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A STUDY OF THE FLIGHT DYNAMICS HELICOPTER CARRYING AN EXTERNAL LOAD USING BIFURCATION THEORY AND CONTINUATION METHODS

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The paper presents a study of the flight dynamics of a helicopter with an articulated rotor, carrying a suspended load. The aircraft model includes rigid body dynamics, individual flap and lag blade dynamics, and inflow dynamics. The load is a point mass with a single suspension point. Results are obtained for load masses of up to 1500 kg, with load-to-helicopter mass ratios up to 33%, and cable lengths up to 55 m. The presence of the external load modifies the flight dynamics and handling quality characteristics of the helicopter because the dynamic and aerodynamic characteristics of the load may make it unstable in certain fight conditions. Maneuvers of this system in wide flight regimes involves non-linear aerodynamics and inertial coupling. In can be stated that the helicopter with the suspended load is a inherently non-linear and time varying system. Theory of dynamical systems provides a methodology for studying such non-linear systems. Bifurcation theory is a part of that theory. It considers changes in the stability of the system which lead to qualitatively different responses of it. In this paper, results from the theory of dynamical systems are used to predict the nature of instabilities caused by bifurcations and the response of the rotorcraft with the suspended load that follow such bifurcations.

Key words: non-linear helicopter dynamics, bifurcation theory, continuation method

1. Introduction

Application of helicopters to transport of heavy and bulky loads creates stability problems, especially in the hovering with hanging loads. Both military and commercial operators have exploited the capability of a helicopter to rapidly move heavy loads to locations where the use of ground based equipment would be impractical or impossible. The external load can modify the flight dynamic characteristics of a helicopter because the load behaves like a pendulum, and can change the natural frequencies and mode shapes of low frequency modes of the helicopter. Also, the aerodynamics of the load may make it unstable in certain flight conditions, with repercussions on the stability and safety of the entire helicopter/load system.

The dynamics of a helicopter with external suspended loads received considerable attention in the late 1960's and early 1970's. There were two reasons for that interest. Firstly, the extensive external load operations in the Vietnam war, and secondly, the Heavy-Lift Helicopter program (HLH). This interest has been renewed recently, prompted by the re-evaluation and extension of the ADS-33 [1] Helicopter Handling Qualities Specifications to transport helicopters, and by the expectation of emerging new cargo helicopter manufacturers.

1.1. Historical outlook

One of the first theoretical studies of the dynamics of a helicopter with a slung load is due to Lucassen and Sterk (1965). A simple 3-degree of freedom model of the hover longitudinal dynamics of the helicopter and the angular displacement of the load was used. A single suspension point was assumed and the aerodynamic forces and moments on the load were neglected. In general, the pole associated with the load pendulum mode was stable; the phugoid remained unstable, but its frequency decreased with the increasing cable length. For some combinations of parameters, the helicopter mode became unstable while the load mode was stabilised. Szustak and Jenney (1971) pointed out that a conventional stability augmentation system was not adequate for precising the hover and load release, and could result in pilot-induced oscillations (PIO). A more effective solution consisted of an inner loop, in which the relative motion of the aircraft and load was fed back to cyclic, and an outer loop, in which the aircraft position above the ground was fed back, again to cyclic. Dukes (1973a) studied the basic stability characteristics of a helicopter with a slung load, and possible feedback stabilisation schemes, and appropriate piloting strategies for various manoeuvres (Dukes, 1973b).

In this work a 3-degree of freedom longitudinal helicopter/load model was used. The positive pitch damping, whether provided by the rotor alone or also by the flight control system, did not necessarily increase the stability of the pendulum mode of the load. This mode, essentially undamped, could become unstable for certain configurations (i.e. cable lengths, load weight, and relative position of the attachment point and gravity center of the aircraft). The pitch damping provided at best a moderate increase in the damping of the mode. A feedback control scheme in which the attachment point was actively and longitudinally moved, proved very effective in the paper, but its practical feasibility was not explored. The previous studies were limited to hover or low speed flight, and therefore the aerodynamics of the suspended load did not play a significant role.

Slung loads are rarely aerodynamically shaped bodies. Typical loads are bluff bodies that may be subject to dynamic instabilities triggered by unsteady aerodynamics. Poli and Cromack (1973) studied the stability in forward flight of a helicopter carrying a container and a circular cylinder. The results indicated that long cables, high speeds, and low weights increased the stability of the loads. A stability study in forward flight by Cliff and Bailey (1975) partially confirmed the results by Pioli and Cromack (1973) because the decrease of the weight improved the stability, but longer cables were found to be destabilising. The differences may be due to different aerodynamics of the load, which was much more idealised by Cliff and Bailey (1975). The lowering of the drag increased the stability. The lateral and longitudinal stability was governed by the same parameters, but the conditions for the lateral stability proved more stringent. The results presented by Łucjanek and Sibilski (1978) confirmed the result obtained by Cliff and Bailey. A few years later, Sibilski and Łucjanek (1983) addressed the stability of a single-point configuration of the load suspension system. The analysed model was much more sophisticated than in any of the previously mentioned studies, and included full non-linear equations for the helicopter motion and dynamics of the 3-degrees of freedom model of the suspended load. The equations were then linearized for the stability analysis, and the effect of several configuration parameters was investigated. The cable length, fore/aft and vertical position of the suspension point, and load weight were all found to affect the stability. Depending on the combination of parameters, some modes could be stabilised and others destabilised, but in all, the instabilities were quite weak. Concurrently, Nagabhushan (1985) addressed in his analysis the low-speed stability of a single-point configurationof the load suspension system. The analysed model included full non-linear equations for the rigid body aircraft motion and rotor flap dynamics. His results confirmed that the cable length, fore/aft and vertical position of the suspension point as well as the load weight affect the stability of the helicopter-suspended load system.

More recently, Cicolani *et al.* (1989) reported the results of flight tests of a UH-60 helicopter, including frequency responses obtained using system identification techniques. While the study focused primarily on the system identification and simulation validation, several conclusions were presented regarding the effect of the loads on the flight dynamics and handling quality characteristics. The increasing load weight reduced the lateral bandwidth; a further increase could reduce the bandwidth to a value below that of the pendulum frequency. The longitudinal stability margins were not very sensitive to the load, but the lateral stability margins were degraded. The effect on the bandwidth and phase delay was highly variable depending on the load configuration. At last, Fusato *et al.* (1999) explored some fundamental aspects of the dynamics of a helicopter with an articulated rotor with an external load suspended from a single attachment point. The results indicated that the external load affects the trim state primarily through the overall increase in the weight of the aircraft, both in straight and in turning flight. The influence of the length of the cable was negligible. Substantial dynamic coupling with the Dutch roll mode occurred. The load mode consisted primarily of the lateral motion of the load. The effect of the load on the phugoid was very small. The suspended load modified the on-axis roll frequency response by adding a notch to the gain curves and a roughly 180-degree jump in the phase curves. The modifications of the frequency response introduced by the load occurred primarily at frequencies lower than those used to determine the bandwidth and phase delay according to the ADS-33 specifications. The phase shifts caused additional crossings of the 135-degree delay line, which at least formally, can reduce the phase bandwidth considerably. If these additional crossings were ignored, the changes in the bandwidth and phase delay would be generally small.

1.2. Subject of the present investigations

A helicopter carrying a suspended load is a inherently non-linear and time varying system. Therefore, a linear model is adequate for basic studies of the flight dynamics of helicopters with suspended loads, but it cannot describe some important practical problems. For example, it is not possible to model, using a linear approach, the sling load "vertical bounce" phenomenon. Another type of problems that linear models cannot capture is the aerodynamic instability due to unsteady inflow, and/or the non-streamlined shape of many suspended loads. These instabilities, described by Gabel and Wilson (1968), Poli and Cromack (1973), Sheldon (1977), Simpson and Flower (1981), and Cicolani *et al.* (1995) often limit the maximum speed of helicopters.

Recent developments in the field of numerical analysis of non-linear equations created a class of computer algorithms known as the continuation methods. Those methods use predictor- corrector techniques to follow solution curves of systems of non-linear equations of motion represented by functions of a number of variables and parameters, respectively. This approach was successfully demonstrated for the flight dynamics analysis a helicopter with a suspended load. The continuation methods are a class of predictor corrector techniques for the solution of systems of non-linear algebraic equations, which are functions of a number of parameters, over a specified range of the parameters. The general technique is to fix all parameters but one, and trace the steady states of the system as a function of this parameter. The stability of each steady state can be determined by calculating the eigenvalues of the linearized system. Any changes in the stability from one steady state to the next will signify bifurcation. Theory dynamical systems has provided a powerful tool for analysis of non-linear phenomena of the aircraft behaviour. In the application of this theory, the numerical continuation methods and bifurcation theory have been used to study roll-coupling instabilities, stall/spin phenomena, and dynamics of high angles of attack is a number of aircraft models. Results of great interest have been reported in several papers (one should mention here papers by Carroll and Mehra (1982), Guicheteau (1990), Jahnke and Culick (1994), Avanzini and de Matteis (1998), Sibilski (1999a,b, 2000), Marusak *et al.* (2000)). The continuation methods are numerical techniques for calculating the steady states of systems of ordinary differential equations, and can be used to study the roll coupling instabilities and high-angle of attack instabilities.

The objectives of this paper are:

- To present formulations and solutions of a mathematical model of a helicopter with an articulated rotor, carrying an external load
- To study the effects of the cable length and load weight on the helicopter/load system dynamic characteristics.

In the present study some fundamental aspects of the nonlinear dynamics of a helicopter with an articulated rotor and with an external suspended load are studied using the continuation and bifurcation methods, by means of checking the stability characteristics related to unstable equilibria. Numerical simulations are used to verify the predictions. The mathematical model of a helicopter used in this study is a nonlinear blade element type model that includes the fuselage, rotor, main rotor inflow, and propulsion system dynamics. The 6-degrees of freedom rigid body motion of the aircraft, 2-degrees of freedom rigid body motion of the slung load, and an articulated, four-bladed main rotor with rigid blades are assumed. The aerodynamic loads are unsteady forces and moments in the direction determined by the local airflow (defined by the suspended load angle of attack and slip angle). Unsteady aerodynamic effects are modeled using the ONERA dynamic inflow model (Tran and Petot, 1981). The state vector has a total of 30 elements: flap and lag displacements and rates for each of the 4 blades (16 states); 9 rigid body velocities, rates, and attitudes; and 2 external load angles with their respective rates. The formalism of analytical mechanics allows one to present dynamic equations of motion of the coupled load/helicopter dynamic system giving an incredibly interesting and comfortable tool for the construction of equations of motion. An example can be the Boltzmann-Hamel equations, which are a generalization of the Lagrange equations of the second kind in quasi-coordinates. These equations are written in the form allowing one to create procedures meant for their automatic formulation, (e.g., by means of the well known commercial software as Mathematica® or Maple V®).

2. Mathematical model of the helicopter – slung load system

2.1. Systems of coordinates

The external load is essentially modeled as a point mass that behaves like a spherical pendulum suspended from a single point. The geometry and the relevant coordinate systems are shown in Fig. 1. The position of the load is described by the three angles Θ_L, Ψ_L, and Φ_L, where Ψ_L is the azimuth angle of the load, Φ_L is measured from the y_3' axis, and Θ_L is measured from the z_4 axis.

The following systems of coordinates (Fig. 1) are used in futher constributions:

- A vertical moving system of co-ordinates $Ox_gy_gz_g$, the Oz_g axis of which is vertical and directed downwards.
- A system of co-ordinates $Oxyz$ attached to the rotorcraft. The origin of this system overlap the rotorcraft centre of mass. The Oxz plane coincides with the symmetry plane of the aircraft, and the Oz axis is directed downwards.
- A system of co-ordinates $O_4x_4y_4z_4$, with the origin O_4 that overlaps the suspension point. All axes are parallel to the slung load axis of inertia, the Ox_4 axis of that system is directed forwards, and the Oz_4 axis is directed downwards.
- A system of co-ordinates $O_5x_5y_5z_5$ attached to the suspended load. The origin of this system overlap the suspended load center of mass. The

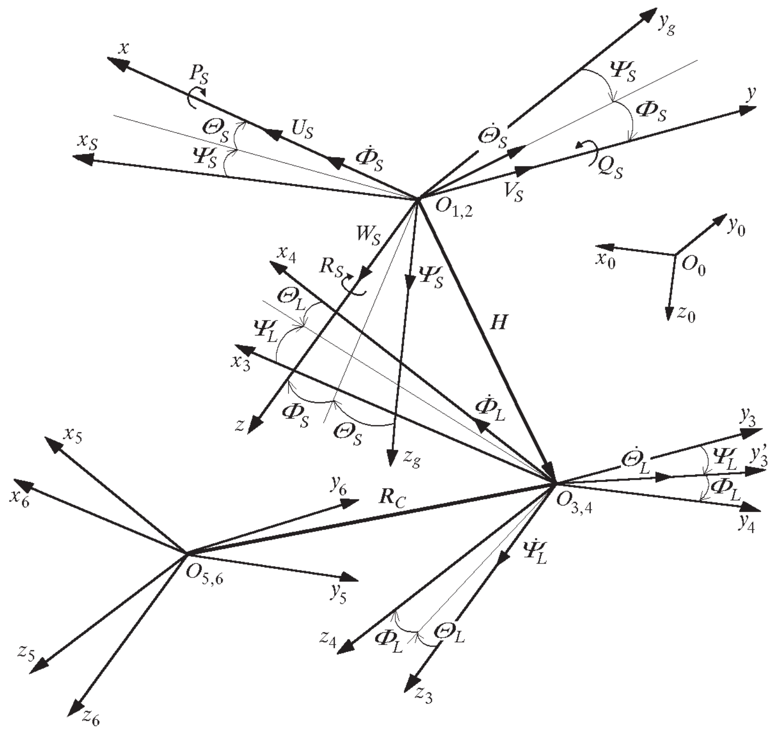

Fig. 1. Systems of co-ordinates

axes of this system are parallel to the axis of the system of co-ordinates $Oxyz$.

– A system of co-ordinates $O_6x_6y_6z_6$ attached to the suspended load. The origin of this system overlap the suspended load center of mass. The axes of this system are parallel to the axis of the system of co-ordinates $O_4x_4y_4z_4$ (the axes of this system overlap the slung load axis of inertia).

The systems of coordinates attached to the main rotor blades are shown in Fig. 2. The following systems of coordinates are used:

– A system of co-ordinates $Ox_{WN}y_{WN}z_{WN}$ with the origin that overlap the centre of the main rotor hub. All axes are parallel to the system of co-ordinates $Oxyz$ attached to the aircraft.
– Systems of co-ordinates of the main rotor hub $Or_is_iz_{WN}$ $(i = 1, 2, ..., n;$ n – number of main rotor blades) attached to the main rotor hub. Those systems turn with the rate Ω of the main.

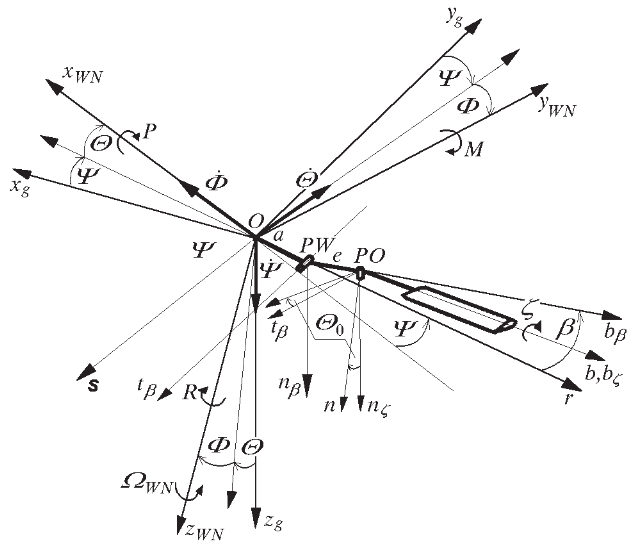

Fig. 2. Systems of co-ordinates attached to the main rotor and main rotor blades

– Systems of co-ordinates attached to the main rotor blades. The rotor blade is mounted to the hub on a universal join – free to flab (flapping hinge PW, the system of co-ordinates $PWt_\beta b_\beta n_\beta$), lead or lag (lag hinge PO, the system of co-ordinates $POt_\zeta b_\zeta n_\zeta$), but fixed in pitch (feathering hinge, the system of co-ordinates $POtbn$).

2.2. Equations of motion

The formalism of theoretical mechanics allows one to present dynamic equations of motion of mechanical systems in quasi-coordinates, giving an incredibly interesting and comfortable tool for the construction of equations of motion of an aircraft. An example can be the Boltzmann-Hamel equations, which are a generalisation of the Lagrange equations of the second kind expressed in quasi-coordinates. The Boltzmann-Hamel equations have the following form (Gutowski, 1972; Arczewski and Pietrucha, 1993)

$$\frac{d}{dt}\Big(\frac{\partial T^*}{\partial \omega_\sigma}\Big) - \frac{\partial T^*}{\partial \pi_\sigma} + \sum_{\mu=1}^{k}\sum_{\lambda=1}^{k} \gamma^{\mu}_{\sigma\lambda}\frac{\partial T^*}{\partial \omega_\mu}\omega_\lambda = Q^*_\sigma \tag{2.1}$$

where T^* is the kinetic energy (function of quasi-coordinates), ω_σ – quasi-velocity, π_σ – quasi co-ordinate, q_λ, q_σ – generalized co-ordinates, $Q^*_\sigma = \sum_{\sigma=1}^{k} Q_\sigma b_{\sigma\mu}$ – component of the generalized force vector, k – number of degrees of freedom of the mechanical system, $\gamma^r_{\mu\alpha}$ – Boltzmann symbols (Gutowski, 1972; Arczewski and Pietrucha, 1993)

$$\gamma^r_{\mu\alpha} = \sum_{r=1}^{k}\sum_{\alpha=1}^{k}\Big(\frac{\partial a_{r\sigma}}{\partial q_\lambda} - \frac{\partial a_{r\lambda}}{\partial q_\sigma}\Big)b_{\sigma\mu}b_{\lambda\alpha} \tag{2.2}$$

where $a_{r\sigma}$, $b_{r\sigma}$ are elements of the transformation matrix.

Relations between the quasi-velocities and generalized velocities are shown in the equations

$$\begin{aligned} \omega_\sigma &= \sum_{\alpha=1}^{k} a_{\sigma\alpha}(q_1, q_2, ..., q_k)\dot{q}_\alpha \\ \dot{q}_\sigma &= \sum_{m=1}^{k} b_{\sigma\mu}(q_1, q_2, ..., q_k)\omega_\mu \qquad\qquad \sigma = 1, ..., k \end{aligned} \tag{2.3}$$

Equations (2.3) can be written in the matrix form

$$\boldsymbol{\Omega} = \mathbf{A}_T\dot{\boldsymbol{q}} \qquad\qquad \dot{\boldsymbol{q}} = \mathbf{A}_T^{-1}\boldsymbol{\Omega} = \mathbf{B}_T\boldsymbol{\Omega} \tag{2.4}$$

where $\boldsymbol{\Omega}$ is the vector of the quasi-velocities and $\boldsymbol{q}$ – vector of the generalized co-ordinates

$$\begin{aligned} \boldsymbol{\Omega} &= [\omega_1, \omega_2, ..., \omega_k]^\top \\ \boldsymbol{q} &= [q_1, q_2, ..., q_k]^\top \end{aligned} \tag{2.5}$$

The construction of the matrix $\mathbf{A}_T$ depends on the explored issue. In the case when we consider a model of a helicopter carrying a suspended load treated as a system containing a rigid fuselage and n rigid blades of the main rotor fixed to the hub by means of three hinges, and three degrees of freedom hinging load, the quasi-velocities and generalized co-ordinates have the following forms

$$\begin{aligned} \boldsymbol{\Omega} &= [u, v, w, p, q, r, \Omega, \dot{\beta}_1, ..., \dot{\beta}_n, \dot{\zeta}_1, ..., \dot{\zeta}_n, \dot{\Psi}_L, \dot{\Theta}_L, \dot{\Phi}_L]^\top \\ \boldsymbol{q} &= [x_s, y_s, z_s, \Phi, \Theta, \Psi, \psi, \beta_1, ..., \beta_n, \zeta_1, ..., \zeta_n, \Psi_L, \Theta_L, \Phi_L]^\top \end{aligned} \tag{2.6}$$

The matrix $\mathbf{A}_T$ is then as follows

$$\mathbf{A}_T = \begin{bmatrix} \mathbf{A}_G & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}_T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{bmatrix} \tag{2.7}$$

The matrices $\mathbf{A}_G$ and $\mathbf{C}_T$ are classical matrices of transformations of kinematic relations, and can be found by Sibilski (1998), the unit matrix $\mathbf{I}$ has dimension $(3n+1) \times (3n+1)$, n is the number of the main rotor blades.

The matrices $\mathbf{D}_i$ can be determined as follows

$$\mathbf{D}_i = \frac{d\boldsymbol{a}_i}{d\boldsymbol{q}} = \begin{bmatrix} \dfrac{\partial a_{11}}{\partial q_1} & \cdots & \dfrac{\partial a_{1k}}{\partial q_k} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial a_{k1}}{\partial q_1} & \cdots & \dfrac{\partial a_{kk}}{\partial q_k} \end{bmatrix} \tag{2.8}$$

where the vector $\boldsymbol{a}_i$ means ith row of the matrix $\mathbf{A}_T$.

In the matrix notation, the Boltzmann symbols can be presented in the form of elements of the block matrix $\boldsymbol{\Gamma}_{(k\times(k\times k))}$ with k being the number of degrees of freedom of the examined dynamic system

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}^1 \\ \boldsymbol{\Gamma}^2 \\ \cdots \\ \boldsymbol{\Gamma}^k \end{bmatrix} = \begin{bmatrix} \mathbf{B}_T^\top(\mathbf{D}_1 - \mathbf{D}_1^\top)\mathbf{B}_T \\ \mathbf{B}_T^\top(\mathbf{D}_2 - \mathbf{D}_2^\top)\mathbf{B}_T \\ \cdots \\ \mathbf{B}_T^\top(\mathbf{D}_k - \mathbf{D}_k^\top)\mathbf{B}_T \end{bmatrix} \tag{2.9}$$

The matrix $\boldsymbol{\Gamma}$ can be presented in the short form

$$\boldsymbol{\Gamma} = \mathbf{B}_T^\top(\mathbf{D} - \mathbf{D}^\top)\mathbf{B}_T \tag{2.10}$$

Finally, the Boltzmann-Hamel equations written in the matrix form can be presented as follows

$$\frac{d}{dt}\Big(\frac{\partial T^*}{\partial \boldsymbol{\Omega}}\Big) + (\boldsymbol{\Gamma}^\top \boldsymbol{\Omega})\frac{\partial T^*}{\partial \boldsymbol{\Omega}} - \mathbf{B}_T^\top \frac{\partial T^*}{\partial \boldsymbol{q}} = \boldsymbol{Q} - \boldsymbol{U} \tag{2.11}$$

The vector $\boldsymbol{Q}$ is the sum of aerodynamic loads and another non-potential forces acting on the helicopter-slung load system, $\boldsymbol{U}$ is the vector of potential forces acting on the helicopter and external hanging load. Equation (2.11) is very comfortable to use in procedures of automatic formulation of equations of motion.

In our case, the subject of considerations is the problem of dynamics of a helicopter with a slung heavy load. The quasi-velocities vector is given by Eq. $(2.6)_1$. The total kinetic energy of the system is the sum of the kinetic energy of the rigid fuselage of the helicopter, rotor blades, and the slung load

$$T^* = T_H^* + \sum_{i=1}^{n} T_{Bi}^* + T_L^* \tag{2.12}$$

According to the general theorem, the kinetic energy of the airframe is (see Sibilski, 1998)

$$T_H^* = \frac{1}{2}m\boldsymbol{V}^2 + \frac{1}{2}\boldsymbol{\Omega}_k^\top \mathbf{J}_A \boldsymbol{\Omega}_k \tag{2.13}$$

The kinetic energy of the ith rotor blade is (see Sibilski, 1998)

$$T_{Bi}^* = \frac{1}{2}m_{Bi}[\boldsymbol{V}+\boldsymbol{\Omega}_k\times\boldsymbol{x}_{Bi}+(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_{Bi})\times\boldsymbol{R}_{Bi}]^2+\frac{1}{2}(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_{Bi})^\top\mathbf{J}_{Bi}(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_{Bi}) \tag{2.14}$$

where $\boldsymbol{x}_{Bi}$ is the vector connecting the centre of gravity of the aircraft with the centre of the main rotor hub, $\boldsymbol{R}_{Bi}$ – vector connecting the centre of the main rotor hub with the blade centre of gravity, $\mathbf{J}_{Bi}$ – moment of inertia of the ith rotor blade with respect to the flapping axis, $\boldsymbol{\omega}_{Bi}$ – vector of the relative angular velocity of the rotor blade

$$\boldsymbol{\omega}_{Bi} = \Omega\boldsymbol{z}_{wn} + \frac{d\beta_i}{dt}\boldsymbol{t}_{\beta_i} + \frac{d\zeta_i}{dt}\boldsymbol{n}_{\zeta_i} + \frac{d\theta_i}{dt}\boldsymbol{b} \tag{2.15}$$

The kinetic energy of the slung load is given by the following formula

$$T_L^* = \frac{1}{2}m_L[\boldsymbol{V}+\boldsymbol{\Omega}_k\times\boldsymbol{H}+(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_L)\times\boldsymbol{R}_C]^2+\frac{1}{2}(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_L)^\top\mathbf{I}_L(\boldsymbol{\Omega}_k+\boldsymbol{\omega}_L) \tag{2.16}$$

where $\boldsymbol{H}$ is the vector connecting the helicopter centre of gravity with the point of suspension of the external load, $\boldsymbol{R}_C$ – vector connecting the point of suspension with the centre of gravity of the external slung load, $\mathbf{I}_L$ – tensor of inertia of the slung load.

After making some transformations, the relation for the kinetic energy can be presented in the form

$$T^* = \frac{1}{2}\boldsymbol{\Omega}^\top\mathbf{E}\boldsymbol{\Omega} \tag{2.17}$$

The matrix $\mathbf{E}$ depends on the mass distribution of the airframe and control surfaces, and has the form

$$\boldsymbol{E} = \begin{bmatrix} \mathbf{M} & -\mathbf{S}_1 & \mathbf{S}_2^{(E)} \\ \mathbf{S} & \mathbf{J}_A & \mathbf{N}^{(E)} \\ (\mathbf{S}_2^{(E)})^\top & (\boldsymbol{N}^{(E)})^\top & \mathbf{I}_S^{(E)} \end{bmatrix} \tag{2.18}$$

Elements of the matrices $\mathbf{M}$, $\mathbf{N}^{(E)}$, $\mathbf{J}^A$, $\mathbf{S}_1$, $\mathbf{S}_2^{(E)}$, $\mathbf{I}_L^{(E)}$, can be found in Sibilski (1980, 1998).

After diferentiation and transformations, we obtain a set of equations describing motion of the helicopter with an articulated rotor, carrying an external suspended load

$$\mathbf{E}\dot{\Omega} + \Big[(\boldsymbol{\Gamma}^\top \boldsymbol{\Omega})\mathbf{E} - \mathbf{B}^\top \boldsymbol{\Omega}^\top \frac{d\mathbf{E}}{d\boldsymbol{q}}\Big]\boldsymbol{\Omega} = \boldsymbol{Q} - \boldsymbol{U} \tag{2.19}$$

Equation (2.19), together with the kinematic relations, create a set of nonlinear ordinary differential equations of the first order that describe motion of the helicopter-suspended load system.

The kinematic relations can be found in Etkin (1972), Sibilski and Łucjanek (1983), Sibilski (1980, 1998). These equations are written in the form allowing one to create procedures meant for their automatic formulation, (e.g., by means of the well known commercial software as Mathematica® or Mathcad®).

2.3. Modeling of aerodynamic loads

Precise description of aerodynamic forces and moments found in the equations of motion is the fundamental source of difficulties. In each phase of the flight, the dynamics and aerodynamics influence each other, which disturbs the precise mathematical description of these processes. The requirement for a method enabling calculation of the aerodynamic load stem both from the flow environment and from the algorithms used in the analysis of the helicopter flight. The bifurcation approach is very fruitful when the sources and nature of aerodynamic phenomena are considered. It is assumed that the airframe model consists of the fuselage, horizontal tail, vertical tail, and landing gear. The fuselage model is based on wind tunnel test data (as a function of the angle of attack α and the slip angle β). The horizontal and vertical tails are treated as aerodynamic lifting surfaces with lift and drag coefficients computed from data tables as functions of the angle of attack α and the slip angle β. The tail rotor is a linear model using the strip-momentum theory with a uniformly distributed inflow. The influence of the rotor dynamic inflow on the airframe and aerodynamic forces and moments of the suspended load are included in the model. The technique used provides the essential effects of the increased interference velocity with an increased rotor load and decreased interference as the rotor wake deflects backward with the increased forward speed, see Narkiewicz (1994). On the basis of the results presented by Pit and Peters (1981), Krotophalli *et al.* (1999), Ermentrout (2001), the effects of changing velocity due to helicopter angular rates are included. Special techniques are proposed to calculate the aerodynamic forces and moments acting on the external load taking into account the dynamics inflow and the interference of the rotor wake on the slung load local inflow velocity and angles of attack and slip (Sibilski, 1980). The aerodynamic data for a NACA 23012 airfoil in the range of the angle of attack $\pm 23°$, and the compressibility effects were included. Those data were blended with suitable low speed data for the remainder of the 360° range to model the reversed flow region and fully stalled retreating blades. Semi-empirical methods, that use differential equations, were used to predict the unsteady aerodynamic loads and dynamic stall effects. The model developed by ONERA (Tran and Petot, 1981) was used. The ONERA model is a semi-empirical, unsteady, non-linear model, which uses experimental data to predict aerodynamic forces on an oscillating airfoil which experiences dynamic stall. State variable formulations of aerodynamic loads allow one to use the existing codes for rotorcraft flight simulation.

3. Dynamical systems theory

In this paper, we will study equations of the following form

$$\dot{x} = f(x,t;\mu) \tag{3.1}$$

and

$$x \mapsto g(x;\mu) \tag{3.2}$$

with $x \in U \subset \Re^n$, $t \in \Re^1$, and $\mu \in V \subset \Re^p$, where U and V are open sets in $\Re^n$ and $\Re^p$, respectively. We view the variables x as a vector of n state variables, the variables μ as a vector of m parameters (or controls), $\dot{x}$ is the time derivative of x, and $f:\ \Re^n \times \Re^m \to \Re^n$ is the smooth vector field (n non-linear functions). Note that both open loop (uncontrolled) and closed loop rigid-body flight dynamical systems can usually be represented in the form of equation (3.1), and referred to, see Ioos and Joseph (1980) as a *vector field* or ordinary differential equation, and to (3.1) as a *map or difference equation*. Both are termed *dynamical systems*.

By a solution to Eq. (3.1) we mean a map, x, from some interval $\Im \subset \Re^1$ into $\Re^n$, which we represent as follows

$$x:\ \ \Im \to \Re^1 \qquad\qquad t \mapsto x(t) \tag{3.3}$$

such that $x(t)$ satisfies (3.1), i.e.

$$\dot{x}(t) = f(x(t),t;\mu) \tag{3.4}$$

The dynamical systems theory (DST) provides a methodology for studying systems of ordinary differential equations. The most important ideas of DST used in the paper will be introduced in the following sections. More information on DST can be found in the book of Wiggins (1990). The first step in the DST approach is to calculate steady states of the system and their stability. The steady states can be found by setting all time derivatives equal to zero and solving the resulting set of algebraic equations. The Hartman-Grobman theorem (see p. 234 in the Wiggins's book) proves that the local stability of a steady state can be determined by linearizing the equations of motion about the steady state and calculating the eigenvalues. The implicit function theorem (Ioos and Joseph, 1980, Chap. 2) proves that any steady state of a system is a continuous function of the parameters of the system if the linearized system is non-singular. A singular linearized system is characterised by a zero eigenvalue. Thus, the steady states of the equations of motion for an aircraft are continuous functions of the control surface deflections and/or vector of the thrust inclinations. Stability changes can occur as the parameters of the system are varied in such a way that the real parts of one or more eigenvalues of the linearized system change their sign. The changes in the stability of a steady state lead to qualitatively different responses of the system and are called bifurcations. Stability boundaries can be determined by searching for the steady states, which have one or more eigenvalues with zero real parts. There are many types of bifurcations, and each has different effects on the aircraft response. Qualitative changes in the response of the aircraft can be predicted by determining how many and what types of eigenvalues have zero real parts at the bifurcation point. The bifurcations for which one real eigenvalue is zero lead to the creation or destruction of two or more steady states. The bifurcations for which one pair of complex eigenvalues has zero real parts can lead to the creation or destruction of periodic motion. The bifurcations for which more than one real eigenvalue or more than one pair of complex eigenvalues has zero real parts lead to very complicated behaviour. The continuation methods are a class of a numerical algorithm used to follow a path of steady states in continuous or discrete dynamical systems as a parameter varies. They make use of the Implicit Function Theorem, which essentially states that if the Jacobian matrix $\mathbf{J}$, Eq. (3.5), of the system linearized at a stationary point is non-singular then this solution is *locally* unique, i.e. it is part of a unique curve of stationary points which is a continuous function of the parameters. The parametric continuation methods are used in numerical application of the bifurcation theory. The associated theorems involve properties of the eigenvalues at steady state solutions points (or Floquet multipliers for periodic orbit solutions), and it is therefore useful in the bifurcation analysis to find all relevant solution branches within a state-parameter space whilst evaluating the eigensystem as the algorithm proceeds. It is this characteristic of the continuation methods that make them suitable for the "global" control law design task at hand: the steady states provide a substantial amount of information about the mechanics governing the system response – including that of the closed-loop controlled system. The Jacobian matrix of an equilibrium point x_0 of the vector field or the fixed point x_0 is the matrix

$$\mathbf{J} = Df(x_0) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdot & \dfrac{\partial f_1}{\partial x_1} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdot & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{3.5}$$

The eigenvalues of the Jacobian matrix are important for the stability analysis. The following notations are used:

Vector Fields: n_0 is the number of eigenvalues of Df_0 with zero real parts; n_+ – number of eigenvalues of Df_0 with positive real parts; n_- – number of eigenvalues of Df_0 with negative real parts

Maps: n_0 in the number of eigenvalues of DF_0 on unit circle; n_+ – number of eigenvalues of DF_0 outside the unit circle; n_- – number of eigenvalues of DF_0 inside the unit circle. An equilibrium or fixed point is called hyperbolic if $n_0 = 0$, that is, it has no overvalues on the imaginary axis.

The continuation method is used to explore the nature of the system steady states as a parameter varies. In this context, the steady states may refer to as standard *equilibria* (stationary trim points such as steady level flight, steady climbs and descents, steady turns and spins) or to *periodic orbits* (limit cycles such as wing rock and oscillatory spin/autorotation). The evolution of branches of equilibria are computed by selecting one of the m controls/parameters as the "free" parameter (or continuation parameter), and then solving for

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \mu) = 0 \qquad \boldsymbol{x} \in \Re^n \qquad \mu \in \Re \tag{3.6}$$

where μ is one of the members of ν. In the work presented here, the continuation method based on that of Keller (1977) is used. The results are plotted as "one-parameter bifurcation diagrams" – two-dimensional plots of each state variable versus μ, with the line-type indicating relevant information about eigenvalue locations. Although dealing solely with the steady states, this information establishes a powerful insight into the mechanics governing the transient responses of the system. When designing the controllers for a manoeuvrable aircraft, the graphical depiction of the underlying dynamics and how this changes in the presence of the controller, is particularly advantageous.

3.1. Poincaré map

In the classical approach to differential equations, the stability of periodic solutions is discussed in terms of the characteristic or Floquet multipliers. The Poincaré map is a more geometrical view, which is in essence equivalent with the classical ideas. To determine the Poincaré map we need three notions: *periodic orbit, flow*, and *vector field*. For the purposes of this paper, it is generally sufficient to regard a differential equation as a system given by equation (3.1). We say that the *vector field* f generates a *flow* $\varPhi_t : U \to \Re^n$, where $\phi_t(x) = \phi(x,t)$ is a smooth function defined for all x in U and t in some interval $\Im = (a,b) \subseteq \Re$, and $\varPhi$ satisfies Eq. (3.1) in the sense that

$$\frac{d}{dt}\Big(\phi(x,t)\Big)_{t=\tau} = f\Big(\phi(x,\tau)\Big) \tag{3.7}$$

for all $x \in U$ and $\tau \in \Im$. Systems of the form (1.1), in which the vector field does not contain time explicitly, are called *autonomous*.

Often we are given an initial condition

$$x(0) = x_0 \in U \tag{3.8}$$

in which case we seek a solution $\phi(x_0,t)$ such that

$$\phi(x_0,t) = x_0 \tag{3.9}$$

In this case $\phi(x_0,t) : \ \Im \to \Re^n$ defines a *solution curve, trajectory*, or *orbit* of differential equation (2.1) *based on* x_0. Let γ be a periodic orbit of some flow $\varPhi_t$ in $\mathcal{F}^n$ arising from a nonlinear vector field $f(x)$. We take a *local cross section* $\varXi \subset \Re^n$, of dimension $n-1$. Denote the point where γ intersects S by p. The *Poincaré map* $P : \ U \to \varSigma$ is defined for a point $q \in U$ by

$$P(q) = \phi_\tau(q) \tag{3.10}$$

where $\tau = \tau(q)$ is the time taken for the orbit $\phi_\tau(q)$ based on q to the first return to S.

3.2. Bifurcation theory

For steady states of aircraft motion, very interesting phenomena appear even if one negative real eigenvalue crosses the imaginary axis when the control vector varies. Two cases can be considered.

- The steady state is regular, i.e. when the implicit function theorem works and the equilibrium curve goes through a limit point. It should be noted that the limit point is structurally stable under uncertainties of the differential system studied.
- The steady state is singular. Several equilibrium curves cross a pitchfork bifurcation point, and the bifurcation point is structurally unstable.

If a pair of complex eigenvalues cross the imaginary axis, when the control vector varies, Hopf's bifurcation appears (Marsden and McCracken, 1976; Keller, 1977; Hassard *et al.*, 1981). Hopf's bifurcation is another interesting bifurcation point. After crossing this point, a periodic orbit appears. Depending of the nature of nonlinearities, this bifurcation may be sub-critical or supercritical. In the first case, the stable periodic orbit appears (even for large changes of the control vector). In the second case the amplitude of the orbit grows in portion to the changes of the control vector.

Other domain of interest concerns the behaviour of the system when periodic orbits loose their stability. Three possibilities can be concerned in this case. Firstly, a real eigenvalue crosses the point $+1$. Limit points are observed in this case. Secondly, a real eigenvalue crosses the point -1. Period doubling bifurcation appears in this case. In the vicinity of this point, the stable periodic orbit of the period T becomes unstable, and a new stable periodic orbit of the period $2T$ appears. This type of stability loss conducts to chaotic motion. Thirdly, two conjugate eigenvalues leave the unit circle. Motion lines on a stable or unstable torus surround the unstable orbit after such a bifurcation.

3.3. Continuation technique

The continuation methods are a direct result of the implicit function theorem, which proves that steady states of a system are continuous functions of the parameters of the system except for steady states at which the linearized system is singular. The general technique is to fix all parameters except for one, and trace the steady states of the system as a function of this parameter. If one steady state of the system is known, a new steady state can be approximated by linear extrapolation from the known steady state, see Doedel and Kernevez (1986). The slope of the curve at the steady state can be determined by taking the derivative of the equation given by setting all time derivatives equal to zero. If two steady states are known, a new steady state can be approximated by linear extrapolation through the two known steady states. The stability of each steady state can be determined by calculating the eigenvalues of the linearized system. Any changes in the stability from one steady state to the next will signify a bifurcation.

3.4. Methodology scheme

Taking into account experience of many researches, one can formulate the following three-step methodology scheme (being based on the bifurcation analysis and continuation technique) for the investigation of nonlinear aircraft behaviour, see Carroll and Mehra (1982), Guicheteau (1990), Jahnke and Culick (1994).

- At the first step it is supposed that all parameters are fixed. The main aim is to search for all possible equilibria and closed orbits, and to analyze their local stability. This study should be as thorough as possible. The global structure of the state space (or *phase portrait*) can be revealed after determining the asymptotic stability regions for all discovered attractors (stable equilibria and closed orbits). An approximate graphic representation plays an important role in the treating of the calculated results.
- At the second step the system behaviour is predicted using the information about the evolution of the portrait with the parameter variations. The knowledge about the type of encountered bifurcation and current position with respect to the stability regions of other steady motions are helpful for the prediction of further motion of the aircraft. The rates of parameter variations are also important for such a forecast. The faster the parameters change, the more is the difference between the steady state solution and transient motion.
- Finally, the numerical simulation is used for checking the obtained predictions and obtaining transient characteristics of the system dynamics for large amplitude disturbances of the state variable and parameter variations.

3.5. Steady state conditions

The bifurcation theory is a set of mathematical results, which aims at the analysis and explanation of unexpected modifications in asymptotic behaviour of non-linear differential systems when parameters are slowly varying. For a fixed control vector $\boldsymbol{u}$, two types of asymptotic states are commonly encountered. The following relation gives the first one

$$f(x,\mu) = 0 \tag{3.11}$$

This relation is named a steady state. The second relation is given by the equation

$$x(T) = x(0) + \int_0^T f(x,\mu)\,dt \tag{3.12}$$

Starting with an approximation of a steady state for given values of parameters, the computer code determines, by a continuation process, the solution curve $x(\mu)$ of the following set of non-linear algebraic equations, and determine the type of bifurcation

$$\begin{cases} \text{Equilibriumpoints}: & f(x,\mu)=0 & \\ \text{Limitpoints}: & f(x,\mu)=0 & \lambda=0 \\ \text{Hopfpoints}: & f(x,\mu)=0 & \lambda_{1,2}=\pm 2i\pi/T \\ \text{Periodicorbits}: & x(T)=x(0)+\int_0^T f(x,\mu)\,dt & \end{cases} \tag{3.13}$$

The continuation process assumes that all functions for (3.11) are continuous and have derivatives. There are several continuation method algorithms. In the present work, the algorithms developed by Doedel and Kernevez (1986), which are based on the work of Keller (1977) were used.

For the examination purposes of this paper the continuation and bifurcation software XPPAUT (Ermentrout, 2001), a WINDOWS® version of the well known AUTO97[1] software, has been used.

4. Results

The helicopter configuration selected for this study is a representative of the PZL W-3 "Sokół". The weight without an external load is 5000 kg, corresponding to $C_T/\sigma = 0.065$. The altitude for all results is 0 m. Most results refer to combinations of cable lengths within the range $0 < l < 55$ m, and

[1] A very useful freeware is available on the Internet:
ftp://ftp.cs.concordia.ca/pub/doedel/auto. It enables one to determine all desired bifurcation points for different values of dynamical system parameters.

load masses within $0 < m < 1500\,\text{kg}$. The external load is assumed as a $2\text{m} \times 2\text{m} \times 5\text{m}$ container.

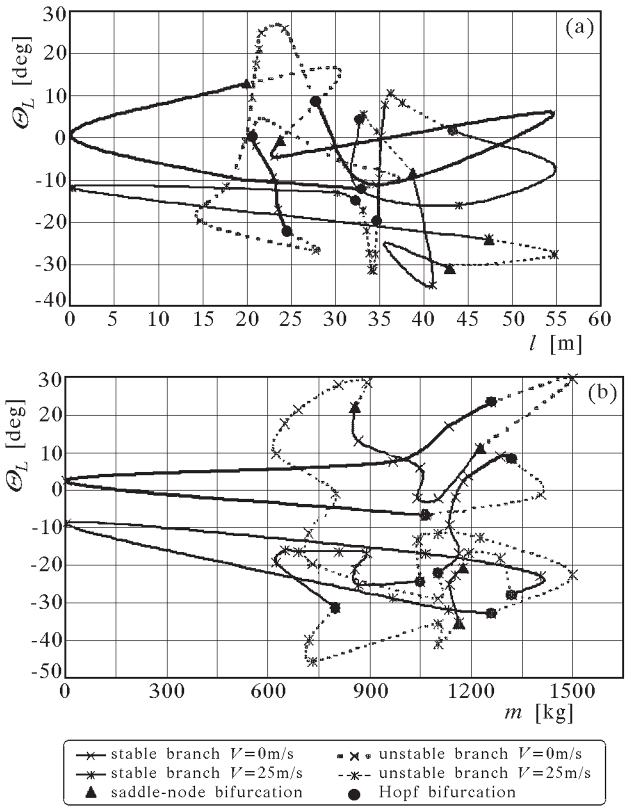

Fig. 3. Steady states for the helicopter carrying the suspended load

The presented results correspond to a bare airframe configuration. Figures 3 and 4 show the steady states for the helicopter with a slung load as a function of the cable length and suspended load mass for two velocities of the system. The first one is the hovering flight $(V = 0\,\text{m/s})$, and the second one is the steady level flight $(V = 25\,\text{m/s})$. The figures show that multiple steady states exist for most cable lengths and suspended load masses. For example, for the hovering flight case, the vertical line representing 15 m of the cable length intersects four steady states. Two of them are stable the others are unstable, so the helicopter could exhibit any of these four steady states. The segment of unstable steady states (for the hovering flight case), contains the trim conditions for the cable length from within $20\text{m} < l < 32\text{m}$ and the slung load mass $850\text{kg} < m_L < 1500\text{kg}$. It occurs, due to six saddle-node or Hopf's

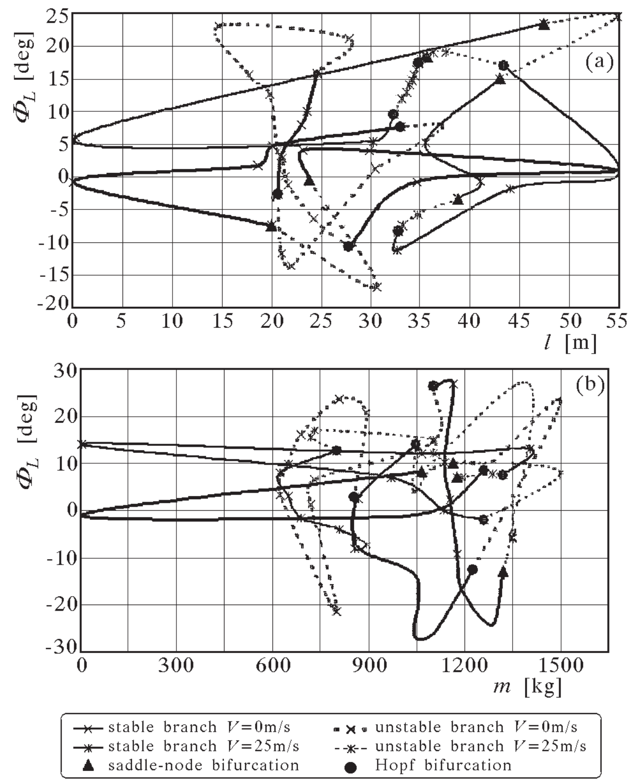

Fig. 4. Steady states for the helicopter carrying the suspended load

bifurcations, for the cable length of 20, 20.5, 24, 25, 28 and 38 meters. These cable lengths and slung load masses from the above mentioned ranges can be regarded as unsafe and dangerous slung load configurations.

Figures 5-13 show the time simulation of motion of the helicopter-suspended load system in which the cable length is assumed $l = 20\,\mathrm{m}$, the suspended load mass $m_L = 1050\,\mathrm{kg}$ (for the hovering flight case), and $l = 32.3\,\mathrm{m}$, $m_L = 800\,\mathrm{kg}$ (for the level flight case). The values of parameters assumed in that simulation put the helicopter with the slung load in the region of unstable steady states. The figures show rapidly developing aircraft oscillations (for the hovering flight case). The slung load oscillations grow slower. While considering the level flight case, the observed rotorcraft and slung load motions are rather stable. The results indicate that the external load affects the helicopter motion. Substantial dynamic coupling occurs with the Dutch roll mode (especially in the hovering flight case). The load mode consists of the lateral and longitudinal motion of the load. The effect of the load on the phugoid is

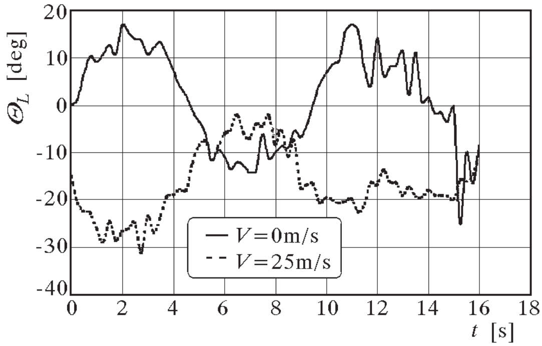

Fig. 5. Longitudinal deflection of the slung load

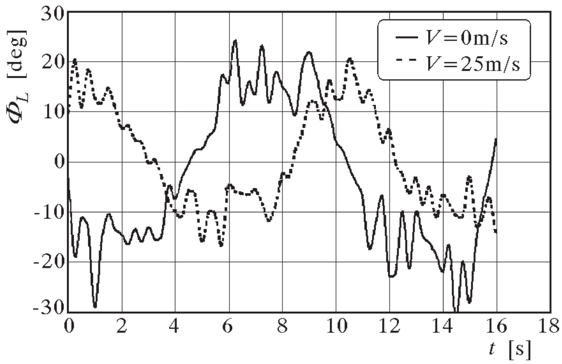

Fig. 6. Lateral deflection of the slung load

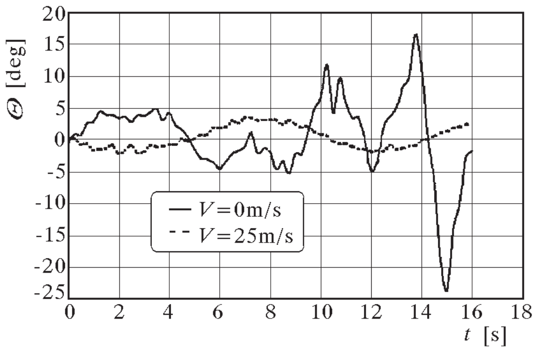

Fig. 7. Pitch angle of the airframe

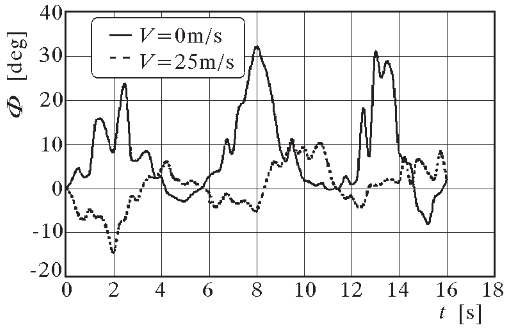

Fig. 8. Roll angle of the airframe

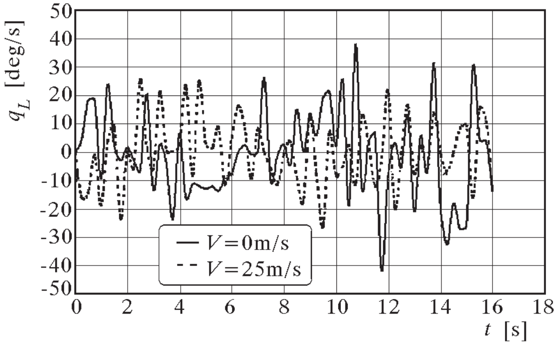

Fig. 9. Longitudinal rate of the slung load

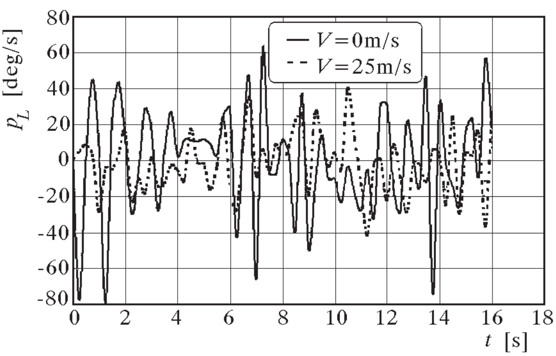

Fig. 10. Lateral deflrction rate of the slung load

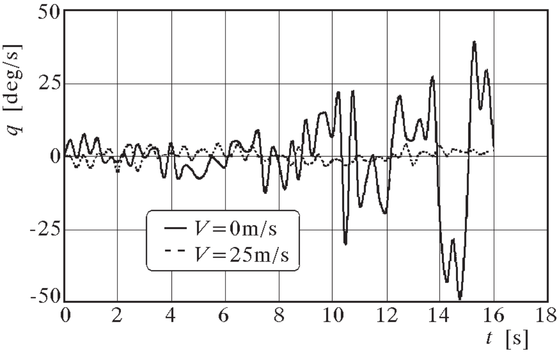

Fig. 11. Pitch rate of the airframe

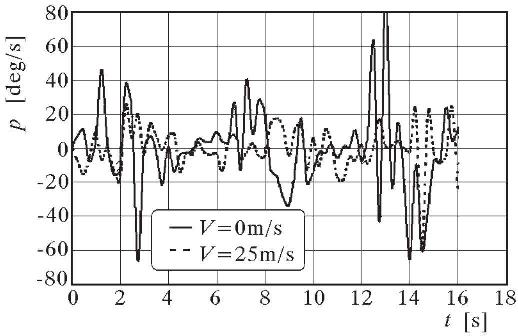

Fig. 12. Roll rate of the airframe

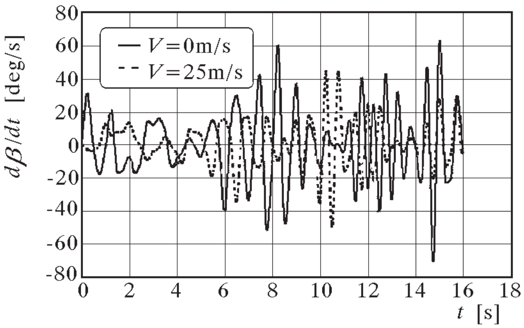

Fig. 13. Flapping rate of the main rotor blade

rather small. The magnitude and frequency of those oscillations are irregular and have a chaotic character.

Figures 14-18 show the Poincaré maps of selected state parameters. It can be stated that taking into consideration the unsteady rotor-blade aerodynamic model and hysteresis of aerodynamic coefficients, one encounters significant irregularities in the solution to equations of motion that are characteristic for chaotic motion. When the condition for the onset of chaotic motion is satisfied, flapping, pitching and rolling motions appear to have chaotic oscillations. The results obtained by Tang and Dowell (1988) confirmed that unsteady aerodynamics, including deep stall phenomena together with a strongly non-linear rotorcraft model, can lead to a chaotic response of the system.

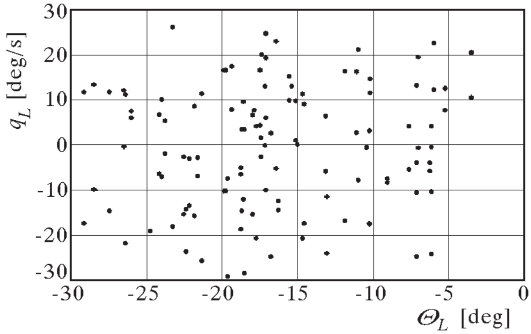

Fig. 14. Longitudinal motion of the slung load ($V = 25\,\mathrm{m/s}$) – Poincaré map

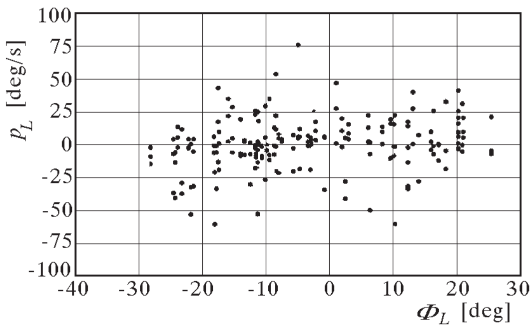

Fig. 15. Lateral motion of the slung load (hovering flight) – Poincaré map

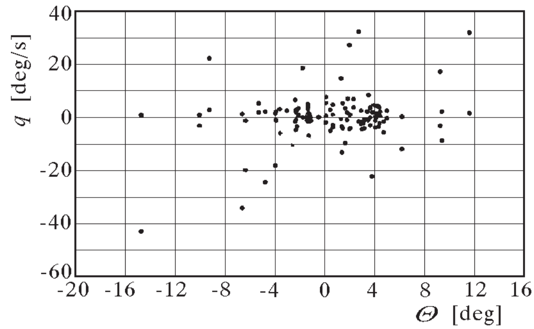

Fig. 16. Airframe phugoid motion – Poincaré map

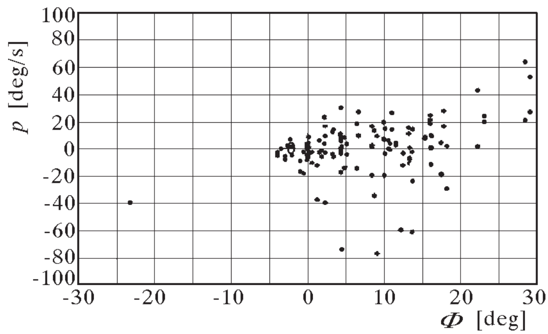

Fig. 17. Airframe Dutch roll – Poincaré map

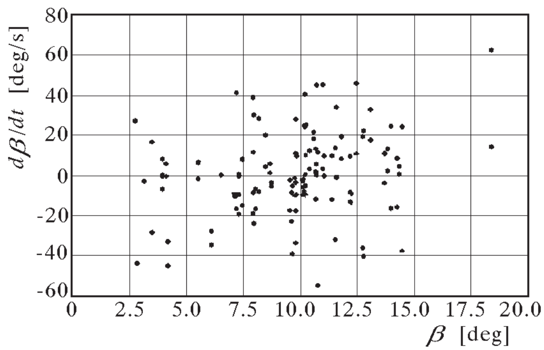

Fig. 18. Flapping motion of the main rotor blade – Poincaré map

5. Summary and conclusions

The paper presented a study of the flight dynamics of a helicopter with an articulated rotor, carrying a suspended load. The aircraft model included the rigid body dynamics, individual flap and lag dynamics of each blade as well as the inflow dynamics. The external load was modeled as a 3-degrees of freedom pendulum suspended from a single point. The aerodynamic load was an unsteady force in the direction determined by the local airflow (defined by the angle of attack and the slip angle of the slung load). The main aim of the study was to apply modern methods of investigation ODE for the prediction of critical configurations of the helicopter-external slung load system.

Based on the carried out investigations, the following conclusions can be drawn:

- The continuation and bifurcation methods prove to be a very useful tool for analysing equations of motion of a rotorcraft carrying an external slung load.
- The efficiency of the methods makes it possible to analyse complicated aerodynamic models using complete equations of motion in the entire range of the system parameters.
- The knowledge of configurations of the helicopter-slung load system, which lead to bifurcation allows one to find unsafe combinations of the hanging load mass and cable length.
- The need for a precise description of aerodynamic loads is the fundamental cause of difficulties in the analysis.
- Substantial dynamic coupling can occur between the Dutch roll mode and the load motion that primarily consists of lateral displacement of the load. Because of this coupling, the Dutch roll damping can decrease with a consequent deterioration of handling characteristics.

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Studium dynamiki lotu śmigłowca z podwieszonym ładunkiem z wykorzystaniem teorii bifurkacji i metod kontynuacyjnych

Streszczenie

W pracy przedstawiono studium dynamiki lotu śmigłowca z przegubowym wirnikiem nośnym, przenoszącego podwieszony pod kadłubem ładunek. W zastosowanym modelu wiropłata uwzględniono stopnie swobody nieodkształcalnego kadłuba, dynamikę wahań i odchyleń łopat wirnika nośnego oraz dynamikę przepływu przez płaszczyznę wirnika nośnego. Założono, że podwieszony ładunek jest punktem materialnym, na który działają siły aerodynamiczne, podwieszonym w jednym punkcie pod kadłubem śmigłowca. Wyniki obliczeń uzyskano dla podwieszonych ładunków o masie do 1500 kg (stosunek masy ładunku do masy śmigłowca do 35%), podwieszonych na linie o długości do 55 m. Obecność podwieszonego ładunku modyfikuje charakterystyki dynamiczne i osiągi śmigłowca ze względu na silne sprzężenia aerodynamiczne i bezwładnościowe pomiędzy jego ruchem a ruchami śmigłowca. Ze względu na fakt, że układ śmigłowiec-podwieszony ładunek jest opisany za pomocą silnie nieliniowych zwyczajnych równań różniczkowych, zastosowanie klasycznej analizy modalnej układu nie zawsze jest możliwe. Doskonałych narzędzi do badania takich równań dostarcza teoria układów dynamicznych i będąca jej częścią teoria bifurkacji. W pracy wykorzystano metodologię teorii układów dynamicznych do prognozowania natury niestabilności spowodowanej występującymi bifurkacjami. Ponadto przeprowadzono symulacje ruchu układu śmigłowiec-podwieszony ładunek po wystąpieniu bifurkacji.

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