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COEFFICIENTS OF SHEAR CORRECTION IN TRANSVERESLY NONHOMOGENEOUS MODERATELY THICK PLATES

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A method of determination of the shear stiffness of an anisotropic nonhomogeneous plate of moderate thickness has been presented in the paper. The set of shear coefficients for the plates has been formulated and determined by the author.

Key words: moderately thick plates, shear coefficients

Dedicated to Professor Czesław Woźniak on the occasion of his 70th birthday

1. Introduction

In the theory of moderately thick plates the following kinematic hypothesis is usually assumed

$$u_\alpha(X^\beta, z) = u_{0\alpha}(X^\beta) - z\phi_\alpha(X^\beta) \qquad u_3(X^\beta, z) = w(X^\beta) \tag{1.1}$$

where $u_{0\alpha}(X^\beta)$, $\phi_\alpha(X^\beta)$ and $w(X^\beta)$ are functions to be found. This hypothesis is associated with the names of Hencky and Bolle. The paper by Hencky (1947) focusses on the bending state of plates. In the same year 1947, Bolle, unaware of the work of Hencky, published two papers (Bolle, 1947a,b) in which kinematic hypothesis (1.1) was assumed for the bending state. Saying it more precisely, Bolle proposed a hypothesis for strains that implies the mentioned hypothesis for displacements.

The strain fields are determined from the formulae

$$\begin{aligned} e_{\alpha\beta}(X^\beta, z) &= e_{0\alpha\beta}(X^\beta) + z\kappa_{\alpha\beta}(X^\beta) \\ e_{\alpha 3}(X^\beta, z) &= \frac{1}{2}\chi_\alpha(X^\beta) \qquad e_{33}(X^\beta, z) = 0 \end{aligned} \tag{1.2}$$

in which

$$e_{0\alpha\beta}(X^\beta) = \frac{1}{2}(u_{0\alpha,\beta} + u_{0\beta,\alpha})$$
$$\kappa_{\alpha\beta}(X^\beta) = -\frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) \qquad \chi_\alpha = w_{,\alpha} - \phi_\alpha \tag{1.3}$$

Upon neglecting the temperature effects the main stress components $S^{\alpha\beta}$ read

$$S^{\alpha\beta} = \widehat{C}^{\alpha\beta\gamma\omega}(e_{0\gamma\omega} + z\kappa_{\gamma\omega}) + \widehat{C}^{\alpha\beta 33}S^{33} \tag{1.4}$$

In the theory of moderately thick plates the strains $e_{\alpha 3}$ do not vanish as in the Kirchhoff theory, but are given by Eq. $(1.2)_2$.

Thus the stresses $S_{\alpha 3}$ could possibly be found from appropriate constitutive equations leading to

$$S^{\alpha 3} = \widehat{C}^{\alpha 3\beta 3}\chi_\beta \tag{1.5}$$

However, the thus derived constitutive stresses (called also extra-stresses) would not satisfy the boundary conditions on the plate faces Π^+ and Π^-.

The normal stresses S_{33} could possibly be determined by the formula

$$S_{K3} = p_K^+ + \int_z^{h_1} b_K\, dz + \Big(\int_z^{h_1} S_K^\alpha\, dz\Big)_{,\alpha} \tag{1.6}$$

for $K = 3$, upon taking into account the stresses $S_{\alpha 3}$ given by constitutive equations (1.5).

The stresses S_{33} found in this manner would satisfy the boundary conditions on the faces, but would simultaneously lead to their incorrect distribution across the plate thickness.

Better predictions of the stresses S_{K3} can be derived from formula (1.6) by finding the stresses $S_{3\alpha}$ and, subsequently, the stresses S_{33}.

Basing upon hypothesis (1.1) one finds the following set of equations of moderately thick plates

— equilibrium equations

$$N^{\beta\alpha}_{,\beta} + p^\alpha + \widehat{b}^\alpha = 0 \qquad M^{\beta\alpha}_{,\beta} - Q^\alpha + m^\alpha + \widehat{m}^\alpha = 0$$
$$Q^\alpha_{,\alpha} + p_3 + \widehat{b}_3 = 0 \tag{1.7}$$

— constitutive equations

$$N_{\alpha\beta} = B^{\gamma\omega}_{\alpha\beta}e_{0\gamma\omega} + H^{\gamma\omega}_{\alpha\beta}\kappa_{\gamma\omega} + s_{0\alpha\beta}$$

$$M_{\alpha\beta} = H^{\gamma\omega}_{\alpha\beta} e_{0\gamma\omega} + D^{\gamma\omega}_{\alpha\beta} \kappa_{\gamma\omega} + s_{\alpha\beta} \tag{1.8}$$

$$Q_\alpha = K^\beta_\alpha \chi_\beta + k^+_{\alpha\beta} h_1 p^{+\beta} + k^-_{\alpha\beta} h_2 p^{-\beta}$$

in which the stress resultants are defined as below

$$N_{\alpha\beta} = \int\limits_{-h_2}^{h_1} S_{\alpha\beta}\, dz \qquad M_{\alpha\beta} = \int\limits_{-h_2}^{h_1} z S_{\alpha\beta}\, dz \qquad Q_\alpha = \int\limits_{-h_2}^{h_1} S_{\alpha 3}\, dz \tag{1.9}$$

the stiffnesses $B_{\alpha\beta\gamma\omega}$, $H_{\alpha\beta\gamma\omega}$, $D_{\alpha\beta\gamma\omega}$ being given by

$$\begin{aligned} B_{\alpha\beta\gamma\omega}(x^\delta) &= \int\limits_{-h_2}^{h_1} \widehat{C}_{\alpha\beta\gamma\omega}(X^\delta, z)\, dz \\ D_{\alpha\beta\gamma\omega}(x^\delta) &= \int\limits_{-h_2}^{h_1} z^2 \widehat{C}_{\alpha\beta\gamma\omega}(X^\delta, z)\, dz \\ H_{\alpha\beta\gamma\omega}(x^\delta) &= \int\limits_{-h_2}^{h_1} z \widehat{C}_{\alpha\beta\gamma\omega}(X^\delta, z)\, dz \end{aligned} \tag{1.10}$$

The transverse shear stiffnesses $K_{\alpha\beta}$ and the quantities $s_{0\alpha\beta}$ and $s_{\alpha\beta}$ are expressed as follows

$$K_{\alpha\beta} = k_{\underline{\alpha\beta}} \widehat{K}_{\alpha\beta} \tag{1.11}$$

$$s_{0\alpha\beta} = \int\limits_{-h_2}^{h_1} S_{33} \widehat{C}_{\alpha\beta 33}\, dz \qquad s_{\alpha\beta} = \int\limits_{-h_2}^{h_1} z S_{33} \widehat{C}_{\alpha\beta 33}\, dz \tag{1.12}$$

where

$$\widehat{K}_{\alpha\beta} = \int\limits_{-h_2}^{h_1} \widehat{C}_{\alpha 3\beta 3}\, dz \tag{1.13}$$

or

$$\begin{aligned} \widehat{K}_{11} &= \int\limits_{-h_2}^{h_1} \frac{G_{13}}{1 - \eta_{45}\eta_{54}}\, dz \qquad \widehat{K}_{22} = \int\limits_{-h_2}^{h_1} \frac{G_{23}}{1 - \eta_{45}\eta_{54}}\, dz \\ \widehat{K}_{12} &= \widehat{K}_{21} = -\int\limits_{-h_2}^{h_1} \frac{\eta_{45} G_{13}}{1 - \eta_{45}\eta_{54}}\, dz = -\int\limits_{-h_2}^{h_1} \frac{\eta_{54} G_{23}}{1 - \eta_{45}\eta_{54}}\, dz \end{aligned} \tag{1.14}$$

The summation does not concern the underlined indices. The quantities h_1 and h_2 represent the distances of the faces to the reference plane.

The method of finding the quantities $s_{0\alpha\beta}$ and $s_{\alpha\beta}$, as certain averaged functions, is explained in the book Jemielita (2001).

Two types of correction factors are introduced into equations (1.8) and (1.11): the shear correction factors $k_{\alpha\beta}$ ($k_{12} = k_{21}$) and other ones $k^+_{\alpha\beta}$, $k^-_{\alpha\beta}$.

The Hencky-Bolle kinematic assumptions imply an incorrect distribution of the transverse shear stresses across the plate thickness.

Introduction of these coefficients enables improving the mathematical description worsened by bad consequencies due to oversimplified kinematic hypothesis in the final equations of the theory.

In the case of unbalanced (i.e. transversely asymmetric) anisotropic plates one should define the set of three coefficients $k_{\alpha\beta}$ and six coefficients $k^+_{\alpha\beta}$, $k^-_{\alpha\beta}$.

Various criteria of assuming the shear correction factors in the transversely homogeneous plates are discussed by Jemielita (1998).

A method of determination of the shear stiffness of an anisotropic nonhomogeneous plate of moderate thickness has been presented in the paper. The set of shear coefficients for the plates has been formulated and determined by the present author.

2. Stiffnesses due to transverse shear

The transverse shear stiffnesses are determined by equating the virtual work of the transverse forces on the averaged strains with the work of the transverse stresses on the transverse strains

$$\delta\mathbf{Q}^\top\mathbf{\Gamma} = \int\limits_{-h_2}^{h_1} \delta\mathbf{S}_S^\top\mathbf{e}_S \, dz = \int\limits_{-h_2}^{h_1} \delta\mathbf{S}_S^\top\mathbf{E}_S\mathbf{S}_S \, dz \tag{2.1}$$

In the above variational equation the following notation is introduced

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \qquad \mathbf{S}_S = \begin{bmatrix} S_{13} \\ S_{23} \end{bmatrix} \qquad \mathbf{E}_S = \begin{bmatrix} \dfrac{1}{G_{13}} & \dfrac{\eta_{54}}{G_{13}} \\ \dfrac{\eta_{45}}{G_{23}} & \dfrac{1}{G_{23}} \end{bmatrix}$$
$$\mathbf{e}_S = \begin{bmatrix} e_{13} \\ e_{23} \end{bmatrix} \qquad \mathbf{\Gamma} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \tag{2.2}$$

The column of the averaged deformations is expressed in the form

$$\boldsymbol{\Gamma} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \mathbf{K}^{-1}\Big[\mathbf{Q} - h_1\mathbf{K}^{+}\mathbf{p}^{+} - h_2\mathbf{K}^{-}\mathbf{p}^{-}\Big] \tag{2.3}$$

where

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \qquad \mathbf{K}^{+} = \begin{bmatrix} k_{11}^{+} & k_{12}^{+} \\ k_{21}^{+} & k_{22}^{+} \end{bmatrix}$$

$$\mathbf{K}^{-} = \begin{bmatrix} k_{11}^{-} & k_{12}^{-} \\ k_{21}^{-} & k_{22}^{-} \end{bmatrix} \qquad \mathbf{p}^{+} = \begin{bmatrix} p_1^{+} \\ p_2^{+} \end{bmatrix} \qquad \mathbf{p}^{-} = \begin{bmatrix} p_1^{-} \\ p_2^{-} \end{bmatrix}$$

Here p_α^{+}, p_α^{-} represent the tangent loadings applied on the plate faces, the matrix $\mathbf{K}$ represents the stiffness matrix due to the transverse shear.

The column of the stresses $\mathbf{S}_S$, in the case of the absence of the volume forces, is represented as follows

$$\mathbf{S}_S = \frac{1}{h}\mathbf{F}\mathbf{Q} - \mathbf{G}^{+}\mathbf{p}^{+} - \mathbf{G}^{-}\mathbf{p}^{-} \tag{2.4}$$

where

$$\mathbf{F}(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{bmatrix} \qquad \mathbf{G}^{+}(z) = \begin{bmatrix} g_{11}^{+}(z) & g_{12}^{+}(z) \\ g_{21}^{+}(z) & g_{22}^{+}(z) \end{bmatrix}$$

$$\mathbf{G}^{-}(z) = \begin{bmatrix} g_{11}^{-}(z) & g_{12}^{-}(z) \\ g_{21}^{-}(z) & g_{22}^{-}(z) \end{bmatrix} \qquad h = h_1 + h_2$$

For the time being, the functions $f_{\alpha\beta}$, $g_{\alpha\beta}^{+}$, $g_{\alpha\beta}^{-}$ are unknown.

Substitution of (2.3) and (2.4) into variational equation (2.1) gives the following formulae for the matrices $\mathbf{K}$, $\mathbf{K}^{+}$ and $\mathbf{K}^{-}$

$$\mathbf{K}^{-1} = \frac{1}{h^2}\int_{-h_2}^{h_1} \mathbf{F}^{\top}\mathbf{E}_S\mathbf{F}\,dz \tag{2.5}$$

$$\mathbf{K}^{+} = \frac{1}{hh_1}\mathbf{K}\int_{-h_2}^{h_1} \mathbf{F}^{\top}\mathbf{E}_S\mathbf{G}^{+}\,dz \qquad \mathbf{K}^{-} = \frac{1}{hh_2}\mathbf{K}\int_{-h_2}^{h_1} \mathbf{F}^{\top}\mathbf{E}_S\mathbf{G}^{-}\,dz \tag{2.6}$$

3. Transverse shear coefficients

The components of the stiffness matrix due to shear can be expressed in form (1.11), from which one obtains the following formulae for the unknown shear coefficients

$$k_{\alpha\beta} = \frac{K_{\alpha\beta}}{\widehat{K}_{\underline{\alpha\beta}}} \tag{3.1}$$

where the stiffnesses $\widehat{K}_{\alpha\beta}$ are defined by (1.13) or (1.14).

In order to make the formulae for the shear coefficients complete one should define the functions $f_{\alpha\beta}$, $g^{+}_{\alpha\beta}$ and $g^{-}_{\alpha\beta}$.

Consider an anisotropic strip of constant stiffnesses. Assume that the deflection of the strip is described by the following functions

$$w = w(X^1) \qquad\qquad \phi_1 = \phi_1(X^1)$$

The strains e_{011} and κ_{11} are functions of the variable X^1.

It is not difficult to prove that by neglecting the stress component S_{33} and the temeperature effect one obtains the following formulae for the stress S_{11} and the stress resultants N_{11}, M_{11}[1]

$$S_{11} = C_{11}(z)(e_{011} + z\kappa_{11}) \qquad\qquad N_{11} = B_{11}e_{011} + H_{11}\kappa_{11}$$

$$M_{11} = H_{11}e_{011} + D_{11}\kappa_{11}$$

The above constitutive equations imply

$$S_{11} = \frac{1}{B_{11}D_{11} - H_{11}^2} C_{11}(z)[N_{11}D_{11} - M_{11}H_{11} - z(N_{11}H_{11} - M_{11}B_{11})] \tag{3.2}$$

where B_{11}, H_{11} and D_{11} represent the in-plane, reciprocal and bending stiffnesses, respectively.

By using the equilibrium equations and formula (3.2) one finds the following formula for the tangent stresses $S_{13} = S_{31}$

$$S_{31}(X^1, z) = S_{13}(X^1, z) = \frac{Q_1(X^1)}{h} f_{11}(z) + p_1^{+}(X^1)g_{11}^{+}(z) + p_1^{-}(X^1)g_{11}^{-}(z) \tag{3.3}$$

[1]In such a strip all the stresses $S_{\alpha K}$ occur. Also the stress resultants are non zero. We give here only those formulae for the stresses and stress resultants which will be used in the sequel.

where

$$f_{11}(z) = \frac{h}{\Delta_1}\Big(H_{11}\int\limits_{-h_2}^{z} C_{11}(z)\;dz - B_{11}\int\limits_{-h_2}^{z} zC_{11}(z)\;dz\Big) \tag{3.4}$$

$$\begin{aligned} g_{11}^{+}(z) &= -\frac{1}{\Delta_1}\Big[(D_{11} - h_1 H_{11})\int\limits_{-h_2}^{z} C_{11}(z)\;dz - \\ &\quad -(H_{11} - h_1 B_{11})\int\limits_{-h_2}^{z} zC_{11}(z)\;dz\Big] \\ g_{11}^{-}(z) &= -\frac{1}{\Delta_1}\Big[(D_{11} + h_2 H_{11})\int\limits_{-h_2}^{z} C_{11}(z)\;dz - \\ &\quad -(H_{11} + h_2 B_{11})\int\limits_{-h_2}^{z} zC_{11}(z)\;dz\Big] + 1 \end{aligned} \tag{3.5}$$

$$\Delta_1 = B_{11}D_{11} - H_{11}^2 \tag{3.6}$$

One notes easily that the thus determined functions satisfy the following boundary conditions

$$f_{11}(h_1) = f_{11}(-h_2) = g_{11}^{+}(-h_2) = g_{11}^{-}(h_1) = 0$$

$$g_{11}^{+}(h_1) = 1 \qquad\qquad g_{11}^{-}(-h_2) = -1$$

and the integral conditions

$$\frac{1}{h}\int\limits_{-h_2}^{h_1} f_{11}(z)\;dz = 1 \qquad\qquad \int\limits_{-h_2}^{h_1} g_{11}^{+}(z)\;dz = \int\limits_{-h_2}^{h_1} g_{11}^{-}(z)\;dz = 0$$

Finding the functions $f_{21}(z)$, $g_{21}^{+}(z)$, $g_{21}^{-}(z)$ is much more difficult. According to the assumed criterion (satisfaction of appropriate constitutive equations or equlibrium equations) we find various forms of these functions, and the obtained coefficients are expressed by complicated formulae.

Since the shear coefficent of unequal indices has minor influence on the final distribution of the stress resultants and displacements, one can assume

$$f_{21}(z) = 0 \qquad\qquad g_{21}^{+}(z) = 0 \qquad\qquad g_{21}^{-}(z) = 0 \tag{3.7}$$

By considering the deflection of the strip that depends only on the variable X^2 one obtains similar expressions. The final formulae for the functions $f_{\alpha\beta}$, $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ are expressed as follows

$$f_{\alpha\underline{\alpha}}(z) = \frac{h}{\Delta_{\underline{\alpha}}}\Big(H_{\alpha\underline{\alpha}} \int\limits_{-h_2}^{z} C_{\underline{\alpha\alpha}}(z)\, dz - B_{\alpha\underline{\alpha}} \int\limits_{-h_2}^{z} zC_{\underline{\alpha\alpha}}(z)\, dz\Big) \tag{3.8}$$

$$\begin{aligned} g^+_{\alpha\underline{\alpha}}(z) &= -\frac{1}{\Delta_{\underline{\alpha}}}\Big[(D_{\alpha\underline{\alpha}} - h_1 H_{\alpha\underline{\alpha}}) \int\limits_{-h_2}^{z} C_{\underline{\alpha\alpha}}(z)\, dz - \\ &\quad -(H_{\alpha\underline{\alpha}} - h_1 B_{\alpha\underline{\alpha}}) \int\limits_{-h_2}^{z} zC_{\underline{\alpha\alpha}}(z)\, dz\Big] \\ g^-_{\alpha\underline{\alpha}}(z) &= -\frac{1}{\Delta_{\underline{\alpha}}}\Big[(D_{\alpha\underline{\alpha}} + h_2 H_{\alpha\underline{\alpha}}) \int\limits_{-h_2}^{z} C_{\underline{\alpha\alpha}}(z)\, dz - \\ &\quad -(H_{\alpha\underline{\alpha}} + h_2 B_{\alpha\underline{\alpha}}) \int\limits_{-h_2}^{z} zC_{\underline{\alpha\alpha}}(z)\, dz\Big] + 1 \end{aligned} \tag{3.9}$$

$$f_{12}(z) = f_{21}(z) = g^+_{12}(z) = g^+_{21}(z) = g^-_{12}(z) = g^-_{21}(z) = 0 \tag{3.10}$$

$$\Delta_\alpha = B_{\alpha\underline{\alpha}} D_{\underline{\alpha\alpha}} - H^2_{\underline{\alpha}\alpha} \tag{3.11}$$

Taking into account relations (3.10) the shear coefficients can be put in an explicit form as follows

$$\begin{aligned} k_{11} &= \frac{h^2 A_{22}}{\widehat{K}_{11} R} \qquad k_{22} = \frac{h^2 A_{11}}{\widehat{K}_{22} R} \\ k_{12} &= k_{21} = -\frac{h^2 A_{12}}{\widehat{K}_{12} R} = -\frac{h^2 A_{21}}{\widehat{K}_{21} R} \end{aligned} \tag{3.12}$$

$$\begin{aligned} k^+_{11} &= \frac{h}{h_1}\frac{A_{22}P^+_{11} - A_{12}P^+_{21}}{R} \qquad k^+_{12} = \frac{h}{h_1}\frac{A_{22}P^+_{12} - A_{12}P^+_{22}}{R} \\ k^+_{21} &= \frac{h}{h_1}\frac{A_{11}P^+_{21} - A_{21}P^+_{11}}{R} \qquad k^+_{22} = \frac{h}{h_1}\frac{A_{11}P^+_{22} - A_{21}P^+_{12}}{R} \end{aligned} \tag{3.13}$$

$$k_{11}^{-} = \frac{h}{h_2}\frac{A_{22}P_{11}^{-} - A_{12}P_{21}^{-}}{R} \qquad k_{12}^{-} = \frac{h}{h_2}\frac{A_{22}P_{12}^{-} - A_{12}P_{22}^{-}}{R}$$
$$k_{21}^{-} = \frac{h}{h_2}\frac{A_{11}P_{21}^{-} - A_{21}P_{11}^{-}}{R} \qquad k_{22}^{-} = \frac{h}{h_2}\frac{A_{11}P_{22}^{-} - A_{21}P_{12}^{-}}{R} \tag{3.14}$$

where

$$R = A_{11}A_{22} - A_{12}A_{21} \tag{3.15}$$

$$A_{\underline{\alpha\alpha}} = \int\limits_{-h_2}^{h_1} \frac{f_{\underline{\alpha\alpha}}^2}{G_{\alpha 3}}\,dz$$
$$A_{12} = A_{21} = \int\limits_{-h_2}^{h_1} \frac{\eta_{45}}{G_{23}} f_{11} f_{22}\,dz = \int\limits_{-h_2}^{h_1} \frac{\eta_{54}}{G_{13}} f_{11} f_{22}\,dz \tag{3.16}$$

$$P_{\underline{\alpha\alpha}} = \int\limits_{-h_2}^{h_1} \frac{f_{\underline{\alpha\alpha}} g_{\underline{\alpha\alpha}}}{G_{\alpha 3}}\,dz$$
$$P_{12} = \int\limits_{-h_2}^{h_1} \frac{\eta_{54}}{G_{13}} f_{11} g_{22}\,dz \qquad P_{21} = \int\limits_{-h_2}^{h_1} \frac{\eta_{45}}{G_{23}} f_{22} g_{11}\,dz \tag{3.17}$$

While finding the quantities $P_{\alpha\beta}^{+}$ and $P_{\alpha\beta}^{-}$ from Eqs (3.17) one should assume the functions $g_{\alpha\beta}^{+}$ and $g_{\alpha\beta}^{-}$.

In the case of orthotropic plates we have $\eta_{45} = \eta_{54} = 0$, which gives the following simple formulae for the shear coefficients

$$k_{\alpha\underline{\alpha}} = \frac{h^2}{\int\limits_{-h_2}^{h_1} G_{\underline{\alpha}3}dz \int\limits_{-h_2}^{h_1} \frac{1}{G_{\underline{\alpha}3}} f_{\underline{\alpha\alpha}}^2\,dz} \tag{3.18}$$

$$k_{\underline{\alpha\alpha}}^{+} = \frac{h \int\limits_{-h_2}^{h_1} \frac{1}{G_{\underline{\alpha}3}} f_{\underline{\alpha\alpha}} g_{\underline{\alpha\alpha}}^{+}\,dz}{h_1 \int\limits_{-h_2}^{h_1} \frac{1}{G_{\underline{\alpha}3}} f_{\underline{\alpha\alpha}}^2\,dz} \qquad k_{\underline{\alpha\alpha}}^{-} = \frac{h \int\limits_{-h_2}^{h_1} \frac{1}{G_{\underline{\alpha}3}} f_{\underline{\alpha\alpha}} g_{\underline{\alpha\alpha}}^{-}\,dz}{h_2 \int\limits_{-h_2}^{h_1} \frac{1}{G_{\underline{\alpha}3}} f_{\underline{\alpha\alpha}}^2\,dz} \tag{3.19}$$

$$k_{12} = k_{21} = k_{12}^{+} = k_{21}^{+} = k_{12}^{-} = k_{21}^{-} = 0 \tag{3.20}$$

In the balanced (or transversely symmetric) orthotropic plates we have $H_{11} = H_{22} = 0$ and the functions depending on the variable z are given by the simple formulae

$$f_{\underline{\alpha\alpha}}(z) = -\frac{h}{D_{\underline{\alpha\alpha}}} \int\limits_{-h_2}^{z} zC_{\alpha\underline{\alpha}}(z)\, dz \tag{3.21}$$

$$\begin{aligned} g^{+}_{\underline{\alpha\alpha}}(z) &= -\Big(\frac{1}{B_{\underline{\alpha\alpha}}} \int\limits_{-h_2}^{z} C_{\alpha\underline{\alpha}}(z)\, dz + \frac{h_1}{D_{\underline{\alpha\alpha}}} \int\limits_{-h_2}^{z} zC_{\alpha\underline{\alpha}}(z)\, dz\Big) \\ g^{-}_{\underline{\alpha\alpha}}(z) &= -\Big(\frac{1}{B_{\underline{\alpha\alpha}}} \int\limits_{-h_2}^{z} C_{\alpha\underline{\alpha}}(z)\, dz - \frac{h_2}{D_{\underline{\alpha\alpha}}} \int\limits_{-h_2}^{z} zC_{\alpha\underline{\alpha}}(z)\, dz\Big) + 1 \end{aligned} \tag{3.22}$$

Formulae (3.18) and (3.21) coincide with those found in the work by Hinton and Owen (1984)[2].

In the case of a homogeneous anisotropic plate equations (3.12)-(3.17) imply

$$\begin{aligned} &k_{11} = k_{22} = k_{12} = k_{21} = \frac{5}{6} \qquad\qquad k^{+}_{11} = k^{+}_{22} = -k^{-}_{11} = -k^{-}_{22} = \frac{1}{6} \\ &k^{+}_{12} = k^{-}_{12} = k^{+}_{21} = k^{-}_{21} = 0 \end{aligned} \tag{3.23}$$

In the case of a transversely isotropic but non-homogeneous plate the transverse forces can be determined from (1.8), where the shear stiffness is defined by the formula

$$K = k \int\limits_{-h_2}^{h_1} G_3\, dz$$

in which

$$k = \frac{h^2}{\int\limits_{-h_2}^{h_1} G_3(z) dz \int\limits_{-h_2}^{h_1} \frac{1}{G_3(z)} f^2(z)\, dz} \tag{3.24}$$

[2]In thas work, page 254, one reads that equations (3.18) can be used to find the shear coefficients in the case of unbalanced plates. This statement is untrue, since we deal with different functions $f_{\underline{\alpha\alpha}}(z)$ in this case, compare Eqs (3.21) with Eqs (3.8).

$$k^{+} = \frac{h \int\limits_{-h_2}^{h_1} \frac{1}{G_3(z)} g^{+}(z) f(z)\, dz}{h_1 \int\limits_{-h_2}^{h_1} \frac{f^2(z)}{G_3(z)}\, dz} \qquad k^{-} = \frac{h \int\limits_{-h_2}^{h_1} \frac{1}{G_3(z)} g^{-}(z) f(z)\, dz}{h_2 \int\limits_{-h_2}^{h_1} \frac{f^2(z)}{G_3(z)}\, dz} \tag{3.25}$$

$$\begin{aligned} f(z) = f_{11}(z) = f_{22}(z) = \frac{h}{\Delta}\Big(H \int\limits_{-h_2}^{z} \frac{E(z)}{1-\nu^2(z)}\, dz - \\ -B \int\limits_{-h_2}^{z} z\frac{E(z)}{1-\nu^2(z)}\, dz\Big) \end{aligned} \tag{3.26}$$

$$\begin{aligned} g^{+}(z) = g^{+}_{11} = g^{+}_{22} = -\frac{1}{\Delta}\Big[(D - h_1 H) \int\limits_{-h_2}^{z} \frac{E(z)}{1-\nu^2(z)}\, dz - \\ -(H - h_1 B) \int\limits_{-h_2}^{z} z\frac{E(z)}{1-\nu^2(z)}\, dz\Big] \\ g^{-}(z) = g^{-}_{11} = g^{-}_{22} = -\frac{1}{\Delta}\Big[(D + h_2 H) \int\limits_{-h_2}^{z} \frac{E(z)}{1-\nu^2(z)}\, dz - \\ -(H + h_2 B) \int\limits_{-h_2}^{z} z\frac{E(z)}{1-\nu^2(z)}\, dz\Big] + 1 \end{aligned} \tag{3.27}$$

where $\Delta = BD - H^2$ and while the stiffnesses B, H, D are given by

$$\begin{aligned} B = \int\limits_{-h_2}^{h_1} \frac{E(z)}{1-\nu^2(z)}\, dz \qquad H = \int\limits_{-h_2}^{h_1} \frac{zE(z)}{1-\nu^2(z)}\, dz \\ D = \int\limits_{-h_2}^{h_1} \frac{z^2 E(z)}{1-\nu^2(z)}\, dz \end{aligned} \tag{3.28}$$

References

1. BOLLE L., 1947, Contribution au probléme linéaire de flexion d'une plaque élastique, *Bull. Tech. Suisse Rom.*, **73**, 21, 281-285

2. BOLLE L., 1947, Contribution au probléme linéaire de flexion d'une plaque élastique, *Bull. Tech. Suisse Rom.*, **73**, 22, 293-298

3. HENCKY H., 1947, Über die Berücksichtigung der Schubverzerrung in ebenen Platten, *Ing. Arch.*, **16**, 72-76

4. HINTON E., OWEN D.R.J., 1984, *Finite Element Software for Plates and Shells*, Swansea UK, Prineridge Press

5. JEMIELITA G., 1998, O współczynnikach ścinania, *Polish-Ukrainian seminar "Theoretical Foundations of Civil Engineering"*, Warszawa, Oficyna Wydawnicza PW, 115-122

6. JEMIELITA, G., 2001, Teorie płyt sprężystych, In: C. Woźniak (edit.) *Mechanika Techniczna*, **8**, Warsaw, Polish Sci. Publ.

Współczynniki ścinania poprzecznie niejednorodnych płyt średniej grubości

Streszczenie

W pracy przedstawiono sposób wyznaczania współczynników ścinania anizotropowych, poprzecznie niejednorodnych płyt średniej grubości o poprzecznrej niejednorodności. Korzystając z założeń kinematycznych Hencky-Bolle'a podano podstawowe równania teorii płyt średniej grubości. Korzystając z zasady równoważności prac wirtualnych wyznaczono wzory na współczynniki ścinania i podano ich jawną postać.

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