26)_2$ holds.

Proof. Here we have: $U_{ad_h}(\Omega) \subset U_{ad}(\Omega)$ and $U_{ad}(\Omega)$ is compact in $U(\Omega)(= C(\overline{\Omega}))$. Hence, there exists a subsequence of $\{e_{\varepsilon_{h_n}}\}_{n\in N}$ such that $(4.26)_1$ holds with $e_\varepsilon \in U_{ad}(\Omega)$. Then, from Lemma 12, we obtain $(4.26)_2$. In the following, we prove that e_ε is a solution to the problem $(\mathcal{P}_\varepsilon)$. Consider any $e \in U_{ad}(\Omega)$ and apply Lemma 14 to obtain $\{o_{h_{n_k}}\}_{k\in N}$, $o_{h_{n_k}} \in U_{ad_{h_{n_k}}}(\Omega)$, such that $o_{h_{n_k}} \to e$ strongly in $U(\Omega)$.

Now, the definition $(\mathcal{P}_\varepsilon)$ implies that

$$\mathcal{L}_{\varepsilon,wg}(e_\varepsilon, \mathbf{M}(e_\varepsilon)) \to \mathcal{L}_{\varepsilon,wg}(e, \mathbf{M}(e)) \tag{4.27}$$

Thus, the element e_ε is the solution to the problem $(\mathcal{P}_\varepsilon)$ which completes the proof.

References

1. ADAMS R.A., 1975, *Sobolev Spaces*, Academic Press, New York
2. BARBU V., 1987, *Optimal Control of Variational Inequalities*, Pitman Advanced Publishing Program, London
3. BOCK I., LOVÍŠEK J., 2001, On the minimum of the work of interaction forces between a pseudoplate and a rigid obstacle, *Mathematica Bohemica*, **126**, 2, 281-292
4. COMODI M.I., 1985, Approximation of bending plate problem with a unilateral inner obstacle, *Numerische Mathematik*, **47**, 435-458
5. GLOWINSKI R., LIONS J.L., TRÉMOLIÉRS R., 1981, *Numerical Analysis of Variational Inequalities*, North Holland, Amsterdam
6. HASLINGER J., NEITTAANMÄKI, 1988, *Finite Element Approximation for Optimal Shape Design. Theory and Applications*, John Wiley, New York
7. HLAVÁČEK I., 1994, A Mixed finite element method for plate bending with a unilateral inner obstacle, *Applications of Mathematics*, **39**, 1, 25-44

8. KHLUDNEV A.M., SOKOLOWSKI J., 1997, *Modelling and Control in Solid Mechanics*, Birkhäuser Verlag, Basel

9. LIONS J.L., 1969, *Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires*, Dunond, Paris

10. LITVINOV W.G., 2000,
it Optimization in Elliptic Problems with Applications to Mechanics of Deformable Bodies and Fluid Mechanics, Birkhäuser Verlag, Basel

11. MYSLINSKI A., SOKOLOWSKI J., 1985, Nondifferentiable optimization problems for elliptic systems, *SIAM J. Control Optimization*, **23**, 632-648

12. MOSCO U., 1971, An introduction to the approximate solutions of variational inequalities. In: *Constructive Aspects of Functional Analysis*, Vol. II, edit. G. Geymant, Centro Internaz. Matem., 497-682

13. ODEN J.T., REDDY J.N., 1976, *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley, New York

14. PANAGIOTOPOULOS P.D., 1985, *Inequality Problems in Mechanics and Applications*, Birkhäuser Verlag, Boston

15. SCHWARTZ L., 1966, *Théorie des Distributions*, Hermann, Paris

Dobór parametrów w zagadnieniu trójwarstwowej płyty z wewnętrzną podporą

Streszczenie

W pracy zajęto się problemem doboru optymalnej grubości trójwarstwowej płyty (z pominięciem naprężeń stycznych w warstwie środkowej) w zbiorze ograniczonych, ciągłych funkcji Lipschitza. Zmienna grubość warstwy zewnętrznej jest optymalizowana poprzez minimalizację ciężaru przy pewnych ograniczeniach narzuconych na maksymalne naprężenia. Zastosowane funkcjonały kosztu reprezentują: 1) ciężar trójwarstwowej płyty, 2) dodatni rozkład (nieujemna miara Radona). Zagadnienie stanu opisano nierównością wariacyjną oraz uwzględniono wpływ zmiennych konstrukcyjnych na współczynniki i zbiór dopuszczalnych funkcji. Udowodniono istnienie optymalnej grubości warstwy i przedstawiono analizę zbieżności przybliżonego problemu optymalizacji metodą funkcji kary. Wykazano istnienie rozwiązania przy minimalizacji ciężaru płyty i minimalizacji pracy sił oddziaływania pomiędzy płytą a podporą na podstawie ogólnego twierdzenia o doborze parametrów w układach opisanych nierównościami wariacyjnymi.

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