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THERMODIFFUSION IN PERIODICALLY LAYERED ELASTIC COMPOSITES

Andrzej Kaczyński

Faculty of Mathematics and Information Science, Warsaw University of Technology
e-mail: akacz@alpha.mini.pw.edu.pl

Stanisław J. Matysiak

Institute of Hydrogeology and Engineering Geology, Faculty of Geology
University of Warsaw
e-mail: matysiak@geo.uw.edu.pl

The paper deals with modelling of thermodiffusion processes in periodically layered elastic composites. The study is based on the homogenized procedure with microlocal parameters and on the relations of the linear coupled theory of diffusion in thermoelastic homogeneous bodies. A useful homogenized model with microlocal parameters accounting for certain local effects of heat and diffusion fluxes, as well as stresses is derived. The equations of the homogenized model are given in terms of unknown macroconcentration, macrotemperature and macrodisplacements and extra unknowns called the microlocal parameters. An illustrating example of the application of the presented model is solved.

Key words: thermodiffusion, elasticity, layered composite, concentration, temperature, displacement

1. Introduction

The diffusion processes in deformable non-porous solids play an important role in many technological applications (for instance: in semi-conductor manufacturing (doping), isolation of radioactive and chemical wastes). Two important factors distinguish the transport in thermoelastic bodies from the classical diffusion:

- coupling between the substance concentration and the deformation of solids,
- coupling between the concentration and the temperature distribution in the bodies.

The penetration of diffused substance is dependent on the state of strain and temperature as well as the diffused substance in solids can lead to strains and stresses. The problems of interactions between thermodiffusion processes and deformations in homogeneous bodies were considered in many papers (see for instance Pidstrygach, 1961; Pidstrygach and Shevthyk, 1969; Nowacki, 1971, 1974; Kubik, 1986; Plavšić and Naerlovic-Veljhovic, 1975, and the monograph by Nowacki and Olesiak, 1991).

The nonhomogeneous bodies with periodically layered structures can be made by man (layered composites) as well as can be found in nature (varved clays, sandstone-slote, sandstone-shale, thin-layered limestone). The problems of modelling of thermomechanical processes in periodically stratified composites are very important in many branches of engineering, geotechnical and geophysical investigations. The thermoelasticity of the composite materials have been given wide attention (see, for example monographs Christensen, 1980; Jones, 1975; Achenbach, 1975; Bakhalov and Panasenko, 1984; Bensonssan at al., 1978; Guz et al., 1982; Tsai and Hahn, 1980; Sanchez-Palenica, 1980; Pobedria, 1984; Vanin, 1985; Broutman and Krock, 1973-76; Woźniak and Woźniak, 1995, and references therein). A variety of exact, approximate and purely numerical methods are available for the solutions of boundary value problems of laminated bodies. However, in the case of composites with a large number of repeated layers, it seems to be suitable to use the homogenized models. One of them is the homogenized model with microlocal parameters given in papers by Woźniak (1987); Matysiak and Woźniak (1988) for elastic and thermoelastic composites. These models have been derived by using the concepts of the nonstandard analysis combined with some postulated *a priori*, heuristic physical assumptions. The governing equations of the homogenized models are expressed in terms of unknown macrodeformations, macrotemperatures and certain extra unknowns called microlocal parameters. The microlocal parameters make it possible to evaluate not only the mean but also local values of deformation and temperature gradients, stresses and heat fluxes in every material component of the composite.

In this paper the problem of modelling of thermodiffusion in elastic periodically layered composites is investigated. The considerations are based on the linear theory of thermodiffusion for elastic homogeneous bodies and the homogenization procedure with microlocal parameters presented in Woźniak (1987);

Matysiak and Woźniak (1988) for linear and nonlinear thermoelastic periodic composites. The derived homogenized model is given in terms of unknown macroconcentration, macrotemperature and macrodisplacements, as well as certain extra unknowns called the microlocal parameters. The macroconcentration, macrotemperature and macrodisplacements represent the averaging values of the concentration, temperature and displacements in the layered composite. The microlocal parameters (diffusional, thermal and kinematical) are related with the periodic structure of the body. The bodies under consideration are assumed to be composed of periodically repeated $(n+1)$ different homogeneous isotropic thermoelastic layers. The perfect contact between the layers being components of the composite is assumed.

Section 2 contains equations of the linear theory of thermodiffusion in elastic solids. The weak form of equations of this theory is also presented.

In Sections 3 equations of the homogenized model is derived by using the homogenization procedure with microlocal parameters. The case of periodic two-layered composites is also discussed. In this case the microlocal parameters are eliminated from the equations of homogenized model. So, the homogenized model is expressed in terms of macroconcentration, macrotemperature and macrodisplacements. Section 4 contains some illustrating example of the obtained homogenized model. In Section 5 some resuming remarks are presented.

2. Basic equations

Consider a thermoelastic nonhomogeneous body, which in a natural (undeformed) configuration is composed of periodically repeated, $(n+1)$ different isotropic homogeneous layers, see Fig. 1. Let the body occupy a regular region B in the Euclidean 3-space referred to a fixed Cartesian coordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$. Let the axis x_3 be normal to the layering. Let $h_1, ..., h_{n+1}$ denote the thickness of layers and $h = h_1 + ... + h_{n+1}$ denote the thickness of each basic, repeated unit of the body. Let $\lambda^{(r)}$, $\mu^{(r)}$, $r = 1, ..., n+1$ denote the Lamé constants, $D^{(r)}$ denote the diffusion coefficients, $\chi^{(r)}$ and $\gamma_c^{(r)}$ denote the coupling coefficients of diffusion and stresses, $k^{(r)}$ denote the coefficients of thermal diffusivities, $\gamma_T^{(r)}$ and $k_e^{(r)}$ denote the coupling coefficients of temperature and stresses, $\beta^{(r)}$ and $k_c^{(r)}$ denote the coupling coefficients of diffusion and temperature. Let t denote time, $\boldsymbol{u}(\boldsymbol{x}, t) = (u_1, u_2, u_3)(\boldsymbol{x}, t)$ denote the displacements vector, $\sigma_{ij}^{(r)}$, $r = 1, ..., n+1$, $i, j = 1, 2, 3$, denote the components of stress tensor, $\eta_i^{(r)}$ denote the components of diffusion fluxes, in

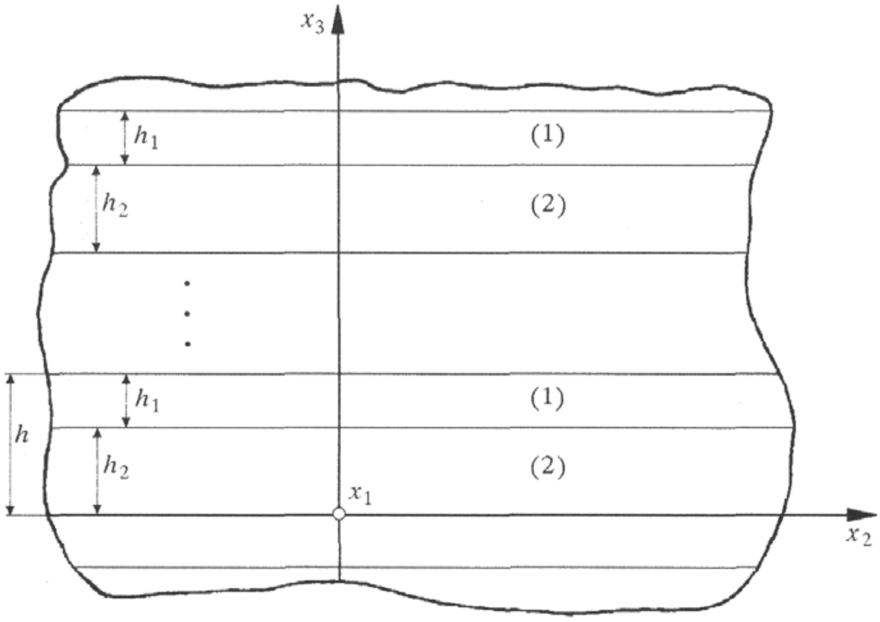

Fig. 1. Scheme of periodically stratiefied elastic body

the rth layer of the fundamental repeated lamina (next called the rth layer). Moreover, let $c = c(\boldsymbol{x}, t)$ be the concentration of the diffusing substance, $\theta = \theta(\boldsymbol{x}, t)$ be the temperature as well as X_i; $i = 1, 2, 3$, denote the components of body forces, σ and W_0 denote the internal diffusive and thermal sources. The layers being the components of composite are isotropic, homogeneous and thermoelastic. The restriction connected with isotropic components of composites can be easily omitted. However, the experimental methods for determination of all material constants of the theory of thermodiffusion for elastic bodies in the case of anisotropy is not known; we confine our attention to isotropic components of the composites. So, the generalized Fick, Fourier and Hooke's laws for the rth layer, $r = 1, ..., n+1$, take the following form (cf. Pidstrygach and Shevthyk, 1969; Nowacki, 1971, 1974; Nowacki and Olesiak, 1991; Olesiak and Pyryev, 1995)

$$\begin{aligned} \eta_i^{(r)} &= -D^{(r)} c_{,i} - \beta^{(r)} \theta_{,i} + \chi^{(r)} u_{k,ki} \\ q_i^{(r)} &= -d^{(r)} c_{,i} - \Lambda^{(r)} \theta_{,i} + \Gamma^{(r)} u_{k,ki} \\ \sigma_{ij}^{(r)} &= 2\mu^{(r)} \varepsilon_{ij} + (\lambda^{(r)} \varepsilon_{kk} - \gamma_T^{(r)} \theta - \gamma_c^{(r)} c) \delta_{ij} \end{aligned} \tag{2.1}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2}$$

and $d^{(r)}$, $\Gamma^{(r)}$, $\Lambda^{(r)}$, $r = 1, ..., n+1$, are constants.

The system of equations governing the processes of thermodiffusion in deformable, elastic rth layer takes the following form (cf. Olesiak and Pyryev (1995))[1]

— generalized Navier equations

$$\mu^{(r)}u_{i,jj} + (\lambda^{(r)} + \mu^{(r)})u_{j,ji} + \rho^{(r)}X_i = \rho^{(r)}\ddot{u}_i + \gamma_T^{(r)}\theta_{,i} + \gamma_c^{(r)}c_{,i} \tag{2.3}$$

— a generalized equation of thermal conductivity

$$k^{(r)}\theta_{,ii} + k_c^{(r)}\dot{c} - k_e^{(r)}\dot{u}_{i,i} - \dot{\theta} = -W_0 \tag{2.4}$$

— a generalized equation of thermodiffusion

$$D^{(r)}c_{,ii} - \chi^{(r)}u_{i,ijj} + \beta^{(r)}\theta_{,ii} - \dot{c} = -\sigma \tag{2.5}$$

Remark. The assumption of perfect bounding, perfect thermal and diffusional contact between the layers implies the continuity of the displacement vector, temperature, concentration, stress vector, heat and diffusion fluxes on the interfaces (planes between layers).

Equations (2.3)-(2.5) with (2.1) can be written in the following weak integral form

$$\begin{aligned}
&\sum_{r=1}^{n=1}\int_{B_r}\Big[\sigma_{ij}^{(r)}v_{i,j} - \rho^{(r)}(X_i - \ddot{u}_i)v_i\Big]\,dB = 0\\
&\sum_{r=1}^{n=1}\int_{B_r}\Big(k^{(r)}\theta_{,ii}v_{,i} - k_c^{(r)}\dot{c}v - k_e^{(r)}\dot{u}_i v_{,i} + \dot{\theta}v - W_0 v\Big)\,dB = 0\\
&\sum_{r=1}^{n=1}\int_{B_r}\Big(D^{(r)}c_{,i}s_{,i} - \chi^{(r)}u_{i,ij}s_{,j} + \beta^{(r)}\dot{\theta}_{,i}s_{,i} + \dot{c}s - \sigma s\Big)\,dB = 0
\end{aligned} \tag{2.6}$$

for all test functions $v_i(\cdot)$, $v(\cdot)$, $s(\cdot)$, such that $v_i(\cdot)|_{\partial B} = 0$, $v(\cdot)|_{\partial B} = 0$, $s(\cdot)|_{\partial B} = 0$, and where B_r, $r = 1, ..., n+1$, denotes the part of the region occupied by the rth material.

Since the body is assumed to be periodic, the material coefficients are h-periodic functions taking constant values in the subsequent layers of the body.

[1] The summation convention holds with respect to all repeated indices and $f_{,i} \equiv \partial f/\partial x_i$, $\dot{f} \equiv \partial f/\partial t$

3. Homogenized model with microlocal parameters

The analysis of thermodiffusion processes in periodically layered elastic composites can be derived by using equations (2.3)-(2.5) and (2.1) with adequate initial and boundary conditions. In the cases of sufficiently large number of repeated layers being the components of the nonhomogeneous bodies, the number of boundary conditions on interfaces is also large and then it seems to be more suitable homogenized models. To obtain a homogenized model of the thermodiffusion processes in periodically layered elastic composites described in Section 2, the microlocal modelling approach will be applied. This method is based on concepts of the nonstandard analysis and some postulated *a priori* physical assumptions, and it has been presented in Woźniak (1987); Matysiak and Woźniak (1988) for periodic thermoelastic composites. Making appeal to the microlocal modelling method, we shall derive a homogenized model of thermodiffusion processes in periodically layered elastic bodies omitting the presentation of mathematical assumptions and detailed calculations.

By analogy to the results of papers by Woźniak (1987); Matysiak and Woźniak (1988), the components of displacement vector $u_i(\cdot)$, the concentration $c(\cdot)$ and the temperature $\theta(\cdot)$ are assumed in the form $(a = 1, ..., n, \ i = 1, 2, 3)$

$$
\begin{aligned}
u_i(\boldsymbol{x}, t) &= U_i(\boldsymbol{x}, t) + \underline{f_a(x_3) W_{ia}(\boldsymbol{x}, t)} \\
c(\boldsymbol{x}, t) &= C(\boldsymbol{x}, t) + \underline{f_a(x_3) G_a(\boldsymbol{x}, t)} \\
\theta(\boldsymbol{x}, t) &= T(\boldsymbol{x}, t) + \underline{f_a(x_3) Q_a(\boldsymbol{x}, t)}
\end{aligned}
\tag{3.1}
$$

where $f_a(\cdot) : R \to R$, $a = 1, ..., n$ are know *a priori* h-periodic functions, called the shape functions (cf. Woźniak, 1987) given by

$$
\begin{aligned}
&f_a(x_3) = \begin{cases} x_3 - \dfrac{1}{2}\delta_a & \text{for} \quad 0 \leqslant x_3 \leqslant \delta_a \\[2ex] \dfrac{\delta_a(x_3 - h)}{\delta_a - h} - \dfrac{1}{2}\delta_a & \text{for} \quad \delta_a \leqslant x_3 \leqslant h \end{cases} \\
&f_a(x_3 + h) = f_a(x_3) \qquad x_3 \in R \\
&\delta_a \equiv h_1 + ... + h_a \qquad a = 1, ..., n \\
&h = h_1 + ... + h_{n+1}
\end{aligned}
\tag{3.2}
$$

The functions U_i, C, T are unknown functions interpreted as the components of macrodisplacement, macroconcentraction and macrotemperature. The additional unknown functions W_{ia}, G_a, Q_a stand for the microlocal parameters for displacements concentration and temperature and they are related with microperiodic structure of the body.

Since $|f_a(x_3)| < h$ for every $x_3 \in R$, then for small h the underlined terms in equations (3.1) are small and will be neglected. However, the derivatives $f'_a(\cdot)$, $a = 1, ..., n$ are not small and the terms involving f'_a cannot be neglected. It leads to the following approximations

$$\begin{aligned} &u_{i,\alpha} \approx U_{i,\alpha} && c_{,\alpha} \approx C_{,\alpha} && \theta_{,\alpha} \approx T_{,\alpha} && \alpha = 1,2 \\ &u_{i,t} \approx U_{i,t} && c_{,t} \approx C_{,t} && \theta_{,t} \approx T_{,t} \\ &u_{i,3} \approx U_{i,3} + f'_a W_{ia} && && c_{,3} \approx C_{,3} + f'_a G_a \\ &\theta_{,3} \approx T_{,3} + f'_a Q_a && && a = 1, ..., n \end{aligned} \tag{3.3}$$

Assuming that the test functions v_i, v take the form

$$\begin{aligned} v_i(\boldsymbol{x}, t) &= V_i(\boldsymbol{x}, t) + \underline{f_a(x_3) Z_{ia}(\boldsymbol{x}, t)} \\ v(\boldsymbol{x}, t) &= V(\boldsymbol{x}, t) + \underline{f_a(x_3) Z_a(\boldsymbol{x}, t)} \\ s(\boldsymbol{x}, t) &= S(\boldsymbol{x}, t) + \underline{f_a(x_3) P_a(\boldsymbol{x}, t)} \end{aligned} \tag{3.4}$$

and substituting (3.1), (3.3) and (3.4) into equations (2.1), (2.2) and (2.6), after some calculations similar to those given by Woźniak (1987), we arrive at the following system of equations

$$\begin{aligned} &\langle\mu\rangle U_{i,jj} + \langle\lambda + \mu\rangle U_{j,ji} + \langle\mu f_{a,i}\rangle W_{aj,j} + \langle\mu f_{a,j}\rangle W_{ai,j} + \langle\lambda f_{a,j}\rangle W_{aj,i} - \\ &-\langle\gamma_c\rangle C_{,i} - \langle\gamma_T\rangle T_{,i} + \langle\rho\rangle X_i - \langle\rho\rangle \ddot{U}_i = 0 \\[2ex] &\langle D\rangle C_{,ii} - \langle D f_{a,i}\rangle G_{a,i} - \dot{C} + \sigma - \langle\chi\rangle U_{k,kii} - \\ &-\langle\chi f_{a,k}\rangle W_{ak,ii} + \langle\beta\rangle T_{,ii} + \langle\beta f_{a,i}\rangle Q_{a,i} = 0 \\[2ex] &\langle k\rangle T_{,ii} + \langle k f_{a,i}\rangle Q_{a,i} + \langle k_c\rangle \dot{C} - \langle k_e\rangle \dot{U}_{i,i} - \dot{T} + W_0 = 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &-\Big[\langle\mu f_{b,j}\rangle (U_{i,j} + U_{j,i}) + \langle\lambda f_{b,i}\rangle U_{k,k}\Big] + \langle\gamma f_{b,i}\rangle C + \langle\gamma_T f_{b,i}\rangle T = \\ &= \langle\mu f_{a,j} f_{b,j}\rangle W_{ai} + \langle(\lambda + \mu) f_{a,j} f_{b,i}\rangle W_{aj} \\[2ex] &\langle D f_{a,i} f_{b,i}\rangle G_a + \langle\beta f_{a,i} f_{b,i}\rangle Q_a - \langle\chi f_{a,i} f_{b,j}\rangle W_{ai,j} = \\ &= -\langle D f_{b,i}\rangle C_{,i} + \langle\chi f_{b,j}\rangle U_{i,ij} - \langle\beta f_{b,i}\rangle T_{,i} \\[2ex] &\langle k f_{a,j} f_{b,i}\rangle Q_a = -\langle k f_{b,i}\rangle T_{,i} + \langle k f_{b,i}\rangle \dot{U}_i \end{aligned} \tag{3.6}$$

where the symbol $\langle\ \rangle$ denotes

$$\langle g \rangle \equiv \frac{1}{h} \int\limits_0^b g(x_3)\, dx_3 \tag{3.7}$$

for any h-periodic integrable function $g(\cdot)$.

Using the formulae (3.7) and (3.2) for an arbitrary h-periodic function $g(\cdot)$ taking a constant value for g_i in a layer of the ith kind, $i = 1, ..., n+1$, we obtain

$$\left.\begin{array}{ll} \langle g \rangle = \sum\limits_{r=1}^{n+1} g_r \eta_r & \langle g f_{a,\beta} \rangle = 0 \\ \langle g f_{a,3} \rangle = \sum\limits_{r=1}^{a} g_r \eta_r - \alpha_a \sum\limits_{r=a+1}^{n+1} g_r \eta_r & \end{array}\right\} \qquad \beta = 1,2$$

$$\langle g f_{a,3} f_{b,3} \rangle = \sum\limits_{r=1}^{b} g_r \eta_r - \alpha_b \sum\limits_{r=b+1}^{a} g_r \eta_r + \alpha_a \alpha_b \sum\limits_{r=a+1}^{n+1} g_r \eta_r \qquad \beta \leqslant a$$

$$\langle g f_{a,\alpha} f_{b,\beta} \rangle = 0 \qquad \langle g f_{a,3} f_{a,\beta} \rangle = 0 \qquad \alpha, \beta = 1,2 \tag{3.8}$$

where

$$\eta_r \equiv \frac{h_r}{h} \qquad \alpha_a \equiv \frac{\eta_1 + ... + \eta_a}{1 - (\eta_1 + ... + \eta_a)} \qquad a = 1, ..., n \tag{3.9}$$

Employing equations (3.8), all material moduli in equations (3.5) and (3.6) can be calculated by substituting for function $g(\cdot)$ the h-periodic functions λ, μ, ρ, γ_T, γ_c, k, k_c, k_e, D, χ, β.

Equations (3.5) and (3.6) constitute the governing equations of the homogenized model of thermodiffusion in periodically stratified elastic bodies. Equations (3.6) stand for a system of $5n$ linear algebraic equations for the microlocal parameters W_{aj}, Q_a, G_a, $a = 1, ..., n$, $j = 1,2,3$. Solving equations (3.6) the microlocal parameters can be determined by macrodisplacements U_i, macroconcentration C, and macrotemperature T. Thus, the microlocal parameters can be eliminated from equations (3.5). It will lead to reduction of the homogenized model to the system of 5 linear partial differential equations with constant coefficients for macrodisplacements $i = 1,2,3$, macroconcentration C and macrotemperature T. These equations should be supplemented by appropriate constitutive relations expressed in terms of macrodisplacements, macroconcentration and macrotemperature, as well as by appropriate boundary and initial conditions similar to those of the linear thermodiffusoelasticity for homogeneous body. Substituting equations (3.1) with (3.2) and (3.3) into (2.1) and using (2.2), the constitutive relations describing the diffusion fluxes $\eta_i^{(r)}$, the head fluxes $q_i^{(r)}$, and the stresses $\sigma_{ij}^{(r)}$, $i,j = 1,2,3$; $r = 1,...,n+1$ in a layer of rth kind can be written in the form

$$
\begin{aligned}
\eta_\alpha^{(r)} &= -D^{(r)}C_{,\alpha} - \beta^{(r)}T_{,\alpha} + \chi^{(r)}(U_{k,k\alpha} + f_a'W_{a3,\alpha})\\
\eta_3^{(r)} &= -D^{(r)}(C_{,3} + f_a'G_a) - \beta^{(r)}(T_{,3} + f_a'Q_a) + \chi^{(r)}(U_{k,k3} + f_a'W_{a3,3})\\
q_\alpha^{(r)} &= -\Lambda^{(r)}T_{,\alpha} - d^{(r)}C_{,\alpha} + \Gamma^{(r)}(U_{k,k\alpha} + f_a'W_{a3,\alpha})\\
q_3^{(r)} &= -\Lambda^{(r)}(T_{,3} + f_a'Q_a) - d^{(r)}(C_{,3} + f_a'G_a) + \Gamma^{(r)}(U_{k,k3} + f_a'W_{a3,3}) \qquad (3.10)\\
\sigma_{\alpha\beta}^{(r)} &= \mu^{(r)}(U_{\alpha,\beta} + U_{\beta,\alpha}) + (\lambda^{(r)}U_{i,i} + \lambda^{(r)}f_a'W_{a3} - \gamma_T^{(r)}T - \gamma_c^{(r)}C)\delta_{\alpha\beta}\\
\sigma_{\alpha3}^{(r)} &= \mu^{(r)}(U_{\alpha,3} + f_a'W_{\alpha3} + U_{3,\alpha})\\
\sigma_{33}^{(r)} &= 2\mu^{(r)}(U_{3,3} + f_a'W_{a3}) + \lambda^{(r)}(U_{i,i} + f_a'W_{a3}) - \gamma_T^{(r)}T - \gamma_c^{(r)}C\\
&\alpha,\beta = 1,2 \qquad i = 1,2,3 \qquad r = 1,...,n+1 \qquad a = 1,...,n
\end{aligned}
$$

4. Thermodiffusion in microperiodic in two-layered elastic bodies

In the case of microperiodic two-layered elastic bodies, the number of equations of the homogenized model obtained in Section 3 will be reduced. Assuming that $n = 1$, the set of shape functions given by (3.2) is reduced to one function

$$
f_1(x_3) = \begin{cases} x_3 - \dfrac{1}{2}h_1 & \text{for} \quad 0 \leqslant x_3 \leqslant h_1 \\[2mm] \dfrac{-\eta_1}{1-\eta_1}x_3 - \dfrac{1}{2}h_1 + \dfrac{h_1}{1-\eta_1} & \text{for} \quad h_1 \leqslant x_3 \leqslant h \end{cases}
$$

where

$$
\eta_1 = \frac{h_1}{h} \tag{4.1}
$$

By using the relation

$$
\eta_2 = \frac{h_2}{h} = 1 - \eta_1
$$

and equations (3.8) and (3.9), we obtain

$$\begin{aligned}
&\langle g\rangle = g_1\eta_1 + g_2\eta_2 = \eta_1 g_1 + (1-\eta_1)g_2 \equiv \widetilde{g}\\
&\langle g f_{1,3}\rangle = g_1\eta_1 - \frac{\eta_1}{1-\eta_1}g_2\eta_2 = \eta_1(g_1-g_2) \equiv [\![g]\!]\\
&\langle g(f_{1,3})^2\rangle = g_1\eta_1 + \frac{\eta_1^2}{(1-\eta_1)^2}g_2\eta_2 = \eta_1 g_1 + \frac{\eta_1^2}{1-\eta_1}g_2 \equiv \widehat{g}
\end{aligned} \tag{4.2}$$

By using (4.2), equations (3.5) and (3.6) for the case of two-layered periodic bodies take the following form

$$\begin{aligned}
&\widetilde{\mu}U_{i,jj} + (\widetilde{\lambda}+\widetilde{\mu})U_{j,ji} + [\![\mu]\!]W_{1j,j}\delta_{i3} + [\![\mu]\!]W_{1i,j}\delta_{j3} + [\![\lambda]\!]W_{1i,j}\delta_{j3} -\\
&-\widetilde{\gamma}_c C_{,i} - \widetilde{\gamma_T}T_{,i} + \widetilde{\rho}X_i - \rho\ddot{U}_i = 0\\
&\widetilde{D}C_{,ii} + [\![D]\!]G_{1,3} - \dot{C} + \sigma - \widetilde{\chi}U_{k,kii} - [\![\chi]\!]W_{13,ii} + \widetilde{\beta}T_{,ii} + [\![\beta]\!]Q_{1,3} = 0\\
&\widetilde{k}T_{,ii} + [\![k]\!]Q_{1,3} + \widetilde{k}_c\dot{C} - \widetilde{k}_e\dot{U}_{i,i} - \dot{T} + W_0 = 0
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\widehat{\mu}W_{1i} + (\widehat{\lambda}+\widehat{\mu})W_{13}\delta_{i3} = -\Big\{[\![\mu]\!](U_{i,3}+U_{3,i}) + [\![\lambda]\!]U_{k,k}\delta_{i3} +\\
&+([\![\gamma_c]\!]C + [\![\gamma_T]\!]T)\delta_{i3}\Big\}\\
&\widehat{D}G_1 + \widehat{\beta}Q_1 - \widehat{\chi}W_{13,3} = -[\![D]\!]C_{,3} + [\![\chi]\!]U_{i,i3} - [\![\beta]\!]T_{,3}\\
&\widehat{k}Q_1 = -[\![k]\!]T_{,3} + [\![k]\!]U_{,3}
\end{aligned} \tag{4.4}$$

Solving equation (4.4) we obtain

$$\begin{aligned}
&W_{11} = -\frac{[\![\mu]\!]}{\widehat{\mu}}(U_{1,3}+U_{3,1}) \qquad\qquad W_{12} = -\frac{[\![\mu]\!]}{\widehat{\mu}}(U_{2,3}+U_{3,2})\\
&W_{13} = -\frac{[\![\lambda]\!]+2[\![\mu]\!]}{\widehat{\lambda}+2\widehat{\mu}}U_{3,3} - \frac{[\![\lambda]\!]}{\widehat{\lambda}+2\widehat{\mu}}U_{k,k} + \frac{[\![\gamma_c]\!]}{\widehat{\lambda}+2\widehat{\mu}}C + \frac{[\![\gamma_T]\!]}{\widehat{\lambda}+2\widehat{\mu}}T\\
&Q_1 = -\frac{[\![k]\!]}{\widehat{k}}T_{,3} + \frac{[\![k]\!]}{\widehat{k}}U_3\\
&G_1 = -\frac{\widehat{\beta}}{\widehat{D}}Q + \frac{\widehat{\chi}}{\widehat{D}}W_{13,3} - \frac{[\![D]\!]}{\widehat{D}}C_{,3} + \frac{[\![\chi]\!]}{\widehat{D}}U_{i,i3} - \frac{\widehat{\beta}}{\widehat{D}}\dot{U}_3
\end{aligned} \tag{4.5}$$

Substituting equations (4.5) into (4.3) we can obtain 5 partial differential equations with constant coefficients for unknowns U_i, C and T.

5. Example

Consider a stationary problem of thermodiffusion in a microperiodic two-layered elastic layer resting on the rigid impermeable foundation. Let the upper plane of the stratified body be subjected to constant temperature θ_0, constant concentration C_0 and be free of loadings. The lower plane of the body is assumed to be fixed to the rigid impermeable thermal and diffusional foundation. The considered problem is dependent on the variable x_3 only, and it is determined by nonzero functions U_3, W_{13}, C, G_1, T and Q_1. We assume that no internal heat and diffusive sources exist.

By using the assumptions presented above, equations (4.3) and (4.4) may be written in the following form

$$
\begin{aligned}
&(\widetilde{\lambda} + 2\widetilde{\mu})U_{3,33} + ([\![\lambda]\!] + 2[\![\mu]\!])W_{13,3} - \widetilde{\gamma}_c C_{,3} - \widetilde{\gamma}_T T_{,3} = 0 \\
&\widetilde{D}C_{,33} + [\![D]\!]G_{1,3} - \widetilde{\chi}U_{3,333} - [\![\chi]\!]W_{13,33} + \widetilde{\beta}T_{,33} + [\![\beta]\!]Q_{1,3} = 0 \\
&\widetilde{k}T_{,33} + [\![k]\!]Q_{1,3} = 0
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&W_{11} = W_{12} = 0 \qquad\qquad Q_1 = -\frac{[\![k]\!]}{\widehat{k}}T_{,3} \\
&W_{13} = -\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}}U_{3,3} + \frac{[\![\gamma_c]\!]}{\widehat{\lambda} + 2\widehat{\mu}}C + \frac{[\![\gamma_T]\!]}{\widehat{\lambda} + 2\widehat{\mu}}T \\
&G_1 = \alpha_1 U_{3,33} + \alpha_2 C_{,3} + \alpha_3 T_{,3}
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
&\alpha_1 = -\frac{\widehat{\chi}}{\widehat{D}}\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}} + \frac{[\![\chi]\!]}{\widehat{D}} \qquad\qquad \alpha_2 = \frac{\widehat{\chi}}{\widehat{D}}\frac{[\![\gamma_c]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - \frac{[\![D]\!]}{\widehat{D}} \\
&\alpha_3 = \frac{\widehat{\beta}}{\widehat{D}}\frac{[\![k]\!]}{\widehat{k}} + \frac{\widehat{\chi}}{\widehat{D}}\frac{[\![\gamma_T]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - \frac{[\![\beta]\!]}{\widehat{D}}
\end{aligned}
\tag{5.3}
$$

Using equations (5.2) we can eliminate microlocal parameters W_{13}, G_1, Q_1 from equations (5.1). It leads to the system of equations

$$
\begin{aligned}
&A_1 U_{3,33} + A_2 C_{,3} + A_3 T_{,3} = 0 \qquad\qquad T_{,33} = 0 \\
&B_1 U_{3,333} + B_2 C_{,33} + B_3 T_{,33} = 0
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
A_1 &= \widetilde{\lambda} + 2\widetilde{\mu} - \frac{([\![\lambda]\!] + 2[\![\mu]\!])^2}{\widehat{\lambda} + 2\widehat{\mu}} & A_2 &= \frac{[\![\gamma_c]\!]([\![\lambda]\!] + 2[\![\mu]\!])}{\widehat{\lambda} + 2\widehat{\mu}} - \widetilde{\gamma}_c \\
A_3 &= \frac{[\![\gamma_T]\!]([\![\lambda]\!] + 2[\![\mu]\!])}{\widehat{\lambda} + 2\widehat{\mu}} - \widetilde{\gamma}_T & B_1 &= [\![D]\!]\alpha_1 - [\![D]\!]\widetilde{\chi} + \frac{[\![\chi]\!]([\![\lambda]\!] + 2[\![\mu]\!])}{\widehat{\lambda} + 2\widehat{\mu}} \\
B_2 &= \widetilde{D} + [\![D]\!]\alpha_2 - \frac{[\![\chi]\!][\![\gamma_c]\!]}{\widehat{\lambda} + 2\widehat{\mu}} & B_3 &= [\![D]\!]\alpha_3 - \frac{[\![\chi]\!][\![\gamma_T]\!]}{\widehat{\lambda} + 2\widehat{\mu}} + \widetilde{\beta} - \frac{[\![\beta]\!][\![k]\!]}{\widehat{k}}
\end{aligned}
\tag{5.5}
$$

The general solution of equation (5.4) takes the form

$$
\begin{aligned}
C(x_3) &= a_1 x_3 + a_2 \qquad\qquad T(x_3) = a_3 x_3 + a_4 \\
U(x_3) &= \frac{1}{2} a_5 x_3^2 + a_6 x_3 + a_7
\end{aligned}
\tag{5.6}
$$

where

$$
a_1 = -\frac{1}{A_2}(A_1 a_5 + A_3 a_3) \tag{5.7}
$$

and $a_2, ..., a_7$ are constants which should be determined from the boundary conditions.

By using equations (3.10) and the assumptions taken in this section, the constitutive relations can be reduced to the following form

$$
\begin{aligned}
\eta_\alpha^{(r)} &= 0 \qquad\qquad q_\alpha^{(r)} = 0 \qquad\qquad \sigma_{\alpha 3}^{(r)} = 0 \\
\eta_3^{(r)} &= -D^{(r)}(C_{,3} + f_1' G_1) - \beta^{(r)}(T_{,3} + f_1' Q_1) + \chi^{(r)}(U_{3,33} + f_1' W_{13,3}) \\
q_3^{(r)} &= -\Lambda^{(r)}(T_{,3} + f_1' Q_1) - d^{(r)}(C_{,3} + f_1' G_1) + \Gamma^{(r)}(U_{3,33} + f_1' W_{13,3}) \\
\sigma_{\alpha\beta}^{(r)} &= \Big[\lambda^{(r)}(U_{3,3} + f_1' W_{13}) - \gamma_T^{(r)} T - \gamma_c^{(r)} C\Big]\delta_{\alpha\beta} \\
\sigma_{33}^{(r)} &= (\lambda^{(r)} + 2\mu^{(r)})(U_{3,3} + f_1' W_{13}) - \gamma_T^{(r)} T - \gamma_c^{(r)} C
\end{aligned}
\tag{5.8}
$$

From equation (3.2) it follows that

$$
f_1' = \begin{cases} 1 & \text{for} \quad r = 1 \\ \dfrac{-\eta_1}{\eta_1 - 1} & \text{for} \quad r = 2 \end{cases} \tag{5.9}
$$

By using equations (5.8), (5.9) and (5.2) it can be shown that the components $\eta_3^{(r)}$, $q_3^{(r)}$, $\sigma_{\alpha 3}^{(r)}$, $\sigma_{33}^{(r)}$ are continuous on interfaces (they are not dependent on r;

$r = 1, 2$). For example, since

$$
\begin{aligned}
&(\lambda^{(1)} + 2\mu^{(1)})\Big(1 - \frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}}\Big) = (\lambda^{(2)} + 2\mu^{(2)})\Big(1 + \frac{\eta_1}{1-\eta_1}\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}}\Big) \equiv e_1 \\
&\frac{\lambda^{(1)} + 2\mu^{(1)}}{\widehat{\lambda} + 2\widehat{\mu}}[\![\gamma_c]\!] - \gamma_c^{(1)} = -\frac{\eta_1}{1-\eta_1}\frac{\lambda^{(2)} + 2\mu^{(2)}}{\widehat{\lambda} + 2\widehat{\mu}}[\![\gamma_c]\!] - \gamma_c^{(2)} \equiv e_2 \qquad (5.10)\\
&\frac{\lambda^{(1)} + 2\mu^{(1)}}{\widehat{\lambda} + 2\widehat{\mu}}[\![\gamma_T]\!] - \gamma_T^{(1)} = -\frac{\eta_1}{1-\eta_1}\frac{\lambda^{(2)} + 2\mu^{(2)}}{\widehat{\lambda} + 2\widehat{\mu}}[\![\gamma_T]\!] - \gamma_T^{(2)} \equiv e_3
\end{aligned}
$$

the components of stress tensor $\sigma_{33}^{(r)}$; $r = 1, 2$, are expressed in the form

$$
\sigma_{33}^{(r)} = e_1 U_{3,33} + e_2 C + e_3 T \qquad (5.11)
$$

The problem considered in this section is determined by the following boundary conditions

$$
\begin{aligned}
&\theta(x_3 = nh) = \theta_0 && \sigma_{33}(x_3 = nh) = 0 \\
&c(x_3 = nh) = c_0 && \frac{\partial\theta}{\partial x_3}(x_3 = 0) = 0 \qquad (5.12)\\
&\frac{\partial c}{\partial x_3}\theta(x_3 = 0) = 0 && u_3(x_3 = 0) = 0
\end{aligned}
$$

where θ_0 is the given constant temperature, c_0 is the given constant concentration and n is a sufficiently large natural number. By using the general solution (5.6), boundary conditions (5.12) and equations (5.11), (5.7) and (5.8), we obtain

$$
\begin{aligned}
&C(x_3) = c_0 \qquad T(x_3) = \theta_0 \qquad U_(x_3) = -\frac{1}{e_1}(e_2 c_0 + e_3\theta_0)x_3 \\
&\eta_i^{(r)}(x_3) = 0 \qquad q_i^{(r)}(x_3) = 0 \qquad \sigma_{\alpha 3}^{(r)}(x_3) = \sigma_{33}^{(r)}(x_3) = \sigma_{12}^{(r)}(x_3) = 0 \\
&\sigma_{11}^{(1)} = \sigma_{22}^{(1)} = d_c^{(1)} c_0 + d_T^{(1)}\theta_0 \qquad \sigma_{11}^{(2)} = \sigma_{22}^{(2)} = d_c^{(2)} c_0 + d_T^{(2)}\theta_0 \qquad (5.13)
\end{aligned}
$$

where

$$
\begin{aligned}
d_c^{(1)} &= \lambda^{(1)}\Big(\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - 1\Big)\frac{e_2}{e_1} + \frac{\lambda^{(1)}[\![\gamma_c]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - [\![\gamma_c^{(1)}]\!] \\
d_T^{(1)} &= \lambda^{(1)}\Big(\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - 1\Big)\frac{e_3}{e_1} + \frac{\lambda^{(1)}[\![\gamma_T]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - [\![\gamma_T^{(1)}]\!] \qquad (5.14)\\
d_c^{(2)} &= -\lambda^{(2)}\Big(1 + \frac{\eta_1}{1-\eta_1}\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}}\Big)\frac{e_2}{e_1} - \frac{\eta_1}{1-\eta_1}\frac{\lambda^{(2)}[\![\gamma_c]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - \gamma_c^{(2)} \\
d_T^{(2)} &= -\lambda^{(2)}\Big(1 + \frac{\eta_1}{1-\eta_1}\frac{[\![\lambda]\!] + 2[\![\mu]\!]}{\widehat{\lambda} + 2\widehat{\mu}}\Big)\frac{e_3}{e_1} - \frac{\eta_1}{1-\eta_1}\frac{\lambda^{(2)}[\![\gamma_T]\!]}{\widehat{\lambda} + 2\widehat{\mu}} - \gamma_T^{(2)}
\end{aligned}
$$

6. Final remarks

The homogenized model of thermodiffusion in periodically stratified elastic composites can be treated as a starting point to some applications in geophysical problems, environmental engineering and engineering of materials.

The applicability of the presented model is limited to composites with components for which the linear theory of thermodiffusoelasticity can be used. The main feature of the homogenized model is that in the modelling of strains, stresses, heat and diffusion fluxes, it describes the microlocal effects, i.e. the effects due to the periodically layered structure of the body by means of microlocal parameters. The presented model constitutes an approximate theory of thermodiffusion processes in periodically stratified composites, in which the conditions of continuity on interfaces of displacement, temperature, concentration as well as of stress vector, normal diffusion and thermal fluxes, are satisfied. Moreover, assuming that the body is homogeneous, so that

$$\begin{aligned}&\Big\{\lambda^{(r)},\mu^{(r)},D^{(r)},\chi^{(r)},\gamma_c^{(r)},k^{(r)},\gamma_T^{(r)},k_e^{(r)},k_c^{(r)},\rho^{(r)},\beta^{(r)}\Big\} =\\ &= \Big\{\lambda^{(s)},\mu^{(s)},D^{(s)},\chi^{(s)},\gamma_c^{(s)},k^{(s)},\gamma_T^{(s)},k_e^{(s)},k_c^{(s)},\rho^{(s)},\beta^{(s)}\Big\} \qquad (6.1)\\ &r \neq s \qquad\qquad r,s \in \{1,...,n+1\}\end{aligned}$$

we obtain from equations (3.8) and (3.6) that

$$W_{ai} = 0 \qquad\qquad G_a = Q_a = 0 \qquad\qquad \begin{aligned}a &= 1,...,n\\ i &= 1,2,3\end{aligned} \qquad (6.2)$$

and equations (3.5) are reduced to the relations of the linear theory of thermodiffusion in homogeneous isotropic elastic bodies, cf. Nowacki and Olesiak (1991).

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Termodyfuzja w periodycznie warstwowych sprężystych kompozytach

Streszczenie

W pracy rozważano zagadnienie modelowania procesu dyfuzji w periodycznie warstwowych sprężystych kompozytach. Rozważania oparto na procedurze homogenizacyjnej z parametrami mikrolokalnymi i na liniowej teorii dyfuzji dla jednorodnych ciał termosprężystych. Wyprowadzony został zhomogenizowany model z parametrami mikrolokalnymi uwzględniający pewne lokalne efekty dla strumieni dyfuzji i ciepła oraz naprężeń. Rozwiązano również przykład ilustrujący zastosowania wyprowadzonego modelu.

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