JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
42, 4, pp. 905-926, Warsaw 2004

VIBRATIONS AND STABILITY OF A TWO-ROD COLUMN LOADED BY A SECTOR OF A ROLLING BEARING

Lech Tomski
Janusz Szmidla
Maria Gołębiowska-Rozanow

Institute of Mechanics and Machine Design Foundations, Technical University of Częstochowa
e-mail: szmidla@imipkm.pcz.pl

In this work, a new type of loading of slender systems, which is a follower force directed towards a positive or a negative pole is presented. Constructional models of loading heads, which realize this type of loading, are also presented. The variant of theoretical investigations concerning formulation of boundary conditions is shown. It results from the energetic formulation. Dependently on constructional variants of both the loading and receiving heads, values of the critical force and courses of the natural frequency as a function of the external loading for the applied geometry and physical constants of the column are determined. Theoretical results are compared with those from an experiment.

Key words: elastic column, divergence instability, natural frequency

1. Introduction

1.1. Euler's and Beck's load. Plane load-natural frequency curves

The loading of a slender system, called Euler's load, has been known since the eighteenth century (compare Euler, 1774). This load is characterised by the fact that a compressive force of a column has a constant point of its application and a constant point of action, which are unchangeable during buckling.

A curve in the plane: load P - natural frequency ω (see Fig. 1a) has always a negative slope, which was proved by Leipholz (1974).

In 1952 Beck (compare Beck, 1953) reported the first solution for columns with a non-conservative load (by follower force). This load is characterised by a force that is tangential to the deflected axis of the column at the loaded end.

The curve in the plane: load-natural frequency is shown in Fig. 1b (change of vibration form takes place at O point). In Fig. 1 the critical force is denoted as P_c, while $M1$ and $M2$ denote the first and second mode of vibrations, respectively.

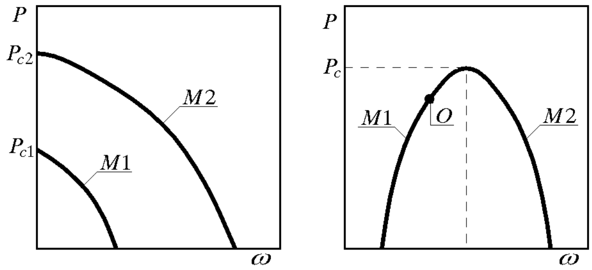

Fig. 1. Frequency curves in the plane: load P - natural frequency ω for divergence (a), and flutter (b) systems

1.2. The generalised load and the condition of its potential

Let us consider a cantilever column shown in Fig. 2. The flexural rigidity is denoted as EJ, mass density as $\overline{\rho}A$ (where E is Young's modulus, J – moment of inertia related to the neutral axis in the bending plane, $\overline{\rho}$ – material density, A – cross-section area). $W(x,t)$ is a transverse displacement of the column, m – concentrated mass at the free end of the column, C_1 – rigidity of rotational spring modelling elasticity of the fastened system. The column is loaded by a longitudinal force P, shearing force H and bending moment M. According to works by Kordas (1963), Gajewski and Życzkowski (1970, 1988), it is assumed that the shearing force H and bending moment M linearly depend on the displacement $W(l,t)$ and deflection angle $[\partial W(x,t)/\partial x]_{x=l}$ of the free end of the column in the following way

$$
\begin{aligned}
H &= P\Big[(1-\mu)\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \gamma W(l,t)\Big] \\
M &= P\Big[\rho\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \nu W(l,t)\Big]
\end{aligned}
\tag{1.1}
$$

where ρ, ν, μ, γ are determined coefficients.

The load is conservative (the external force has a potential) if rotation of the gradient of its vector field is equal to zero, which leads to the relation (Kordas, 1963; Gajewski and Życzkowski, 1970, 1988; Tomski *et al.*, 1996)

$$
\frac{\partial H}{\partial W(x,t)}\Big|_{x=l} = \frac{\partial M}{\frac{\partial W(x,t)}{\partial x}\big|_{x=l}}
\tag{1.2}
$$

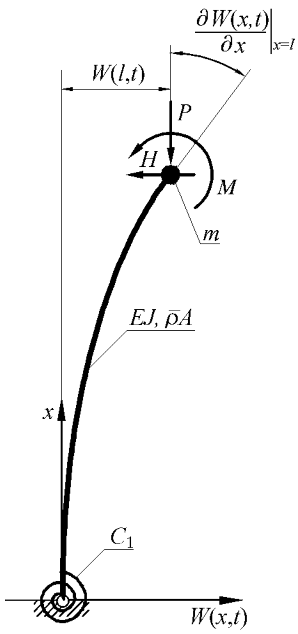

Fig. 2. A scheme of the cantilever column subjected to the generalised load

For the generalised load, relation (1.2) gives

$$\nu + \mu - 1 = 0 \tag{1.3}$$

A physical interpretation of this condition according to the field theory is given in the work by Tomski *et al.* (1996).

Apart from Euler's, Beck's and the generalised loads, the following loads are referred to in the literature:

- the load developed by a force directed towards the positive pole – the fixed point through which the direction of the force action passes is placed below the free end of the column (Gajewski and Życzkowski, 1969; Timoshenko and Gere, 1961)

- the load developed by a force directed towards the negative pole – the fixed point through which the direction of the force action passes is placed above the free end of the column (Gajewski and Życzkowski, 1969; Dąbrowski, 1984).

1.3. Authors' research

1.3.1. Specific load. Load-natural frequency curves in the plane

In 1994 Tomski *et al.* described a new system of a loaded column and planar frame (Tomski *et al.*, 1995). The vast results of theoretical and experimental research on a column subjected to a specific load, are presented in a paper by Tomski *et al.* (1996). A specific load can be called a generalised load with the force directed towards the positive pole. Such a load can give an uncommon course to the curve in the plane: load P - natural frequency ω (see Tomski *et al.*, 1996).

The function $P(\omega)$ (Fig.3) for this system has the following course:

- for $P \in < 0, P_c)$ (P_c is the critical load) the angle of the curve tangent to $P(\omega)$ can take a positive, zero or negative value
- for $P \approx P_c$ load, the slope of the curve in the $P(-\omega)$ plane is always negative
- change of the free vibration form (from the first to the second and inversely) takes place along the curve which determines the $P(\omega)$ function for the basic frequency ($M1$, $M2$ denote the first and second forms of vibrations, respectively).

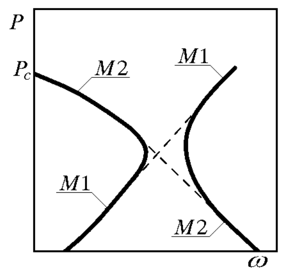

Fig. 3. Frequency curves in the plane: load-natural frequency for the divergence-pseudo-flutter type system

The system realising such a course was named the divergence-pseudo-flutter type, Bogacz *et al.* (1998), as opposed to the already known separate systems: divergence and flutter ones.

Further results of theoretical and experimental investigations, concerning the specific load are presented in the following publications:

- a follower load with the force directed towards the positive pole, Tomski *et al.* (1998)

- a generalised load with the force directed towards the negative pole and a follower load with the force directed towards the negative pole, Tomski *et al.* (1999).

An attempt to optimise a column subjected to a generalised load by a force directed towards the positive pole was made by Bogacz *et al.* (1998). The results of experimental investigations, connected with changes of the free vibration form along the load-natural frequency curve, were presented by Bogacz *et al.* (1998). It was proved that the course of the curve was in good agreement with that shown in Fig. 3.

1.3.2. *The extended condition for the load potential*

In this paper, the condition for the potential of a generalised load has been worked out according to the property that if the potential of external forces exists (Levinson, 1966; Wallerstein, 2002), then

$$\delta V = -\delta L \tag{1.4}$$

where δV is a variation of the potential energy and δL – variation of the potential work

$$\begin{aligned} \delta V &= \frac{1}{2}\Big(\delta M \frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + M\delta\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + \delta H W(l,t) + H\delta W(l,t)\Big) \\ \delta L &= -\Big(M\delta\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} + H\delta W(l,t)\Big) \end{aligned} \tag{1.5}$$

while (Levinson, 1966)

$$\delta M = \delta H = 0 \tag{1.6}$$

The energy of the force P is neglected because it is a potential load.

Taking into account (1.1), (1.4)–(1.6), the extended condition for the external load potential is

$$(\mu + \nu - 1)\Big(\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\delta W(l,t) - W(l,t)\delta\frac{\partial W(x,t)}{\partial x}\Big|_{x=l}\Big) = 0 \tag{1.7}$$

If $W(l,t)$ and $[\partial W(x,t)/\partial x]_{x=l}$ are linearly independent, the second factor in relationship (1.7) is set to zero. The same condition for the external load potential was obtained in the paper by Tomski *et al.* (1996), in which the self-adjointness condition of differential operators describing free vibrations of the column was borne in mind.

2. Statement of the problem

New heads loading and receiving the column are presented in this paper. These heads are to realise a load with an *a priori* unknown force direction and point of application. The value of the column bending moment depends on the point of force application P.

The boundary conditions are calculated according to the Hamilton principle with the manifestation of the load in two intersections arising from design of the forcing and loading heads. The formulation of the load potential is used here through the second factor of condition (1.7).

The results of numerical calculations, concerning the load-natural frequency curves, are verified by experimental investigations.

It must be noted that every specific load, i.e.:

- generalised load with a force directed towards positive or negative poles
- load by a follower force directed towards positive or negative poles, can be realised with the use of loading and receiving heads in a few alternative designs, Tomski *et al.* (1996, 1998, 1999).

3. Structural schemes of the heads realising the load

The loading systems, which the column is subjected to, are presented in Fig. 4. These systems are composed of an enclosure (1) and end with rolling guides (2). The outer race (3) (Figs 4a,b) and internal rings (Fig. 4c) of a roll bearing (ball bearing) are mounted on the enclosure (1). The internal ring (5) (Figs 4a-b) and the outer race (Fig. 4c) of the roll bearing are placed in an element (4). The element (4) is connected to a block (7) by means of a lock (6). Two rods (8) of the column are mounted in the block. It is assumed that the elements of length l_0 (lock (6), element (4), block (7)) are infinitely rigid (this relates to constructional considerations). The elements (1, 3) make up the loading head, while (4, 5, 6, 7) – the receiving head.

The column consists of two rods (8.1, 8.2) with the bending rigidity $(EJ)_1$ and $(EJ)_2$, respectively, and the mass per unit length $(\overline{\rho}A)_1$ and $(\overline{\rho}A)_2$ (and $(EJ)_1 = (EJ)_2$, $(\overline{\rho}A)_1 = (\overline{\rho}A)_2$, $(EJ)_1 + (EJ)_2 = EJ$, $(\overline{\rho}A)_1 + (\overline{\rho}A)_2 = \overline{\rho}A$). The rods of the column have the same cross-sections and are made of the same material. The rods and their physical and geometrical parameters are distinguished by 1, 2 indexes, which are only needed to calculate symmetrical

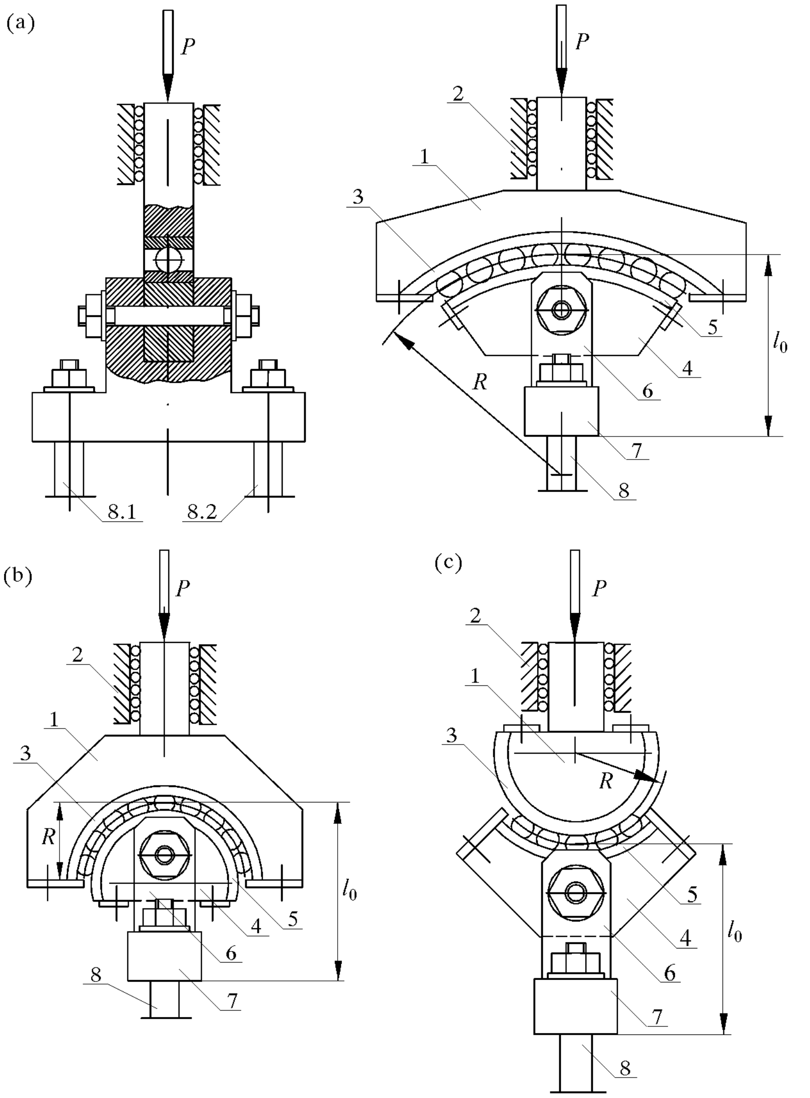

Fig. 4. Structural schemes of heads realising the column load: (a) for $R > 0$, $R - l_0 > 0$; (b) for $R > 0$, $R - l_0 < 0$; (c) for $R < 0$, $l_0 > 0$

natural frequencies and to determine corresponding forms of vibration. Hence, we can assume a global bending rigidity EJ and elementary mass of the column $\overline{\rho}A$ in the following considerations in this paper.

4. The physical model of the system

The physical model of the considered system, in the constructional variant shown in Fig. 4a, is presented in Fig. 5. The systems, shown in Figs 4b,c, are a specific case of the system presented in Fig. 4a. The load is manifested in I-I and II-II sections. Three-component parts X, Y, Z, which fulfil the determined aims, can be distinguished by taking into account the system shown in Fig. 5.

The system X creates external load 1 and internal forces 3a, which are balanced by head 2 and rolling guides. The system Y consists of head 4 and balanced forces 3a and 5a (Fig. 5c). The system Z is determined by column 6 and external forces 5b and column mount 7.

The manifestation of the load in the Y and Z systems makes it possible to specify boundary conditions on the basis of mechanical energy balance (vibration problem) or potential energy balance (statics criterion). The manifestation of the load in the Z system in section II-II makes it possible to specify boundary conditions on the basis of mechanical energy or potential energy, and also on the dependence of external load 5b on the internal forces in column 6.

It should be underlined that if the generalised load is taken into account, then the force direction and its point of application are *a priori* unknown for the considered structure. As a result, the coefficient of the follower force η_1 and coefficient of the bending moment η_2 (Fig. 5) are assumed in the following considerations.

Geometrical dependences between elements of the structure and load placement (force P and bending moment M) lead to the following relationship

$$W(l,t) = (R - l_0)\frac{\partial W(x,t)}{\partial x}\Big|_{x=l} \qquad \eta_2 = 1 - \eta_1 \tag{4.1}$$

4.1. Mechanical energy of the system and Hamilton's principle. Boundary conditions

Total potential energy of the system depicted in Fig. 5 is examined with respect to the place of its manifestation (Table 1):

— energy of the elastic strain

$$V_1 = \frac{EJ}{2}\int\limits_0^l \Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big]^2\,dx \tag{4.2}$$

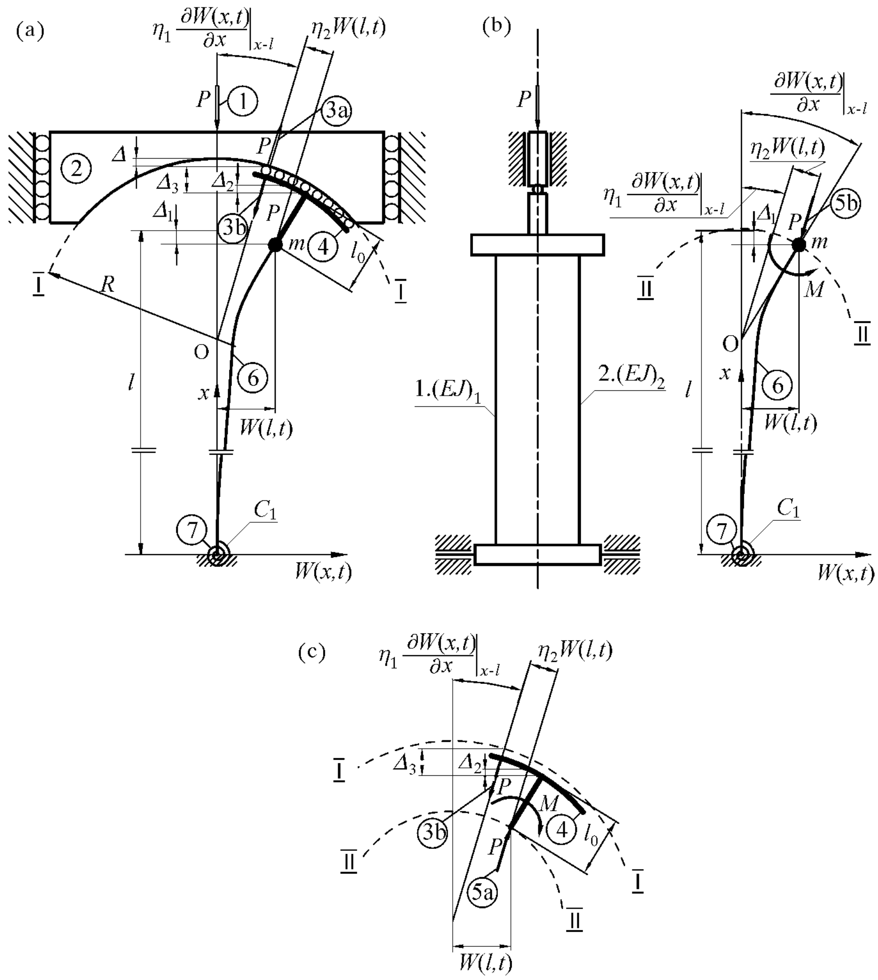

Fig. 5. The physical model of the object

— potential energy of elasticity of the fastening

$$V_2 = \frac{1}{2}C_1\Big[\frac{\partial W(x,t)}{\partial x}\Big|_{x=0}\Big]^2 \tag{4.3}$$

— potential energy of the system V_{kn} (Table 1 – where $k = 3,4,5$, $n = *, **, ***$)

Table 1. Potential energy of the system

Manifestation of the load in section		
I-I	II-II	I-I and II-II
Potential energy of the vertical component of the force P		
$V_2^* = -P\varDelta_1 - P\varDelta_2 + P\varDelta_3$	$V_2^{**} = -P\varDelta_1$	V_2^{***}
Potential energy of the horizontal component of the force P		
$V_3^* = PR\eta_1^2\varDelta_l^2/2$	$V_3^{**} = P\eta_1 W(l,t)\varDelta_l/2$	V_3^{***}
Potential energy of the bending moment		
$V_4^* = 0$	$V_4^{**} = -V_4^{***}$	V_4^{***}

where

$$V_2^{***} = -P\varDelta_2 + P\varDelta_3$$

$$V_3^{***} = \frac{P}{2}[R\eta_1^2\varDelta_l^2 - \eta_1 W(l,t)\varDelta_l]$$

$$V_4^{***} = -\frac{P}{2}\eta_2 W(l,t)\varDelta_l$$

$$\varDelta_l = \frac{\partial W(x,t)}{\partial x}\Big|_{x=l} \qquad \varDelta_1 = \frac{1}{2}\int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial x}\Big]^2\, dx$$

$$\varDelta_2 = \frac{1}{2}l_0\varDelta_l^2 \qquad \varDelta_3 = \frac{1}{2}R(1-\eta_1^2)\varDelta_l^2$$

Kinetic energy for the considered system is as follows

$$T = \frac{1}{2}\overline{\rho}A\int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial t}\Big]^2\, dx + \frac{m}{2}\Big[\frac{\partial W(x,t)}{\partial t}\Big|_{x=l}\Big]^2 \tag{4.4}$$

In this paper, the formulation of the problem is carried out with the use of Hamilton's principle (Goldstein, 1950)

$$\delta\int\limits_{t_1}^{t_2}(T-V)\, dt = 0 \tag{4.5}$$

The commutation of integration (with respect to x and t) and variation calculation is used within Hamilton's principle (4.5). The equation of motion, after taking into account the commutation of variation and differentiation operators and after integrating kinetic and potential energies of the system, is obtained in the form

$$EJ\frac{\partial^4 W(x,t)}{\partial x^4} + P\frac{\partial^2 W(x,t)}{\partial x^2} + \overline{\rho}A\frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{4.6}$$

and after giving consideration to conditions (4.1), the following boundary condition at the free end of the system is imposed

$$\frac{\partial^3 W(x,t)}{\partial x^3}\Big|_{x=l} - \frac{1}{R-l_0}\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=l} - \frac{m}{EJ}\frac{\partial^2 W(x,t)}{\partial t^2}\Big|_{x=l} = 0 \tag{4.7}$$

The conditions for the fastening are as follows

$$W(0,t) = 0 \qquad EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|_{x=0} - C_1\frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{4.8}$$

Condition (4.7) is independent of the follower coefficients η_1 and η_2.

The loading of columns, for which the boundary conditions at the free end ($x = l$) are defined by relations $(4.1)_1$ and (4.7), depending on a constructional variant of the loading head (sign of the radius of curvature R (Fig. 4), assume the name of:

- follower force directed towards the positive pole ($R > 0$ – Fig. 4a,b)
- follower force directed towards the negative pole ($R < 0$ – Fig. 4c)

independently of the length of the rigid element l_0 of the head receiving the load.

5. Solution to the boundary problem

The equations of motion for the considered column, with the function of transverse vibration $W_i(x,t)$ predicted in the form

$$W_i(x,t) = y_i(x)\cos(\omega t) \qquad i = 1,2 \tag{5.1}$$

are as follows

$$(EJ)_i y_i^{IV}(x) + S_i y_i''(x) - (\overline{\rho}A)_i\omega^2 y_i(x) = 0 \qquad \sum_{i=1}^{2} S_i = P \tag{5.2}$$

where a symmetrical distribution of the bending rigidity and mass per unit length is assumed.

The boundary conditions at the fixed and free end of the column, with regard to relationships $(4.1)_1$ and (4.7), take the following form

$$
\begin{aligned}
& y_1(0) = y_2(0) = 0 && y_1'(0) = y_2'(0) \\
& y_1(l) = y_2(l) && y_1'(l) = y_2'(l) \\
& y_1''(0) + y_2''(0) - c_1^* y_1'(0) = 0 && y_1(l) = (R - l_0) y_1'(l) \\
& y_1'''(l) + y_2'''(l) - \frac{1}{R - l_0}[y_1''(l) + y_2''(l)] + \frac{m\omega^2}{(EJ)_1} y_1(l) = 0
\end{aligned}
\tag{5.3}
$$

where

$$
c_1^* = \frac{C_1}{(EJ)_1}
$$

A general solution to Eqs. (5.2) is

$$
y_i(x) = C_{1i}\cosh(\alpha_i x) + C_{2i}\sinh(\alpha_i x) + C_{3i}\cos(\beta_i x) + C_{4i}\sin(\beta_i x) \tag{5.4}
$$

where C_{ji} are integration constants $(j = 1, 2, 3, 4)$, and

$$
\alpha_i^2 = -\frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^2 + \Omega_i^{*2}} \qquad\qquad \beta_i^2 = \frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^2 + \Omega_i^{*2}}
$$

while

$$
\Omega_i^{*2} = \frac{(\overline{\rho}A)_i\omega^2}{(EJ)_i} \qquad\qquad k_i = \sqrt{\frac{S_i}{(EJ)_i}}
$$

Substitution of solutions (5.4) into boundary conditions (5.3) yields a transcendental equation for eigenvalues of the considered system.

6. Experimental stand

The experimental stand for the examination of free vibrations of the considered columns is shown in Fig. 6. It consists of head (1) which can be horizontally shifted along guides (2). The load is applied to the tested column by means of screw systems belonging to the head. The loading force is measured by dynamometer (3). Column (5) is clamped to supports 4(1) and 4(2). Support 4(1) enables fixing of loading head (6), see Fig. 4. Tests of normal frequencies were performed with the use of a two-channel vibration analyser made by Brüel and Kjaer (Denmark).

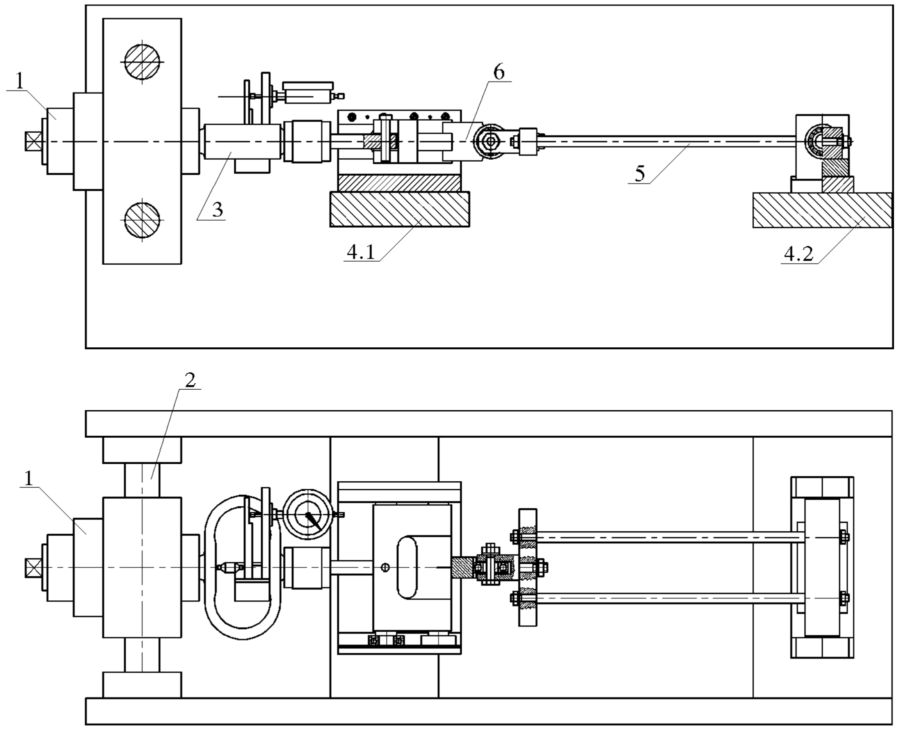

Fig. 6. The test rig for experimental research on the considered column

7. Numerical and experimental results

For the considered column numerical computations were accomplished on the basis of the solution to the boundary value problem. Then, the course of natural frequencies in relation to the external loads (for systems whose physical and geometrical parameters are given in Table 2) was experimentally verified on the stand (Fig. 6).

The parameters of loading and receiving heads are also included in Table 2. Systems $K2$, $K7$, $K8$ correspond with the head variant presented in Fig. 4c, systems $K5$, $K6$ – Fig. 4a, system $K1$ – Fig. 4b depending on the head curvature and reciprocal relation between the rigid element l_0 and radius R.

Columns $K3$, $K4$, for which $R = l_0$, are the specific variant. The free end of the system is at the non-deformed axis of the column for such a relation.

Table 2. Geometrical and physical parameters of the considered columns

Columns	EJ [Nm^2]	$\overline{\rho}A$ [kg/m]	l [m]	R [m]	l_0 [m]	m [kg]
$K1$	152.68	0.631	0.7	0.0285	0.091	0.25
$K2$	152.68	0.631	0.7	−0.0285	0.091	0.4
$K3, K4$	152.68	0.631	0.71	0.058	0.058	0.335
$K5, K6$	152.68	0.631	0.71	0.058	0.025	0.25
$K7, K8$	152.68	0.631	0.71	−0.058	0.025	0.35

The boundary conditions for $x = l$ can be stated as follows

$$y_1(l) = y_2(l) = 0 \qquad y_1'(l) = y_2'(l) \qquad y_1''(l) + y_2''(l) = 0 \tag{7.1}$$

The results, obtained from experiments (points) and numerical computations (lines), are presented in Fig. 7 - Fig. 11, while columns $K4$, $K6$, $K8$ are characterised by $c_1^* = 0$ joint attachment for $x = 0$. The rigid attachment $(1/c_1^* = 0)$ was applied in the remaining cases. The results are limited to the first three basic natural frequencies ($M1$, $M2$, $M3$) and two additional frequencies ($M2^e$, $M3^e$) characterised by symmetry of vibrations (compare Tomski *et al.*, 1998). Both numerical and experimental results are in good agreement.

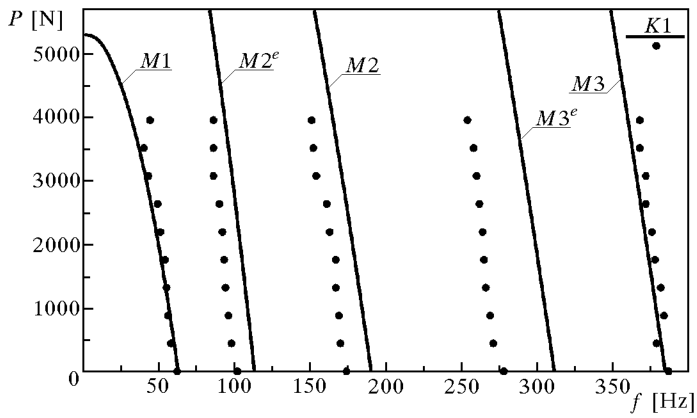

Fig. 7. Frequency curves in the load-natural frequency fplane for column $K1$

Additional results, connected with changes in the critical load and natural frequency, were obtained by taking into account the correctness of the assumed mathematical model describing the variations. The rigid attachment of the system $1/c_1^* = 0$ was also taken into consideration.

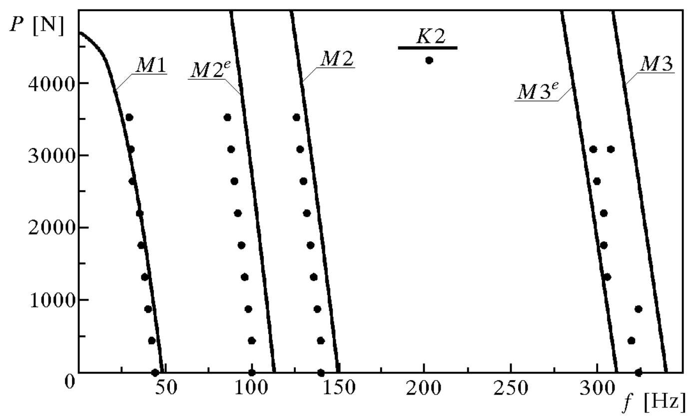

Fig. 8. Frequency curves in the load-natural frequency fplane for column $K2$

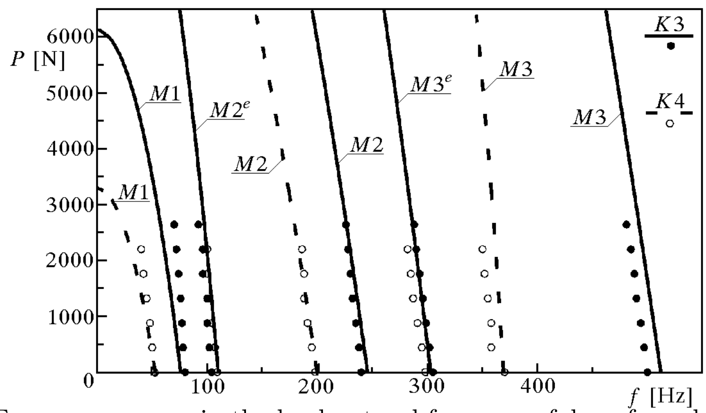

Fig. 9. Frequency curves in the load-natural frequency fplane for columns $K3$ and $K4$

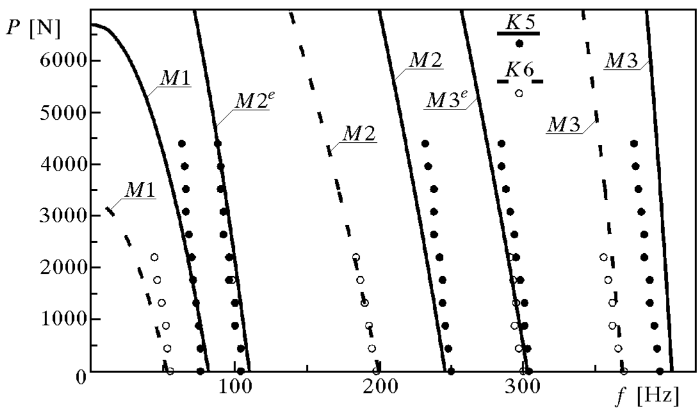

Fig. 10. Frequency curves in the load-natural frequency fplane for columns $K5$ and $K6$

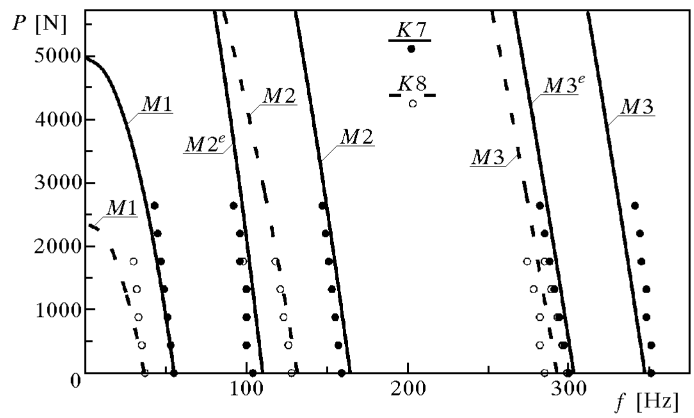

Fig. 11. Frequency curves in the load-natural frequency fplane for columns $K7$ and $K8$

The change in the critical load parameter is presented in Fig. 12 in the full range of the radius R of the loading head for three lengths A, B, C of the rigid element carrying the load. The value of the critical force P_c is related to the overall length of the system

$$\lambda_c = \frac{P_c l_1^2}{EJ} \tag{7.2}$$

while $R^* = R/l_1$, $l_0^* = l_0/l_1$, $l_1 = l_0 + l =$ const and l_0 – length of the rigid element, l – length of the column.

The curves A, B, C represent the value of the critical load parameter from point $A_{-\infty}$, $B_{-\infty}$, $C_{-\infty}$ to point A_0, B_0, C_0 for the column loaded by the follower force towards the negative pole $R^* - l_0^* < 0$ and positive pole $R^* > 0$ with $R^* - l_0^* < 0$.

In the remaining range of the radius R^* $((A_0, A_\infty)$, (B_0, B_∞), $(C_0, C_\infty))$ the system is loaded by the follower force towards the positive pole when $R^* - l_0^* > 0$. For the considered values of the radius R^* of the loading head, changes of every curve of the critical load are characterised by occurrence of the maximum value of the critical load parameter λ_{max} (Fig. 13).

The extreme value for every R^* and l_0^* fulfils the dependence

$$\frac{R^* - l_0^*}{1 - l_0^*} = \frac{1}{2} \tag{7.3}$$

The value of the critical load parameter corresponding with $R^* \to \pm\infty$ is specified by lines 1, 2, 3 (Fig. 12). The points A_0, B_0, C_0 describe the value

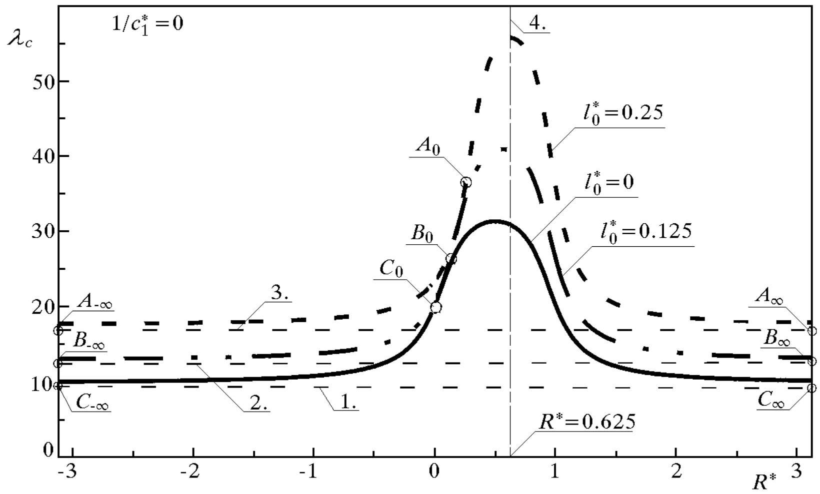

Fig. 12. The change of the critical load parameter λ_c as a function of radius R^* of the loading head

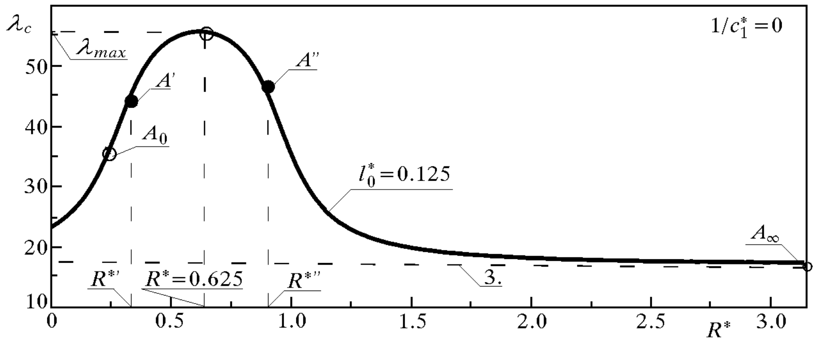

Fig. 13. The change of the critical load parameter λ_c in relation to positive values of the radius R^* of the loading head

of the critical force for the column with the joint attachment at the free ends (compare $K3$, $K4$ – Table 2).

The range $R^* \in (R^{*'}, R^{*''})$ of positive values of the radius R^* of the loading head exists for every length of the rigid element l_0^*, see Fig. 13. The considered column is of divergence-pseudo-flutter type $(A'$, A'' point) in the above range. The considered system is of divergence type for the remaining positive and negative values of R^*.

The numbering of the considered column as one of the two types of systems is associated with the course of the natural frequency in relation to the external load, which is shown in Fig. 14 and Fig. 15.

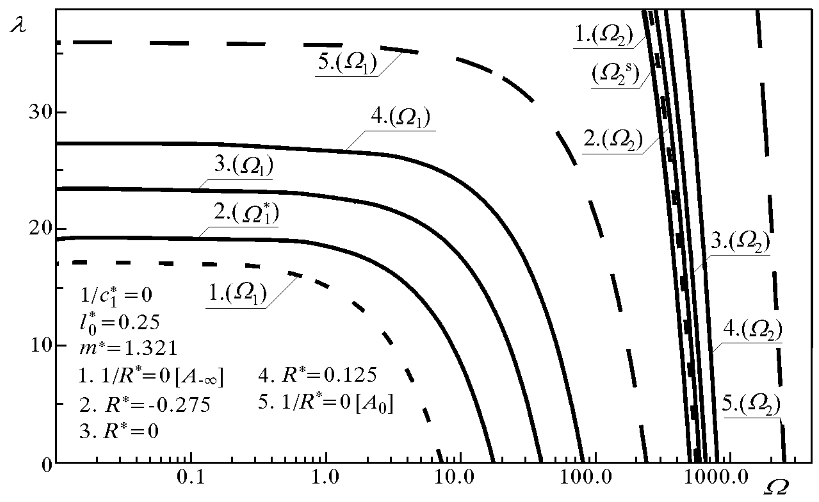

Fig. 14. Frequency curves in the load-natural frequency plane for $R^* - l_0^* \leqslant 0$

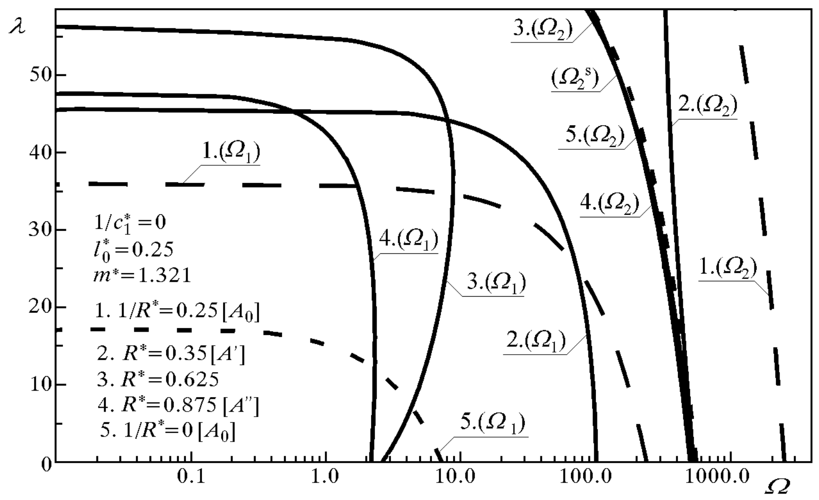

Fig. 15. Frequency curves in the load-natural frequency plane for $R^* - l_0^* \geqslant 0$

The presentation was limited to the first two basic natural frequencies in a dimensionless form Ω_i and to an additional symmetric natural frequency Ω_2^s in relation to the dimensionless loading parameter λ, while

$$\lambda = \frac{Pl_1^2}{EJ} \qquad \Omega_i = \frac{\overline{\rho}A\omega^2 l_1^4}{EJ} \qquad m^* = \frac{m}{\overline{\rho}Al_1} \tag{7.4}$$

The slope of the eigenvalue curves (Fig. 14) is always negative for the column loaded by a follower force directed towards the negative pole. That slope can be positive (curve 3 in Fig. 15), negative or zero (curves 2, 4) for the system loaded by a follower force directed towards the positive pole.

The discussed curves were sketched for a constant length l_0^* of the rigid element of the loading head and concentrated mass at the free end of the column m^*. The value of the critical load $\Omega = 0$ stays in accordance with curve A in Fig. 12.

The change of natural frequencies in relation to the dimensionless loading parameter for a constant radius R^* of the loading head (line 4 in Fig. 12) is shown in Fig. 16.

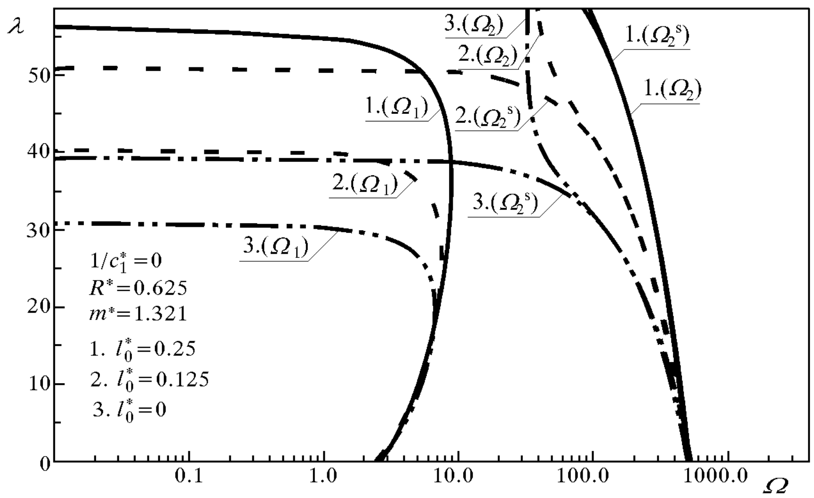

Fig. 16. Frequency curves in the load-natural frequencies plane for $R^* = \text{const}$

The considered column is of divergence-pseudo-flutter type independently of the length l_0^* of the loading element for the chosen geometrical and physical parameters of the system.

8. Conclusions

On the basis of experiments and carried out numerical simulations for the presented variant of the specific load by the follower force towards the negative or positive pole, one can state that:

- correct boundary conditions for the considered structure can be determined on the basis of the minimum of potential energy (static problem) or on the basis of the minimum of mechanical energy (Hamilton's principle)
- the considered system can be of divergence or divergence-pseudo-flutter type with regard to the design of loading and receiving heads, (value of the radius R and length l_0 of the rigid element of the head receiving the load)
- there are such values of geometrical parameters of the loading and receiving heads for which the maximum of the critical load is obtained
- the system is conservative according to the extended principle of potential, which was described by Tomski *et al.* (1996) and resulted from the self-adjointness of differential operators, while in this paper the conservative system is determined from the relationship between the potential energy and work of the potential system.

Acknowledgements

This research has been supported by the State Committee for Scientific Research, Warsaw, Poland, under grant No. 7T07C03218.

References

1. Beck M., 1953, Die kniclast des einseitig eingespannten tangential gedruckten Stabes, *Zeitschrift für angewangte Mathematik und Physik*, **4**, 225-228, 476-477 (errata)
2. Bogacz R., Imiełowski S., Tomski L., 1998, Optymalization and stability of columns on example of conservative and nonconservative systems, *Machine Dynamics Problems*, **20**, 35-47
3. Dąbrowski R., 1984, Stability of a freestanding column loaded through a roller bearing, *Ingenieur-Archiv*, **54**, 16-24
4. Euler L., 1774, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. Appendix: De curvis elasticis, Lausanne and Geneva, from *Opera omnia*, **1**
5. Gajewski A., Życzkowski M., 1969, Optimal shaping of a rod compressed by a force directed towards the pole (in Polish), *Rozprawy Inżynierskie*, **17**, 2, 299-329

6. Gajewski A., Życzkowski M., 1970, Optimal design of elastic columns subject to the general conservative behavior of loading, *Zeitschrift für angewangte Mathematik und Physik*, **21**, 806-818

7. Gajewski A., Życzkowski M., 1988, *Optimal Structural Design under Stability Constraints*, Dordrecht: Kluwer Academic Publishers, 103-107

8. Goldstein H., 1950, *Classical Mechanics*, Addison-Wesley Publishing Company, Inc., Cambridge, Mass, 38-40

9. Kordas Z., 1963, Stability of the elastically clamped compressed bar in the general case of behaviour of the loading, *Bulletin de L'Academie Polonaise des Sciences*, **XI**, 419-427

10. Leipholz H.H.E., 1974, On conservative elastic systems of the first and second kind, *Ingenieur-Archiv*, **43**, 255-271

11. Levinson M., 1966, Application of the Galerkin and Ritz Methods to nonconservative problems of elastic stability, *Zeitschrift für angewangte Mathematik und Physik*, **21**, 806-818, **17**, 431-442

12. Timoshenko S.P., Gere J.M., 1961, *Theory of Elastic Stability*, New York: McGraw-Hill

13. Tomski L., Gołębiowska-Rozanow M., Szmidla J., 1994, Free vibration of a column loaded by a force and bending moment, *Proceedings of the XVIth Symposium "Vibrations in Physical Systems"*, Poznań, 317-318

14. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1995, Stability and vibration of a two-member frame under generalized load, *Proceedings of European Session of International Colloquium "Stability of Steel Structures"*, Budapest, I/409-I/416

15. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1996, Vibration and stability of an elastic column subject to a generalized load, *Archive of Applied Mechanics*, **67**, 105-116

16. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1998, Vibration and stability of a cantilever column subject to a follower force passing through a fixed point, *Journal of Sound and Vibration*, **214**, 1, 67-81

17. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1999, Vibration and stability of columns to a certian type of generalised load, *Journal of Theoretical and Applied Mechanics*, **37**, 2, 283-289

18. Wallerstein D.V., 2002, *A Variational Approach to Structural Analysis*, (chapter 5), A Wiley-Intersciense Publication, John Wiley and Sons INC.

Drgania i stateczność dwuprętowej kolumny obciążonej poprzez wycinek łożyska tocznego

Streszczenie

W pracy prezentuje się nowe obciążenie układów smukłych, które jest obciążeniem siłą śledzącą skierowaną do bieguna dodatniego lub ujemnego. Przedstawia się rozwiązania konstrukcyjne głowic obciążonych, które realizują to obciążenie. Prezentuje się rozważania teoretyczne dotyczące sformułowania warunków brzegowych na podstawie całkowitej energii układu. W zależności od rozwiązania konstrukcyjnego głowicy obciążającej i przejmującej obciążenie określa się wartość siły krytycznej oraz przebieg częstości drgań własnych w funkcji obciążenia zewnętrznego dla zadanej geometrii i stałych fizycznych kolumny. Wyniki badań teoretycznych porównuje się z wynikami badań eksperymentalnych.

Manuscript received March 23, 2004; accepted for print April 24, 2004