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ON THE ASYMPTOTIC BEHAVIOR OF THERMAL TAYLOR DISPERSION PROCESSES IN PERIODIC MEDIA

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Using the generalized method of moments and a central limit theorem, we shall describe a large class of thermal dispersion phenomena occurring in some macrohomogeneous systems. We shall be interested in computing the macroscale coefficients in terms of the microscale coefficients and the system geometry. Also, the functional dependence of the effective coefficients on the velocity and the spatial scale parameters is analyzed.

Key words: macrotransport equation, macroscale coefficients, scale parameters

1. Introduction

The field of macrotransport processes constitutes a natural extension of classical Taylor dispersion theory for unidirectional, rectilinear flows (see Taylor, 1953) to a large class of flow and dispersion problems. It is well-known by now that G.I. Taylor used rather intuitive semi-analytical arguments to prove the Fickian nature of the mean axial dispersion of a diffusing solute injected into a viscous fluid flowing through a circular cylindrical tube. He showed that, asymptotically, in the long-time limit, such a dispersion process is described by a one-dimensional convective-diffusive equation and the dispersion coefficient characterizing this axial macrotransport equation governing the cross-sectional mean solute concentration is

$$D^* = D + \frac{a^2\overline{U}^2}{48D} \tag{1.1}$$

where a is the radius of the tube, $\overline{U}$ is the mean velocity of flow and D is the coefficient of molecular diffusion. This new coefficient combines the microscopically distinct effects of radial molecular diffusion and axial convective solute flow. Dispersion is caused by the radial inhomogeneity of the Poiseuille velocity field which interacts with the lateral diffusion of solute molecules. In fact, all macrotransport processes combine such a Brownian (stochastic) diffusive transport mechanism with an inhomogeneous, convective (deterministic) transport mechanism. The stochastic dispersion is assumed to act over a period of time large enough to allow the sampling of all such velocities by molecular diffusion across the streamlines of the flow.

Taylor's technique, later extended to the case of solute dispersion in turbulent flows, provided the first framework for the generic phenomenon referred to in the literature as "Taylor dispersion". In 1956, Aris (see Aris, 1956) extends these results for cylinders of noncircular cross-section and develops a rigorous theory, based on a *method of moments* scheme. Also, he analyzes the effects of time-periodic convection on dispersion (see Aris, 1960). In 1971 Horn (see Horn, 1971) made another important step to put the foundations of the so-called *generalized Taylor dispersion theory*. He extended the classical theory to multidimensional phase spaces. A decisive step was done then by Brenner (see Brenner, 1980) who developed a paradigmatic dispersion theory for very general and complex systems. In 1993, recognizing the analogy existing between the mass transport and other modes of transport phenomena, Brenner and Edwards extended this theory to non-material transport processes (see Brenner and Edwards, 1993).

Hence, based upon a rigorous description of microtransport processes occurring in heterogeneous systems, macrotransport theory, alternatively known in the literature as the generalized Taylor dispersion theory, allows us to describe a large class of material and nonmaterial dispersive phenomena occurring in macrohomogeneous systems.

Applications of the macrotransport theory are presently recognized in many fields of scientific and engineering research. Various other methods have been developed for obtaining the macroscale behavior and properties of some heterogeneous complex systems. These include homogenization techniques (see Bensoussan *et al.*, 1978, Sanchez-Palencia, 1980), statistical and volume-averaging methods (see Koch and Brady, 1985) and probabilistic methods based on central limit theorems (see Bhattacharya *et al.*, 1989).

In this paper, we shall be especially interested in getting a macrotransport paradigm for a class of thermal transport phenomena occurring in some complex multidimensional adiabatic systems. We shall deduce simple macrotransport equations (and effective coefficients appearing therein) which apply, for long times, at a coarse-grained level of description of our systems.

Our analysis is based on two alternative methods: the above mentioned generalized method of moments and a probabilistic method based on a central limit theorem for Markov processes.

We are also interested in getting the asymptotic behavior of the macrodispersion coefficients as functions of the velocity and the spatial scale parameters which characterize our transport processes.

Specific examples are given in Section 3 to illustrate the computation of these macrotransport coefficients as functions of the prescribed microscale data.

2. A macrotransport paradigm for thermal dispersion phenomena in adiabatic systems

Let us introduce two distinctly different classes of independent coordinate variables which characterize a generic microtransport process. These will be designated as *global* and *local* variables and denoted by $\boldsymbol{Q}$ and $\boldsymbol{q}$

$$\boldsymbol{Q} = [Q_1, Q_2, ..., Q_r] \qquad\qquad \boldsymbol{q} = [q_1, q_2, ..., q_s] \tag{2.1}$$

Together, the vectors $(\boldsymbol{Q}, \boldsymbol{q})$ define a multidimensional phase space $\boldsymbol{Q}_\infty \oplus \boldsymbol{q}_0$ within which our transport processes occur. The global subspace $\boldsymbol{Q}_\infty$, representing the domain of the values taken by $\boldsymbol{Q}$, will be unbounded, while the subspace $\boldsymbol{q}_0$ $(\boldsymbol{q} \in \boldsymbol{q}_0)$ will, generally, be bounded. The global coordinate $\boldsymbol{Q}$ properly corresponds to a long-time scale, while the local variable corresponds to a short-time scale. They are also called *slow* and, respectively, *fast* variables (see Sanchez-Palencia, 1980).

The generic microtransport equation governing the evolution of the temperature field $T = T(\boldsymbol{Q}, \boldsymbol{q}, t)$ in continuous adiabatic systems may be represented as

$$\rho C_p \frac{\partial T}{\partial t} + \nabla_{\boldsymbol{Q}} \cdot \boldsymbol{J} + \nabla_{\boldsymbol{q}} \cdot \boldsymbol{j} = 0 \tag{2.2}$$

where the constitutive equations for the global and the local internal energy flux-density vectors are

$$\begin{aligned} \boldsymbol{J} &= \rho(\boldsymbol{q}) C_p(\boldsymbol{q}) \boldsymbol{U}(\boldsymbol{q}) T - \mathsf{K}_T(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{Q}}(T) \\ \boldsymbol{j} &= \rho(\boldsymbol{q}) C_p(\boldsymbol{q}) \boldsymbol{u}(\boldsymbol{q}) T - \mathsf{k}_T(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{q}}(T) \end{aligned} \tag{2.3}$$

Here, $(\mathbf{K}_T, \mathbf{k}_T)$ denote the global and the local-space thermal conductivities and $(\boldsymbol{U}, \boldsymbol{u})$ the comparable velocity vectors. Here, ρ and C_p are positive functions and $\mathbf{K}_T$ and $\mathbf{k}_T$ are positive definite tensors.

This system of equations is subjected to the following conditions

$$\begin{aligned}
&\boldsymbol{n}\cdot\boldsymbol{j} = 0 \qquad \boldsymbol{n}\cdot\mathbf{K}_T\cdot\nabla_{\boldsymbol{q}}T = 0 && \text{on}\quad \partial\boldsymbol{q}_0\\
&|\boldsymbol{Q}|^m\{T,\boldsymbol{J}\}\to\{0,\mathbf{0}\}\quad\text{as}\quad|\boldsymbol{Q}|\to\infty && \begin{array}{l}\forall(\boldsymbol{q}, t\geqslant 0)\\ m=0,1,...\end{array}\\
&T(\boldsymbol{Q},\boldsymbol{q},0) = T_0(\boldsymbol{Q},\boldsymbol{q})
\end{aligned}\tag{2.4}$$

with the right-hand side a prescribed function.

Note that the energy dissipation and kinetic energy contribution are neglected in the microtransport equation.

We shall limit ourselves to the class of problems for which $\boldsymbol{n}\cdot\boldsymbol{u} = 0$ on $\partial\boldsymbol{q}_0$ and $\boldsymbol{u}\cdot\nabla_{\boldsymbol{q}}C_p = 0$. Also, it will be supposed that the thermal properties are everywhere nonnegative definite.

It proves useful to reformulate linear microscale problem (2.2) in terms of a Green's function. In this context, let us define a quantity $P = P(\boldsymbol{Q},\boldsymbol{q},t\,|\,\boldsymbol{Q}',\boldsymbol{q}',t')$ such that $\rho C_p P$ to be interpreted as the conditional probability density of having the temperature $T(\boldsymbol{Q},\boldsymbol{q},t)$ at the position $(\boldsymbol{Q},\boldsymbol{q})$ at the moment t if we had the temperature $T(\boldsymbol{Q}',\boldsymbol{q}',t')$ at $(\boldsymbol{Q}',\boldsymbol{q}')$ at the earlier moment t'. P will generally depend only on the differences $\boldsymbol{Q}-\boldsymbol{Q}'$ and $t-t'$ and so, choosing $\boldsymbol{Q}'=\mathbf{0}$ and $t'=0$, we can consider, without loss of generality, that $P = P(\boldsymbol{Q},\boldsymbol{q},t\,|\,\boldsymbol{q}')$.

Since we are modeling the transport of conserved entities, we have

$$\int\limits_{Q_\infty}\int\limits_{q_0}\rho C_p P\; d\boldsymbol{q}d\boldsymbol{Q} = 1 \qquad t\geqslant 0$$

and

$$P = 0 \qquad t<0$$

Also, the relationship between the temperature field T and the Green function P is

$$T(\boldsymbol{Q},\boldsymbol{q},t) = \int\limits_{Q'_\infty}\int\limits_{\boldsymbol{q}'_0}\rho(\boldsymbol{q}')C_p(\boldsymbol{q}')P(\boldsymbol{Q},\boldsymbol{q},t\,|\,\boldsymbol{q}')T(\boldsymbol{Q}',\boldsymbol{q}',0)\; d\boldsymbol{q}'d\boldsymbol{Q}' \tag{2.5}$$

and the microtransport equation of energy dispersion in continuous systems may be represented as

$$\rho C_p\frac{\partial P}{\partial t} + \nabla_{\boldsymbol{Q}}\cdot\boldsymbol{J}_P + \nabla_{\boldsymbol{q}}\cdot\boldsymbol{j}_P = \delta(\boldsymbol{Q})\delta(\boldsymbol{q}-\boldsymbol{q}')\delta(t)$$

where

$$\boldsymbol{J}_P = \rho(\boldsymbol{q})C_p(\boldsymbol{q})\boldsymbol{U}(\boldsymbol{q})P - \mathsf{K}_T(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{Q}}(P)$$
$$\boldsymbol{j}_P = \rho(\boldsymbol{q})C_p(\boldsymbol{q})\boldsymbol{u}(\boldsymbol{q})P - \mathsf{k}_T(\boldsymbol{q}) \cdot \nabla_{\boldsymbol{q}}(P)$$

Defining the macroscale Green's function

$$\overline{P}(\boldsymbol{Q}, t \,|\, \boldsymbol{q}') = \frac{1}{\overline{\rho C_p}^*} \int\limits_{q_0} \rho(\boldsymbol{q})C_p(\boldsymbol{q})P(\boldsymbol{Q}, \boldsymbol{q}, t \,|\, \boldsymbol{q}')\, d\boldsymbol{q} \tag{2.6}$$

this will become asymptotically independent of $\boldsymbol{q}'$

$$\overline{P}(\boldsymbol{Q}, t \,|\, \boldsymbol{q}') \cong \overline{P}(\boldsymbol{Q}, t)$$

and, hence, following a moment analysis (see Brenner and Edwards, 1993; Timofte, 1996), we are led to the following macrotransport equation

$$\overline{\rho C_p}^* \Big(\frac{\partial \overline{P}}{\partial t} + \overline{\boldsymbol{U}}^* \cdot \nabla_{\boldsymbol{Q}} \overline{P} \Big) = \overline{\mathsf{k}_T}^* : \nabla_{\boldsymbol{Q}} \nabla_{\boldsymbol{Q}} \overline{P} + \delta(\boldsymbol{Q})\delta(t) \tag{2.7}$$

subjected to the conditions

$$\begin{aligned} &\overline{P} = 0 && t < 0 \\ &\overline{P} \to 0 && \text{when} \quad |\boldsymbol{Q}| \to \infty \end{aligned} \tag{2.8}$$

Here, the macroscale coefficients $\overline{\rho C_p}^*$ and $\overline{\boldsymbol{U}}^*$ are given by

$$\overline{\rho C_p}^* = \frac{1}{\tau_0} \int\limits_{q_0} \rho C_p \, d\boldsymbol{q} \tag{2.9}$$

where

$$\tau_0 = \int\limits_{q_0} d\boldsymbol{q} \tag{2.10}$$

and by

$$\overline{\boldsymbol{U}}^* = \int\limits_{q_0} \rho C_p P_0^{\infty} \boldsymbol{U} \, d\boldsymbol{q} \tag{2.11}$$

The effective thermal conductivity dyadic has the expression:

$$\overline{\mathsf{k}_T}^* = \overline{\mathsf{k}}^M + \overline{\mathsf{k}}^C \tag{2.12}$$

where

$$\overline{\mathsf{k}}^M = \frac{1}{\tau_0} sim \int_{q_0} \mathsf{K}_T \; d\boldsymbol{q} \tag{2.13}$$

is the "molecular" contribution and

$$\overline{\mathsf{k}}^C = sim \int_{q_0} \Big[\Big(P_0^\infty - \frac{1}{\tau_0 \overline{\rho C_p}^*} \Big) \mathsf{K}_T + \rho C_p P_0^\infty \boldsymbol{B}(\boldsymbol{U} - \overline{\boldsymbol{U}}^*) \Big] \; d\boldsymbol{q} \tag{2.14}$$

is the convective contribution.

These phenomenological coefficients are to be obtained after solving the associated local problems for $P_0^\infty(\boldsymbol{q})$ and $\boldsymbol{B}(\boldsymbol{q})$

$$\begin{aligned} &\nabla_{\boldsymbol{q}} \cdot \boldsymbol{j}_0^\infty = 0 \\ &\boldsymbol{j}_0^\infty = \rho C_p \boldsymbol{u} P_0^\infty - \mathsf{k}_T \cdot \nabla_{\boldsymbol{q}} P_0^\infty \\ &\boldsymbol{n} \cdot \mathsf{K}_T \cdot \nabla_{\boldsymbol{q}} P_0^\infty = 0 \qquad \text{on} \quad \partial \boldsymbol{q}_0 \\ &\int_{q_0} \rho C_p P_0^\infty \; d\boldsymbol{q} = 1 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\boldsymbol{j}_0^\infty \cdot \nabla_{\boldsymbol{q}} \boldsymbol{B} - \nabla_{\boldsymbol{q}} \cdot (P_0^\infty \mathsf{k}_T \cdot \nabla_{\boldsymbol{q}} \boldsymbol{B}) = \rho C_p P_0^\infty (\boldsymbol{U} - \overline{\boldsymbol{U}}^*) \\ &\rho C_p P_0^\infty \boldsymbol{n} \cdot \mathsf{k}_T \cdot \nabla_{\boldsymbol{q}} \boldsymbol{B} = 0 \qquad \text{on} \quad \partial \boldsymbol{q}_0 \end{aligned} \tag{2.16}$$

More, we shall require that P_0^∞ is nonnegative for all $\boldsymbol{q} \in \boldsymbol{q}_0$. Also, P_0^∞ and $\boldsymbol{B}$ must be single-valued for all $\boldsymbol{q} \in \boldsymbol{q}_0$.

So, we can express the macrotransport coefficients $\overline{\rho C_p}^*$, $\overline{\boldsymbol{U}}^*$ and $\overline{\mathsf{k}_T}^*$ in terms of the prescribed microscale data and the system geometry. It is worthwhile to notice that, in fact, ρC_p, which is inhomogeneous in $\boldsymbol{q}$, acts like a biasing potential. This causes a redistribution of the internal energy in the local space, which is finally reflected in the magnitude of the macrotransport coefficients $\overline{\boldsymbol{U}}^*$ and $\overline{\mathsf{k}_T}^*$.

The coarse-grained macroscale temperature field

$$\overline{T}(\boldsymbol{Q}, t) = \frac{1}{\tau_0} \int_{Q'_\infty} \int_{q'_0} \rho(\boldsymbol{q}') C_p(\boldsymbol{q}') \overline{P}(\boldsymbol{Q} - \boldsymbol{Q}', t \,|\, \boldsymbol{q}') T(\boldsymbol{Q}', \boldsymbol{q}', 0) \; d\boldsymbol{q}' d\boldsymbol{Q}' \tag{2.17}$$

will satisfy the *macrotransport equation*

$$\overline{\rho C_p}^{*}\Big(\frac{\partial \overline{T}}{\partial t} + \overline{\boldsymbol{U}}^{*} \cdot \nabla_{\boldsymbol{Q}} \overline{T}\Big) = \overline{\mathbf{k}_T}^{*} : \nabla_{\boldsymbol{Q}} \nabla_{\boldsymbol{Q}} \overline{T} \tag{2.18}$$

subjected to appropriate initial and boundary conditions (see Brenner and Edwards, 1993; Timofte, 1996).

A similar analysis can be done for the problem of thermal dispersion in discontinuous adiabatic systems.

The geometrical structure of the discontinuous medium is idealized as being spatially periodic (porous media, composite materials, laminated media). The periodic medium is represented as a spatially periodic array in $\mathbb{R}^3$, composed of topologically indistinguishable unit cells of periodicity, having the same shape, orientation, volume and "content".

If we denote by τ_0 the volume of such an elementary cell and by τ_p the volume of the solid part, we have

$$\tau_p = \tau_0 - \tau_f$$

where τ_f is the interstitial fluid volume within such a cell.

Arbitrarily designating one of the elementary cells as being the *zeroth cell*, it is convenient to measure the position vector $\boldsymbol{R}$ of any point in space relative to the centroid of this cell. Denoting by $\boldsymbol{R_n}$ the position vector of the centroid of the $\boldsymbol{n}$-th cell relative to the centroid of the zeroth cell and by $\boldsymbol{r} \in \tau_0\{\boldsymbol{n}\}$ the local position vector for any point within the $\boldsymbol{n}$-th cell relative to an origin at its center, we have

$$\boldsymbol{R} = \boldsymbol{R_n} + \boldsymbol{r} \tag{2.19}$$

If we suppose that at the particle-fluid interface S_p we have a local equilibrium described by a linear partitioning relationship, using similar notations as in the continuous case and introducing microscale Green's function $P = P(\boldsymbol{R_n}, \boldsymbol{r}, t \,|\, \boldsymbol{r}')$, this will obey the following system of cellular-level equations (see Brenner and Edwards, 1993; Timofte, 1996)

$$\begin{aligned}
&\rho(\boldsymbol{r}) C_p(\boldsymbol{r}) \frac{\partial P}{\partial t} + \nabla \cdot \boldsymbol{J} = \delta_{\boldsymbol{nn}'} \delta(\boldsymbol{r} - \boldsymbol{r}') \delta(t) \\
&\boldsymbol{J} = \rho(\boldsymbol{r}) C_p(\boldsymbol{r}) \boldsymbol{U}(\boldsymbol{r}) P - \mathbf{K}_T(\boldsymbol{r}) \cdot \nabla P \\
&\boldsymbol{\nu} \cdot \varDelta_{S_p} \boldsymbol{J} = 0 \qquad \text{on } S_p \\
&\left.\begin{array}{l} |\boldsymbol{R_n} - \boldsymbol{R_{n'}}|^m P \to 0 \\ |\boldsymbol{R_n} - \boldsymbol{R_{n'}}|^m \boldsymbol{J} \to 0 \end{array}\right\} \qquad \text{as} \qquad \left\{\begin{array}{l} \{\boldsymbol{n} - \boldsymbol{n}'\} \to \infty \\ m = 0, 1, \dots \end{array}\right.
\end{aligned} \tag{2.20}$$

In $(2.20)_3$, for an arbitrary tensor field $\boldsymbol{f}$, Δ_{S_p} defines the "jump" of $\boldsymbol{f}$, across the discontinuous phase surface S_p and $\boldsymbol{\nu}$ is a unit vector which is normal to this surface.

The thermophysical properties ρ, C_p, K_T and the local fluid velocity are regarded as being spatially periodic.

Following a moment-matching scheme and considering macroscale Green's function $\overline{P}$

$$\overline{P}(\boldsymbol{R_n}, t \,|\, \boldsymbol{r}') = \frac{1}{\overline{\rho C_p}^*} \int\limits_{\tau_0} \rho(\boldsymbol{r}) C_p(\boldsymbol{r}) P(\boldsymbol{R_n}, r, t \,|\, \boldsymbol{r}')\; d^3\boldsymbol{r} \cong \overline{P}(\overline{\boldsymbol{R}}, t) \tag{2.21}$$

we get the following *macrotransport equation*

$$\overline{\rho C_p}^* \Big(\frac{\partial \overline{P}}{\partial t} + \overline{\boldsymbol{U}}^* \cdot \nabla \overline{P} \Big) = \overline{\mathsf{k}_T}^* : \nabla\nabla\overline{P} + \delta(\overline{\boldsymbol{R}})\delta(t) \tag{2.22}$$

with

$$\overline{P} \to 0 \qquad \text{as} \qquad |\overline{\boldsymbol{R}}| \to \infty \tag{2.23}$$

Here, $\overline{\boldsymbol{R}}$ is the macroscale (Darcy) position vector of a lattice point relative to an origin O arbitrarily chosen in the unit periodic cell.

The macroscale coefficients $\overline{\rho C_p}^*$, $\overline{\boldsymbol{U}}^*$ and $\overline{\mathsf{K}_T}^*$ are given by the following formulas

$$\begin{aligned} \overline{\rho C_p}^* &= \frac{1}{\tau_0} \int\limits_{\tau_0} \rho(\boldsymbol{r}) C_p(\boldsymbol{r})\; d^3\boldsymbol{r} \\ \overline{\boldsymbol{U}}^* &= \int\limits_{\tau_0} \boldsymbol{J}_0^\infty(\boldsymbol{r})\; d^3\boldsymbol{r} \\ \overline{\boldsymbol{\alpha}}^* &= \frac{\overline{\mathsf{K}_T}^*}{\overline{\rho C_p}^*} = \int\limits_{\tau_0} P_0^\infty(\boldsymbol{r}) (\nabla \boldsymbol{B})^\top(\boldsymbol{r}) \cdot sim\mathsf{K}_T(\boldsymbol{r}) \cdot \nabla \boldsymbol{B}(\boldsymbol{r})\; d^3\boldsymbol{r} \end{aligned} \tag{2.24}$$

The fields $P_0^\infty(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ satisfy, for $\boldsymbol{r} \in \boldsymbol{\tau}_0$, the following boundary-value problems

$$\begin{aligned} &\nabla \cdot \boldsymbol{J}_0^\infty = 0 \\ &\boldsymbol{J}_0^\infty = \rho C_p \boldsymbol{U} P_0^\infty - \mathsf{K}_T \cdot \nabla P_0^\infty \\ &\boldsymbol{\nu} \cdot \Delta_{S_p} \boldsymbol{J}_0^\infty = 0 \qquad \Delta_{S_p} P_0^\infty = 0 \qquad \text{on } S_p \\ &\|P_0^\infty\| = 0 \qquad \|\nabla P_0^\infty\| = 0 \qquad \text{on } \partial\tau_0 \\ &\int\limits_{\tau_0} \rho(\boldsymbol{r}) C_p(\boldsymbol{r}) P_0^\infty(\boldsymbol{r})\; d^3\boldsymbol{r} = 1 \end{aligned} \tag{2.25}$$

and

$$
\begin{aligned}
&\nabla \cdot (P_0^\infty \mathsf{K}_T \cdot \nabla \boldsymbol{B}) - \boldsymbol{J}_0^\infty \cdot \nabla \boldsymbol{B} = \rho C_p P_0^\infty \overline{\boldsymbol{U}}^* \\
&\boldsymbol{B} \quad \text{is continuous across} \ \ S_p \\
&\boldsymbol{\nu} \cdot \varDelta_{S_p}(\mathsf{K}_T \cdot \nabla \boldsymbol{B}) = 0 \qquad \text{on} \ \ S_p \\
&\|\boldsymbol{B}\| = -\|\boldsymbol{r}\| \qquad\qquad \|\nabla \boldsymbol{B}\| = 0 \qquad \text{on} \ \ \partial\tau_0
\end{aligned}
\tag{2.26}
$$

Here, for any tensor-valued field F, $\|\mathsf{F}\|$ defines the "jump" in the value of F between the equivalent points lying on opposite pairs of the cell faces.

In this manner, we can obtain a macrotransport paradigm for a class of thermal dispersion phenomena occurring in periodic media.

Moreover, for this case, introducing two positive scalars U_0 and a, we can express the fluid velocity $\boldsymbol{U}$ in the form

$$
\boldsymbol{U}(\boldsymbol{r}) = U_0 \boldsymbol{V}(\boldsymbol{r}/a) \tag{2.27}
$$

U_0 and a will be interpreted as being the velocity and the spatial scale parameters which characterize our transport processes.

We are interested in getting the functional dependence of the asymptotic dispersion coefficients $\overline{\alpha}^*$ in terms of these two parameters.

Using a central limit theorem for Markov processes, it can be proved (see Timofte, 1999; Bhattacharya *et al.*, 1989) that for a special case of thermal dispersion phenomena in periodic media, the macroscale coefficients $\overline{\alpha}^*_{ij}$ depend only on the product aU_0, the result being in accordance with all the experimental studies that have been done. In fact

$$
\overline{\boldsymbol{\alpha}}^*(a, U_0) = \overline{\boldsymbol{\alpha}}^*(U_0, a) = \overline{\boldsymbol{\alpha}}^*(aU_0, 1) \tag{2.28}
$$

This interchangeability of the velocity and spatial scale parameters in the large-scale dispersion matrix enables us to consider, if needed, that the spatial scale parameter a is held fixed at $a = 1$, while the velocity parameter U_0 is allowed to vary.

A more precise analysis of the asymptotic behavior of the dispersion coefficients $\overline{\alpha}^*_{ij}$ can be done if the thermophysical properties ρ, C_p, K_T are supposed to be constant and if we make more restrictive assumptions about the velocity field (see Timofte, 1999).

3. Applications

As a first example, we shall consider the problem of internal energy dispersion in an incompressible viscous fluid moving under laminar flow conditions between two parallel, insulated porous plates separated by a distance h. The upper plate moves at a velocity $\boldsymbol{U}_0$ parallel to it in the $\boldsymbol{x}$-direction. Simultaneously, there exists a uniform flow across the channel (in the negative $\boldsymbol{y}$-direction) at a constant velocity $\boldsymbol{v}_0$. In this case, the fluid velocity field $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \frac{U_0 y}{h}\boldsymbol{i} - v_0\boldsymbol{j} \tag{3.1}$$

At $t = 0$, an amount of heat is instantaneously added into our system over some region of the infinite domain between the plates in the form of some initial temperature distribution $T_0(\boldsymbol{x})$.

Assuming that the thermophysical properties ρ, C_p and K_T are constant, the evolution of the temperature $T(t, \boldsymbol{x})$ will be governed by the following equation

$$\frac{\partial T}{\partial t} + \frac{U_0 y}{h}\frac{\partial T}{\partial x} - v_0\frac{\partial T}{\partial y} = \alpha \Delta T \tag{3.2}$$

with the initial condition $T(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$ and with $\alpha = K_T/(\rho C_p)$.

Introducing the dimensionless parameter

$$\beta = \frac{v_0 h}{\alpha} \tag{3.3}$$

and considering the incomplete gamma function

$$\gamma(n+1, \beta) = \int_0^\beta \xi^n \exp(-\xi)\, d\xi \qquad n = 0, 1, 2, ... \tag{3.4}$$

the macroscale thermal velocity $\overline{\boldsymbol{U}}$ is given by

$$\overline{\boldsymbol{U}}^* = \overline{U}^* \boldsymbol{i} \tag{3.5}$$

where

$$\overline{U}^* = \frac{U_0 \gamma(2, \beta)}{\beta \gamma(1, \beta)} \tag{3.6}$$

If we consider the mean axial fluid velocity

$$\overline{V} = \frac{U_0}{2} \tag{3.7}$$

we get

$$\frac{\overline{U}^*}{\overline{V}} = \frac{2\gamma(2,\beta)}{\beta\gamma(1,\beta)} \tag{3.8}$$

So, the thermal velocity $\overline{U}^*$ is different from the mean axial fluid velocity $\overline{V}$.

As the cross flow velocity $v_0 \to 0$, corresponding to $\beta \to 0$, it is easy to see that

$$\lim_{\beta\to 0} \frac{\overline{U}^*}{\overline{V}} = 1$$

If $v_0 \to \infty$, then $\beta \to \infty$ and

$$\lim_{\beta\to \infty} \frac{\overline{U}^*}{\overline{V}} = 0$$

Using the general formulas given by the above method of moments, we see that the only component of the effective thermal dispersivity dyadic $\overline{\boldsymbol{\alpha}}^*$ which is different from zero is

$$\overline{\alpha}^*_{11} = \alpha + k(\beta)\frac{h^2\overline{V}^2}{\alpha} \tag{3.9}$$

with

$$k(\beta) = \frac{4}{\beta^4}\Big[\frac{\gamma(2,\beta)}{\gamma(1,\beta)}\Big]^2\Big[2\frac{\gamma(2,\beta)}{\gamma(1,\beta)} + 3\frac{\gamma(3,\beta)}{\gamma(2,\beta)}\Big] \tag{3.10}$$

We notice that if $v_0 = 0$ we get a formula which is similar to classical formula (1.1) for the case of Taylor's solute dispersion.

As a second example, let us consider the problem of internal energy dispersion in a layered periodic porous medium, saturated with a viscous incompressible fluid. We shall choose as a periodic cell the parallelepiped τ_0 having the sides l_x, l_y and l_z. Let us suppose that the thermophysical properties ρ, C_p, K_T are constant and the velocity field $\boldsymbol{U}$ is periodic, with the period l_z

$$\boldsymbol{U} = \Big[U_0\Big(1 + \sin\frac{2\pi z}{l_z}\Big), U_0\sin\frac{2\pi z}{l_z}, U_0\omega\Big] \tag{3.11}$$

Here, U_0 and ω are given real parameters (see Timofte, 1996).

Initially, the medium has an uniform temperature T_0 (we can choose $T_0 = 0$). At $t = 0$, an amount of heat Q is instantaneously introduced into the system as the initial distribution of temperature $T(0,\boldsymbol{x}) = T_0(\boldsymbol{x})$.

With $\alpha = K_T/(\rho C_p)$, the evolution of the temperature $T(t,\boldsymbol{x})$ will be governed by the following equation

$$\frac{\partial T}{\partial t} = \alpha\Delta T - U_0\Big(1 + \sin\frac{2\pi z}{l_z}\Big)\frac{\partial T}{\partial x} - U_0\sin\frac{2\pi z}{l_z}\frac{\partial T}{\partial y} - U_0\omega\frac{\partial T}{\partial z} \tag{3.12}$$

subjected to the initial condition $T(0,\boldsymbol{x}) = T_0(\boldsymbol{x})$.

Obviously

$$\overline{\boldsymbol{U}}^* = (U_0, 0, U_0\omega) \tag{3.13}$$

Following the general scheme offered by the above method of moments, we can compute the macroscale coefficients $\overline{\alpha}^*_{ij}$

$$\begin{aligned}
&\overline{\alpha}^*_{11} = \overline{\alpha}^*_{22} = \alpha + \frac{\alpha l_z^2 U_0^2}{2[(2\pi\alpha)^2 + (U_0 l_z \omega)^2]} \\
&\overline{\alpha}^*_{33} = \alpha \\
&\overline{\alpha}^*_{12} = \overline{\alpha}^*_{21} = \frac{\alpha l_z^2 U_0^2}{2[(2\pi\alpha)^2 + (U_0 l_z \omega)^2]} \\
&\overline{\alpha}^*_{13} = \overline{\alpha}^*_{31} = \overline{\alpha}^*_{23} = \overline{\alpha}^*_{32} = 0
\end{aligned} \tag{3.14}$$

It is simple to see that for small values of $l_z U_0$, $\overline{\alpha}^*_{ij}$ depend quadratically on $l_z U_0$. However, as $l_z U_0 \to \infty$, each $\overline{\alpha}^*_{ij}$ becomes asymptotically constant.

As the final example, we shall consider the problem of internal energy dispersion in a periodic porous medium, saturated with an incompressible viscous fluid having the velocity field $\boldsymbol{U}(\boldsymbol{x}) = U_0 \boldsymbol{V}(\boldsymbol{x})$ given by

$$\boldsymbol{V} = [0, 2 + \sin 2\pi x, 2 + \cos 2\pi x \, \cos 2\pi y] \tag{3.15}$$

We assume that the spatial scale parameter a is fixed at $a = 1$ and the phenomenological coefficients ρ, $, C_p$ and K_T are strictly positive constants. Obviously

$$\overline{\boldsymbol{U}}^* = [0, 2U_0, 2U_0] \tag{3.16}$$

It is easy to see that in this case

$$\begin{aligned}
&\overline{\alpha}^*_{11} = \alpha \\
&\overline{\alpha}^*_{12} = \overline{\alpha}^*_{21} = \overline{\alpha}^*_{13} = \overline{\alpha}^*_{31} = \overline{\alpha}^*_{23} = \overline{\alpha}^*_{32} = 0
\end{aligned} \tag{3.17}$$

For this example, closed-form solutions of the macrotransport coefficients $\overline{\alpha}^*_{22}$ and $\overline{\alpha}^*_{33}$ cannot be obtained. However, the analytical theory developed by Timofte (1999) and Bhattacharya *et al.* (1989) shows that, as $U_0 \to \infty$, $\overline{\alpha}^*_{22} = \alpha + \mathcal{O}(U_0^2)$ and $\overline{\alpha}^*_{33} = \alpha + \mathcal{O}(1)$.

This example reflects the influence of the geometry of the flow curves on the asymptotic behavior of the macrotransport coefficients.

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O asymptotycznej naturze procesu termicznej dyspersji Taylora w ośrodkach periodycznych

Streszczenie

W pracy dokonano przeglądu szerokiej klasy zjawisk związanych z termiczną dyspersją w wybranych układach jednorodnych w skali makro. W opisie wykorzystano uogólnioną metodę momentów i twierdzenie o granicy centralnej. Jako szczególnie

interesujący przedstawiono problem obliczania współczynników makroskali w funkcji współczynników mikroskali i geometrii badanego układu. Ponadto przeanalizowano funkcjonalną zależność współczynników efektywnych od pola prędkości i przestrzennych parametrów skali.

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