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LAYOUT OPTIMIZATION OF TWO ISOTROPIC MATERIALS IN ELASTIC SHELLS

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The two-phase layout problem within the thin plate theory was solved by Gibiansky and Cherkaev in 1984. The same problem in the plane-stress formulation was solved by the same authors in 1987 and eventually cleared up by Allaire and Kohn in 1993. In the thin shell theory both these formulations are coupled, which is clearly seen in the homogenization formulae found by Lewiński and Telega in 1988, Telega and Lewiński in 1998, and in a general setting of the layout problem presented in the book by the same authors. The aim of the present paper is to set this problem within the Mushtari-Donnell-Vlasov approximation. The main result of the present examination is the lower bound of the complementary energy found by using the translation method. The translation matrix involves off-diagonal components, which leads to the effective complementary potential of a specific coupled form, expressible in terms of invariants of the stress and couple resultants.

Key words: homogenization, minimum compliance problem, relaxation by homogenization

1. Introduction

Let us imagine a spatial curve γ_1 along which a loading is prescribed and a supporting curve γ_2. Let two isotropic materials (1) and (2) of given amounts are at our disposal. Our aim is forming of the stiffest transversely homogeneous shell (of a given constant thickness h) transmitting the loading on γ_1 to the support γ_2. We note that two different sets of design variables are present in the formulation: the design variables defining the middle surface of the shell and the characteristic function of the domain occupied by material (2). Both sets of the design variables are of different nature. Necessity of handling the middle surface as a compound variable invokes the methods of the minimal surface problem, see e.g. Nitsche (1975). Although challenging, this geometric aspect of the problem will not be dealt with. In the present paper, the middle surface will be held fixed, while the distribution of material (2) will play the role of the design variable.

To pose the problem correctly, we draw upon the known results concerning the plane elasticity, see Allaire and Kohn (1993), and the thin plate theory, see Gibiansky and Cherkaev (1984, 1987) and Lipton (1994). They teach us that the problem must be reformulated to a relaxed form by admitting microstructures of properties governed by the homogenization formulae.

A general scheme for relaxing the layout optimization problem for thin shells was given in Lewiński and Telega (2000, Sec. 28.2). This scheme will be specified here for Mushtari-Donnell-Vlasov shells. In this shell model, the measures $\kappa_{\alpha\beta}$ of change of curvature are approximate. It is assumed that the tangent displacements have a negligible influence on $\kappa_{\alpha\beta}$. Consequently, the first two local equilibrium equations do not involve the couple resultants. This approximation is sometimes recommended for shallow shells.

In the optimal layout problem in its relaxed form this approximation is essential, because it simplifies the homogenization formulae for effective stiffnesses of shells with highly oscillating material properties, see Lewiński and Telega (2000, Sec. 17.1). Thus, Mushtari-Donnell-Vlasov approximation concerns both the levels: macro- and microscopic. Its role at the micro-level is even more important, since it results in decoupling of the homogenized constitutive relations, see Lewiński and Telega (2000, Sec. 17.4). Having these relations, one can apply the translation method to rearrange the general formula of the effective potential of the relaxed problem to an explicit form, expressible in terms of invariants of the stress and couple resultants. Finding the explicit form of this potential is the main objective of the present paper.

The result obtained is somewhat paradoxical. In the case of shape design, the effective potential turns out to involve the terms coupling $\mathbf{N}$ and $\mathbf{M}$, namely $\operatorname{tr}(\mathbf{NM})$ and $\operatorname{tr}\mathbf{N}\operatorname{tr}\mathbf{M}$. Moreover, the definitions of regimes, see (5.2) and Table 1, couple these tensors.

The relaxation by homogenization has at least two aims. First, it rearranges the initial problem to the well-posed form. Secondly, it admits considering the degenerated case of shape design, which is understood now as mixing a given material with voids. Passing to zero with the values of the elastic moduli of the weaker material can be performed, which leads to a non-degenerated relaxed form of the shape design problem. In the present paper we show that this passage can also be done in the context of the shell theory.

2. Mushtari-Donnell-Vlasov equations

Consider a shell of constant thickness h with a middle surface S being an image of a plane domain Ω; a point $\boldsymbol{\xi} = (\xi^1, \xi^2) \in \Omega$ is mapped on S and ξ^1, ξ^2 parametrize the surface S. Deformation of the shell is fully determined by the displacement fields (u_1, u_2, w) of its middle surface; u_α is a displacement tangent to the ξ^α coordinate, while w is normal to S. The strain measures are assumed according to the Mushtari-Donnell-Vlasov approximation

$$
\begin{aligned}
\epsilon_{\alpha\beta}(\boldsymbol{u}, w) &= \frac{1}{2}(u_{\alpha\|\beta} + u_{\beta\|\alpha}) - b_{\alpha\beta} w \\
\kappa_{\alpha\beta}(w) &= -w_{\|\alpha\beta}
\end{aligned}
\tag{2.1}
$$

here $(\cdot)_{\|\alpha}$ represents the αth covariant derivative and $(b_{\alpha\beta})$ is the curvature tensor. The shell is considered as elastic and the constitutive relations are given by

$$
N^{\alpha\beta} = A^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{u}, w) \qquad\qquad M^{\alpha\beta} = D^{\alpha\beta\lambda\mu} \kappa_{\lambda\mu}(w) \tag{2.2}
$$

where $\mathbf{A}$ and $\mathbf{D}$ are tensors of membrane and bending stiffnesses, respectively. The stress resultants satisfy the equilibrium equations whose variational form is

$$
\int_S [N^{\alpha\beta} \epsilon_{\alpha\beta}(\boldsymbol{v}, v) + M^{\alpha\beta} \kappa_{\alpha\beta}(v)] \, dS = f(\boldsymbol{v}, v) \tag{2.3}
$$

for all kinematically admissible $(\boldsymbol{v}, v)$.

Here $\boldsymbol{v} = (v_\alpha)$ are trial tangent displacements, and v represents a trial normal displacement. The linear form $f(\boldsymbol{v}, v)$ expresses the work of the loading on the trial displacements. The displacements are viewed as kinematically admissible if they are sufficiently regular and fulfil kinematic boundary conditions.

3. Compliance minimization of a two-phase shell

The shell is considered as made of two isotropic materials of moduli $(\widetilde{k}_1, \widetilde{\mu}_1)$ and $(\widetilde{k}_2, \widetilde{\mu}_2)$ such that $\widetilde{k}_2 > \widetilde{k}_1$, $\widetilde{\mu}_2 > \widetilde{\mu}_1$. The tensors of membrane and bending stiffnesses have the following representations

$$
\mathbf{A}_\alpha = 2k_\alpha \mathbf{I}_1 + 2\mu_\alpha \mathbf{I}_2 \qquad\qquad \mathbf{D}_\alpha = \frac{h^2}{12} \mathbf{A}_\alpha \tag{3.1}
$$

where $k_\alpha = h\widetilde{k}_\alpha,\ \mu_\alpha = h\widetilde{\mu}_\alpha$ and

$$\begin{aligned} I_1^{\alpha\beta\lambda\mu} &= \frac{1}{2} g^{\alpha\beta} g^{\lambda\mu} \\ I_2^{\alpha\beta\lambda\mu} &= \frac{1}{2}(g^{\alpha\lambda} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\lambda} - g^{\alpha\beta} g^{\lambda\mu}) \end{aligned} \tag{3.2}$$

Here $(g^{\alpha\beta})$ are components of the metric tensor on S.

Let $\mathbb{S}$ be the set of statically admissible stress resultants $(\mathbf{N}, \mathbf{M})$. According to Castigliano's theorem the compliance C, defined as $f(\boldsymbol{u}, w)$, can be represented by

$$C = \min_{(\mathbf{N},\mathbf{M}) \in \mathbb{S}} \int_S [\mathbf{N} : (\mathbf{A}^{-1}\mathbf{N}) + \mathbf{M} : (\mathbf{D}^{-1}\mathbf{M})]\, dS \tag{3.3}$$

where $\mathbf{A}$ and $\mathbf{D}$ assume the form $\mathbf{A}_1$ and $\mathbf{D}_1$ or $\mathbf{A}_2$ and $\mathbf{D}_2$. Let χ_α be the characteristic function of the domain occupied by the αth material. The materials are distributed transversely homogeneous, hence the integrals

$$\int_S \chi_\alpha \, dS = \mathcal{A}_\alpha \tag{3.4}$$

determine the volumes occupied by the materials.

The layout optimization problem can be put in the following form

$$\begin{aligned} \inf\Big\{ \int_\Omega [\mathbf{N} : (\mathbf{A}^{-1}\mathbf{N}) + \mathbf{M} : (\mathbf{D}^{-1}\mathbf{M})]\sqrt{g}\, d\boldsymbol{\xi} \ \Big|\ (\mathbf{N}, \mathbf{M}) \in \mathbb{S}, \\ \chi_2 \in L^\infty(\Omega, \{0,1\}), \quad \int_\Omega \chi_2 \sqrt{g}\, d\boldsymbol{\xi} = \mathcal{A}_2 \Big\} \end{aligned} \tag{3.5}$$

where $g = \det(g_{\alpha\beta})$. We know that this problem requires relaxation, see Lewiński and Telega (2000, Sec. 26.1), which, roughly speaking, introduces an underlying microstructure. To each point $\boldsymbol{\xi} \in \Omega$ we assign a cell $Y = (0, l_1) \times (0, l_2)$ in which the distribution of the αth material is determined by the characteristic function $\chi_\alpha^Y(\boldsymbol{y}),\ \boldsymbol{y} = (y_1, y_2) \in Y,\ \chi_1^Y = 1 - \chi_2^Y$. Let the averaging over Y be denoted by $\langle \cdot \rangle$ or

$$\langle f \rangle = \frac{1}{|Y|} \int_Y f(\boldsymbol{y})\, d\boldsymbol{y} \tag{3.6}$$

Let us recall the definitions of the sets of statically admissible stress and couple resultants defined for the cells Y

$$
\begin{aligned}
\mathbb{S}_1^{per}(Y) &= \Big\{ \boldsymbol{n} \in L^2(Y, \mathbb{E}_2^s) \,\Big|\, \langle \boldsymbol{n}(\boldsymbol{y}) \rangle = \boldsymbol{0}, \quad \frac{\partial n^{\alpha\beta}}{\partial y_\beta} = 0 \ \text{ in } Y, \\
&\qquad n^{\alpha\beta}\nu_\beta \ \text{ take opposite values at the opposite sides of } Y \Big\} \\
\mathbb{S}_2^{per}(Y) &= \Big\{ \boldsymbol{m} \in L^2(Y, \mathbb{E}_2^s) \,\Big|\, \langle \boldsymbol{m}(\boldsymbol{y}) \rangle = \boldsymbol{0}, \quad \frac{\partial^2 m^{\alpha\beta}}{\partial y_\alpha \partial y_\beta} = 0 \ \text{ in } Y, \\
&\qquad m_\nu = m^{\alpha\beta}\nu_\alpha\nu_\beta \ \text{take equal values at the opposite sides of } Y, \\
&\qquad q = \nu_\alpha \frac{\partial m^{\alpha\beta}}{\partial y_\beta} + \frac{\partial (m^{\alpha\beta}\nu_\alpha\tau_\beta)}{\partial s} \ \text{take opposite values at the} \\
&\qquad \text{opposite sides of } Y \Big\}
\end{aligned}
\tag{3.7}
$$

Here $\boldsymbol{\nu} = (\nu_\alpha)$ represents a unit outward vector normal to ∂Y, and $\mathbb{E}_2^s$ is a set of symmetric 2×2 matrices.

The distribution of stiffnesses within Y is given by

$$
\begin{aligned}
\mathbf{A}(y) &= \mathbf{A}_1 \chi_1^Y(\boldsymbol{y}) + \mathbf{A}_2 \chi_2^Y(\boldsymbol{y}) \\
\mathbf{D}(y) &= \mathbf{D}_1 \chi_1^Y(\boldsymbol{y}) + \mathbf{D}_2 \chi_2^Y(\boldsymbol{y})
\end{aligned}
\tag{3.8}
$$

and, consequently,

$$
\begin{aligned}
\mathbf{A}^{-1}(y) &= \mathbf{A}_1^{-1} \chi_1^Y(\boldsymbol{y}) + \mathbf{A}_2^{-1} \chi_2^Y(\boldsymbol{y}) \\
\mathbf{D}^{-1}(y) &= \mathbf{D}_1^{-1} \chi_1^Y(\boldsymbol{y}) + \mathbf{D}_2^{-1} \chi_2^Y(\boldsymbol{y})
\end{aligned}
\tag{3.9}
$$

We introduce the notation

$$
\boldsymbol{\sigma} = (\boldsymbol{n}, \boldsymbol{m})^\top \qquad\qquad \mathfrak{a} = \begin{bmatrix} \mathbf{A}^{-1}(\boldsymbol{y}) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1}(\boldsymbol{y}) \end{bmatrix}
\tag{3.10}
$$

and define the effective complementary potential

$$
\begin{aligned}
2W^*(\mathbf{N}, \mathbf{M}, \rho) &= \min \Big\{ \langle \boldsymbol{\sigma} : (\mathfrak{a}\boldsymbol{\sigma}) \rangle \,\Big|\, \boldsymbol{\sigma} \in \mathbb{S}_1^{per}(Y) \times \mathbb{S}_2^{per}(Y), \\
&\qquad \langle \boldsymbol{\sigma} \rangle = (\mathbf{N}, \mathbf{M})^\top, \quad \langle \chi_2^Y \rangle = \rho \Big\}
\end{aligned}
\tag{3.11}
$$

for $\rho \in [0, 1]$.

The relaxation rearranges problem (3.5) to the form

$$\min\Big\{2\int\limits_{\Omega} W^*(\mathbf{N},\mathbf{M},m_2)\sqrt{g}\,d\boldsymbol{\xi}\,\Big|\,m_2\in L^\infty(\Omega;[0,1]),\qquad \int\limits_{\Omega} m_2\sqrt{g}\,d\boldsymbol{\xi}=\mathcal{A}_2,\qquad (\mathbf{N},\mathbf{M})\in\mathbb{S}\Big\} \tag{3.12}$$

4. The lower bound of W^*

The expression for W^* is not explicit. It is the translation method (see Cherkaev, 2000) which makes it possible to express W^* in terms of the invariants of $\mathbf{N}$ and $\mathbf{M}$. This method consists of two stages. The first one finds a lower bound of W^*. The final stage is to prove that the bound is sharp. This means showing a microstructure realizing this lower bound. The present paper concerns the first stage of the method.

To make the further computations possibly simple, we introduce the vectorial basis

$$\begin{aligned}\boldsymbol{a}_1&=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\\ \boldsymbol{a}_2&=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1-\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\\ \boldsymbol{a}_3&=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)\end{aligned} \tag{4.1}$$

where $\boldsymbol{e}_i$, $i=1,2,3$, are unit vectors of the Cartesian basis; $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$. We introduce representations

$$\mathbf{N}=\sum_{i=1}^{3}N^i\boldsymbol{a}_i\qquad\qquad \mathbf{M}=\sum_{i=1}^{3}M^i\boldsymbol{a}_i \tag{4.2}$$

Note that

$$\det\mathbf{N}=-\frac{1}{2}\mathbf{N}^\top\mathbf{T}\mathbf{N} \tag{4.3}$$

with

$$\mathbf{T}=\operatorname{diag}(-1,1,1) \tag{4.4}$$

or $\mathbf{T}=\mathbf{I}_2-\mathbf{I}_1$, see Lewiński and Telega (2000).

Application of the translation method for the similar thin plate problem is described with all details in Lewiński and Telega (2000, Sec. 26). Thus the technique of the translation method will not be explained here.

To take into account the membrane-bending coupling we choose the translation matrix in the form

$$\boldsymbol{\mathcal{T}} = \left[\begin{array}{c|c} \alpha\mathbf{T} & \gamma\mathbf{T} \\ \hline \gamma\mathbf{T} & \beta\mathbf{T} \end{array}\right]_{6\times 6} \tag{4.5}$$

with $\alpha, \gamma, \beta \in \mathbb{R}$, $\beta \geqslant 0$.

Gibiansky and Cherkaev (1987, 1997) proved that

$$\begin{aligned} \langle \boldsymbol{n} : \mathbf{T}\boldsymbol{n} \rangle &= \langle \boldsymbol{n} \rangle : \mathbf{T}\langle \boldsymbol{n} \rangle \qquad \forall\, \boldsymbol{n} \in \mathbb{S}_1^{per}(Y) \\ \langle \boldsymbol{m} : \mathbf{T}\boldsymbol{m} \rangle &\geqslant \langle \boldsymbol{m} \rangle : \mathbf{T}\langle \boldsymbol{m} \rangle \quad \forall\, \boldsymbol{m} \in \mathbb{S}_2^{per}(Y) \end{aligned} \tag{4.6}$$

In this paper we note that

$$\langle \boldsymbol{n} : \mathbf{T}\boldsymbol{m} \rangle = \langle \boldsymbol{n} \rangle : \mathbf{T}\langle \boldsymbol{m} \rangle \qquad \forall\, \boldsymbol{n} \in \mathbb{S}_1^{per}(Y) \qquad \forall\, \boldsymbol{m} \in \mathbb{S}_2^{per}(Y) \tag{4.7}$$

The proof of (4.7) will be published elsewhere. Let us stress only that the differential conditions concealed in the definitions of sets (3.7) are crucial for properties (4.6) and (4.7).

Equality (4.7) links $\boldsymbol{n}$ with $\boldsymbol{m}$, which can be viewed as curious: two fields $\boldsymbol{n}$ and $\boldsymbol{m}$ turn out to be linked, although they seem to be independent, see (3.7). In the theory considered the stress couples do not intervene to the membrane equilibrium equations. None the less $\boldsymbol{n}$ and $\boldsymbol{m}$ are linked by (4.7) only because they obey some differential constraints.

To make the dimensions of $\boldsymbol{n}$ and $\boldsymbol{m}$ equal we introduce $\widetilde{\boldsymbol{m}} = (\sqrt{12}/h)\boldsymbol{m}$ and $\widetilde{\mathbf{M}} = (\sqrt{12}/h)\mathbf{M}$ and put $\widetilde{\gamma} = \gamma(h/\sqrt{12})$, $\widetilde{\beta} = \beta(h^2/12)$. We have

$$\boldsymbol{\sigma} : \mathfrak{a}\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}} : \widetilde{\mathfrak{a}}\widetilde{\boldsymbol{\sigma}} \tag{4.8}$$

where $\widetilde{\boldsymbol{\sigma}} = (\boldsymbol{n}, \widetilde{\boldsymbol{m}})^\top$, $\widetilde{\mathfrak{a}} = \operatorname{diag}[\mathbf{A}^{-1}(y), \mathbf{A}^{-1}(y)]$ or

$$\widetilde{\mathfrak{a}} = \operatorname{diag}\Big(\frac{1}{2}K(y), \frac{1}{2}L(y), \frac{1}{2}L(y), \frac{1}{2}K(y), \frac{1}{2}L(y), \frac{1}{2}L(y)\Big) \tag{4.9}$$

and

$$\begin{aligned} K(\boldsymbol{y}) &= K_1\chi_1^Y(\boldsymbol{y}) + K_2\chi_2^Y(\boldsymbol{y}) \\ L(\boldsymbol{y}) &= L_1\chi_1^Y(\boldsymbol{y}) + L_2\chi_2^Y(\boldsymbol{y}) \end{aligned} \tag{4.10}$$

where

$$K_\alpha = \frac{1}{k_\alpha} \qquad\qquad L_\alpha = \frac{1}{\mu_\alpha} \tag{4.11}$$

Similarly: $\widetilde{\boldsymbol{\mathcal{T}}}$ is given by (4.5) with replacing $\widetilde{\gamma}$ for γ and $\widetilde{\beta}$ for β.

The translation method requires positive definiteness of $(\widetilde{\mathfrak{a}} - \widetilde{\boldsymbol{\mathcal{T}}})$ or the positive definiteness of the matrices

$$\left[\begin{array}{ccc|ccc}
\frac{1}{2}K_\sigma + \alpha & & & \widetilde{\gamma} & & \\
 & \frac{1}{2}L_\sigma - \alpha & & & -\widetilde{\gamma} & \\
 & & \frac{1}{2}L_\sigma - \alpha & & & -\widetilde{\gamma} \\
\hline
\widetilde{\gamma} & & & \frac{1}{2}K_\sigma + \widetilde{\beta} & & \\
 & -\widetilde{\gamma} & & & \frac{1}{2}L_\sigma - \widetilde{\beta} & \\
 & & -\widetilde{\gamma} & & & \frac{1}{2}L_\sigma - \widetilde{\beta}
\end{array}\right] \tag{4.12}$$

where $\sigma = 1, 2$.

We remember that $K_1 > K_2$ and $L_1 > L_2$, see Sec. 3. One can prove that matrices (4.12) are positive definite provided that $(\alpha, \widetilde{\beta}, \widetilde{\gamma})$ belong to the set Z

$$\begin{gathered}
Z \;=\; \Big\{(\alpha,\beta,\gamma)\;\Big|\; -\frac{1}{2}K_\sigma \leqslant \alpha \leqslant \frac{1}{2}L_\sigma, \qquad 0 \leqslant \beta \leqslant \frac{1}{2}L_\sigma, \qquad \sigma = 1,2, \\
4\gamma^2 \leqslant (K_\sigma + 2\alpha)(K_\sigma + 2\beta), \qquad 4\gamma^2 \leqslant (L_\sigma - 2\beta)(L_\sigma - 2\alpha)\Big\}
\end{gathered} \tag{4.13}$$

The translation method with the shifting matrix given by (4.5) results in the following estimate

$$\begin{aligned}
&W^*(\mathbf{N}, \widetilde{\mathbf{M}}, \rho) \geqslant \overline{W}(\mathbf{N}, \widetilde{\mathbf{M}}, \rho) \\
&2\overline{W}(\mathbf{N}, \widetilde{\mathbf{M}}, \rho) = \max_{(\alpha,\beta,\gamma)\in Z} \big[\boldsymbol{\sigma} : \big(\widehat{\mathfrak{a}}(\alpha,\beta,\gamma)\boldsymbol{\sigma}\big)\big]
\end{aligned} \tag{4.14}$$

with $\boldsymbol{\sigma} = (\mathbf{N}, \widetilde{\mathbf{M}})^\top$ and

$$\widehat{\mathfrak{a}}(\alpha,\beta,\gamma) = \Big\langle (\widetilde{\mathfrak{a}} - \widetilde{\boldsymbol{\mathcal{T}}})^{-1} \Big\rangle^{-1} + \widetilde{\boldsymbol{\mathcal{T}}} \tag{4.15}$$

To find $\widehat{\mathfrak{a}}$ we twice come across the algebraic task of inverting the 6×6 matrix of a form

$$\mathbf{X} = \left[\begin{array}{ccc|ccc} a & & & -c_1 & & \\ & b & & & c_2 & \\ & & b & & & c_2 \\ \hline -c_1 & & & d & & \\ & c_2 & & & e & \\ & & c_2 & & & e \end{array}\right] \tag{4.16}$$

The matrix $\mathbf{X}$ can be easily inverted. The result is surprisingly simple

$$\mathbf{X}^{-1} = \left[\begin{array}{ccc|ccc} a' & & & -c_1' & & \\ & b' & & & c_2' & \\ & & b' & & & c_2' \\ \hline -c_1' & & & d' & & \\ & c_2' & & & e' & \\ & & c_2' & & & e' \end{array}\right] \tag{4.17}$$

with

$$\begin{aligned} a' &= \frac{d}{f} & c_1' &= -\frac{c_1}{f} & f &= ad - (c_1)^2 \\ b' &= \frac{e}{g} & c_2' &= -\frac{c_2}{g} & g &= be - (c_2)^2 \\ d' &= \frac{a}{f} & e' &= \frac{b}{g} \end{aligned} \tag{4.18}$$

Thus, both the matrices $\mathbf{X}$ and $\mathbf{X}^{-1}$ have the same structure. Consequently, the lower bound $\overline{W}$ has the form of $(4.14)_2$ with

$$\begin{aligned} \boldsymbol{\sigma} : [\widehat{\mathfrak{a}}(\alpha,\beta,\gamma)\boldsymbol{\sigma}] &= \widehat{a}(N^1)^2 + \widehat{b}[(N^2)^2 + (N^3)^2] - \\ &-2\widehat{c}_1\widetilde{M}^1 N^1 + 2\widehat{c}_2(\widetilde{M}^2 N^2 + \widetilde{M}^3 N^3) + \widehat{d}(\widetilde{M}^1)^2 + \widehat{e}[(\widetilde{M}^2)^2 + (\widetilde{M}^3)^2] \end{aligned} \tag{4.19}$$

where $\widehat{a}$, $\widehat{b}$, $\widehat{c}_1$, $\widehat{c}_2$, $\widehat{d}$, $\widehat{e}$ depend on ρ, α, β, γ, K_σ, L_σ. These relations are closed-form but complicated, and they will not be reported here.

Potential (4.19) corresponds to the linear shell theory with a membrane-bending coupling. Let us note, however, that performing maximization in $(4.14)_2$ rearranges the potential $\overline{W}$ to a non-linear hyperelastic form, depending upon the invariants

$$\begin{array}{ccc} (\operatorname{tr}\mathbf{N})^2 & \|\operatorname{dev}\mathbf{N}\|^2 & (\operatorname{tr}\widetilde{\mathbf{M}})^2 \\ \|\operatorname{dev}\widetilde{\mathbf{M}}\|^2 & \operatorname{tr}\mathbf{N}\operatorname{tr}\widetilde{\mathbf{M}} & \operatorname{tr}(\mathbf{N}\widetilde{\mathbf{M}}) \end{array} \tag{4.20}$$

where dev means deviator and

$$\|\text{dev}\,\mathbf{N}\| = \Big[(N^2)^2 + (N^3)^2\Big]^{\frac{1}{2}} \tag{4.21}$$

is its norm.

Performing the maximization in $(4.14)_2$ is not an easy task, because many parameters enter the formulae. Further consideration will be confined to a specific case of shape design.

5. Bounding W^* for the shape design

Relaxed formulation (3.11) comprises the case of shape design or the case of $k_1 \to 0$, $\mu_1 \to 0$, $K_1 \to +\infty$, $L_1 \to +\infty$. For simplicity we put $K_2 = K$, $L_2 = L$, $m_2 = \theta$ and find

$$\begin{aligned} 2\overline{W}(\mathbf{N}, \widetilde{\mathbf{M}}, \theta) &= \frac{1}{2\theta}\Big\{K(N^1)^2 + L[(N^2)^2 + (N^3)^2]\Big\} + \\ &+\alpha\frac{1-\theta}{\theta}[(N^1)^2 - (N^2)^2 - (N^3)^2] + \frac{1}{2\theta}\Big\{K(\widetilde{M}^1)^2 + L[(\widetilde{M}^2)^2 + (\widetilde{M}^3)^2]\Big\} + \\ &+\widetilde{\beta}\frac{1-\theta}{\theta}[(\widetilde{M}^1)^2 - (\widetilde{M}^2)^2 - (\widetilde{M}^3)^2] + 2\widetilde{\gamma}\frac{1-\theta}{\theta}[\widetilde{M}^1 N^1 - \widetilde{M}^2 N^2 - \widetilde{M}^3 N^3] \end{aligned} \tag{5.1}$$

The parameters $\alpha, \widetilde{\beta}, \widetilde{\gamma}$ are chosen according to Table 1. The regimes $1 \div 6$ are defined by (5.2).

Table 1. Values of $\alpha, \widetilde{\beta}, \widetilde{\gamma}$ for regimes $1 \div 6$ defined by (5.2)

Regimes	2α	$2\widetilde{\beta}$	$2\widetilde{\gamma}$
2 and 4 and 6	$-K$	0	$-\sqrt{L(L+K)}$
2 and 3 and 6	$-K$	L	0
1 and 4 and 6	L	0	$-\sqrt{K(K+L)}$
1 and 3 and 6	L	L	$-(K+L)$
2 and 4 and 5	$-K$	0	$\sqrt{L(L+K)}$
2 and 3 and 5	$-K$	L	0
1 and 4 and 5	L	0	$\sqrt{K(K+L)}$
1 and 3 and 5	L	L	$K+L$

Regime number

$$
\begin{array}{ll}
1: \det \mathbf{N} \geqslant 0 & 4: \det \widetilde{\mathbf{M}} \leqslant 0 \\
2: \det \mathbf{N} \leqslant 0 & 5: \operatorname{tr} \mathbf{N} \cdot \operatorname{tr} \widetilde{\mathbf{M}} \geqslant \operatorname{tr}(\mathbf{N}\widetilde{\mathbf{M}}) \\
3: \det \widetilde{\mathbf{M}} \geqslant 0 & 6: \operatorname{tr} \mathbf{N} \cdot \operatorname{tr} \widetilde{\mathbf{M}} \leqslant \operatorname{tr}(\widetilde{\mathbf{N}}\mathbf{M})
\end{array}
\tag{5.2}
$$

Let us extract the potential of a homogeneous shell

$$
2W^{\circ}(\mathbf{N},\mathbf{M}) = \frac{1}{2}\Big[\frac{1}{2}K(\operatorname{tr}\mathbf{N})^2 + L\|\operatorname{dev}\mathbf{N}\|^2\Big] + \frac{1}{2}\Big[\frac{1}{2}K(\operatorname{tr}\widetilde{\mathbf{M}})^2 + L\|\operatorname{dev}\widetilde{\mathbf{M}}\|^2\Big]
\tag{5.3}
$$

from (5.1), and rearrange (5.1) to the form

$$
\overline{W}(\mathbf{N},\widetilde{\mathbf{M}},\theta) = W^{\circ}(\mathbf{N},\mathbf{M}) + \frac{1-\theta}{2\theta}G(\mathbf{N},\widetilde{\mathbf{M}})
\tag{5.4}
$$

where

$$
\begin{aligned}
G(\mathbf{N},\widetilde{\mathbf{M}}) &= 2W^{\circ}(\mathbf{N},\mathbf{M}) + \Big|\alpha\Big[\frac{1}{2}(\operatorname{tr}\mathbf{N})^2 - \|\operatorname{dev}\mathbf{N}\|^2\Big]\Big| + \\
&+\Big|\widetilde{\beta}\Big[\frac{1}{2}(\operatorname{tr}\widetilde{\mathbf{M}})^2 - \|\operatorname{dev}\widetilde{\mathbf{M}}\|^2\Big]\Big| + 2|\widetilde{\gamma}[\operatorname{tr}\mathbf{N}\operatorname{tr}\widetilde{\mathbf{M}} - \operatorname{tr}(\mathbf{N}\widetilde{\mathbf{M}})]|
\end{aligned}
\tag{5.5}
$$

and $\alpha, \widetilde{\beta}, \widetilde{\gamma}$ are piece-wise constant quantities, according to Table 1. Note that $G(\mathbf{N},\widetilde{\mathbf{M}})$ is non-negative.

Let us replace W^* in (3.12) by $\overline{W}$ to consider the shape design problem in the form

$$
\begin{aligned}
\min\Big\{2\int\limits_{\Omega} \overline{W}(\mathbf{N},\widetilde{\mathbf{M}},\theta)\sqrt{g}\,d\boldsymbol{\xi}\;\Big|\;\theta \in L^{\infty}(\Omega;[0,1]),\\
\int\limits_{\Omega} \theta\sqrt{g}\,d\boldsymbol{\xi} = \mathcal{A}_2, \quad (\mathbf{N},\mathbf{M}) \in \mathbb{S}\Big\}
\end{aligned}
\tag{5.6}
$$

Now, we introduce the Lagrangian multiplier λ associated with the isoperimetric condition and interchange the order of the minimum in (5.6) and maximum over λ. For a fixed λ we find

$$
\min\Big\{\int\limits_{\Omega} F_{\lambda}(\mathbf{N},\mathbf{M})\sqrt{g}\,d\boldsymbol{\xi}\;\Big|\;(\mathbf{N},\mathbf{M}) \in \mathbb{S}\Big\}
\tag{5.7}
$$

where

$$
F_{\lambda}(\mathbf{N},\mathbf{M}) = \min_{0\leqslant\theta\leqslant 1}[2\overline{W}(\mathbf{N},\widetilde{\mathbf{M}},\theta) + \lambda\theta]
\tag{5.8}
$$

The result of Allaire and Kohn (1993) concerning the plane elasticity case applies here. Thus, the function F_λ can be expressed as

$$F_\lambda(\mathbf{N},\mathbf{M}) = 2W^\circ(\mathbf{N},\mathbf{M}) + \begin{cases} 2\sqrt{\lambda G(\mathbf{N},\widetilde{\mathbf{M}})} - G(\mathbf{N},\widetilde{\mathbf{M}}) & \text{for } G(\mathbf{N},\widetilde{\mathbf{M}}) \leqslant \lambda \\ \lambda & \text{otherwise} \end{cases} \tag{5.9}$$

6. Case of small volume

If the quantity $\mathcal{A}_2$ is a small number, then λ is big and the main part of problem (5.7) reduces to

$$\min\Big\{\int_\Omega \sqrt{G(\mathbf{N},\widetilde{\mathbf{M}})}\sqrt{g}\, d\boldsymbol{\xi} \,\Big|\, (\mathbf{N},\widetilde{\mathbf{M}}) \in \mathbb{S}\Big\} \tag{6.1}$$

Let us note that the function $\sqrt{G(\mathbf{N},\widetilde{\mathbf{M}})}$ is homogeneous of rank 1. Thus, there exists a closed set B such that the problem dual to (6.1) assumes the form

$$\max\{f(\boldsymbol{v},v) \,|\, (\boldsymbol{\epsilon}(\boldsymbol{v},v),\boldsymbol{\kappa}(v)) \in B\} \tag{6.2}$$

where $\boldsymbol{\epsilon}(\boldsymbol{v},v)$, $\boldsymbol{\kappa}(v)$ are defined by (2.1) and $f(\boldsymbol{v},v)$ represents the work of the loading. Here, the loading must be applied to the boundary; the surface loadings are excluded. The passage from (6.1) to (6.2) has been thoroughly explained in Lewiński and Telega (2001) for the thin plate problem. Similar arguments link (6.2) with (6.1).

The set B is called a locking locus, and problem (6.2) is, in fact, a locking problem.

7. Final remarks

Although estimate $(4.14)_1$ is correct, its sharpness has not been shown. It means that the underlying microstructure has not been disclosed and the method cannot be alternatively applied by controlling the properties of the microstructure. Two questions are crucial here: do the higher rank laminates suffice and do they possess oblique fibres or are they all mutually orthogonal?

The results of the paper apply not only to shells but to a general plate problem as well. If a flat plate is subjected both to an in-plane and transverse loading, its optimal design reflects the simultaneous work in the two planes. The optimal layout becomes a compromise between the optimal in-plane and bending type designs.

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Optymalizacja rozmieszczenia dwu materiałów w powłokach sprężystych

Streszczenie

Zagadnienie optymalnego rozkładu dwóch materiałów izotropowych w sprężystych płytach cienkich rozwiązali Gibianskij i Czerkajew w roku 1984. Minimalizacji podlegała podatność płyty. Analogiczne zadanie dotyczące teorii tarcz rozwiązali ci sami autorzy w 1987 r. Sformułowanie to uzupełnili i uściślili Allaire i Kohn w roku 1993. W zadaniu dotyczącym powłok cienkich oba te sformułowania są ze sobą sprzężone, co jasno jest widoczne w formułach homogenizacji znalezionych w pracach Lewińskiego i Telegi z roku 1988 oraz w pracach Telegi i Lewińskiego z roku 1998; ogólne, niejawne sformułowanie tego zadania optymalizacji omówiono w książce tych samych autorów. Celem niniejszej pracy jest sformułowanie tego zadania w sposób jawny w zakresie technicznej teorii powłok Musztariego-Donnella-Własowa. W pracy wyprowadzamy w sposób jawny dolne oszacowanie energii komplementarnej z wykorzystaniem metody translacji. Macierz translacji zawiera tutaj składniki pozadiagonalne. Ta postać macierzy translacji prowadzi do zastępczego potencjału o specyficznej postaci sprzężonej, wyrażalnej za pomocą niezmienników sił wewnętrznych w powłoce.

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