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APPLICATION OF THE MODIFIED METHOD OF FINITE ELEMENTS TO IDENTIFICATION OF TEMPERATURE OF A BODY HEATED WITH A MOVING HEAT SOURCE

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A transient 2-dimensional problem of heat flow with a moving heat source is considered. It is assumed that the heat source is moving along one of the sides of a rectangle with a constant velocity. The unknown is the temperature distribution in a flat rectangular area. The paper presents an approximate solution which is based on the finite element method with a modified basis function. The space-time basis functions are combinations of heat polynomials which strictly satisfy the heat equation. Cartesian coordinates were used to solve the problem.

Key words: moving heat source, 2D heat conduction, FEM, heat polynomials

1. Introduction

Among approximate methods of solving problems related to heat conduction specified in references Ciałkowski and Magnucki (1982), Gdula (1984), Szargut (1992), the noteworthy ones include:

- finite element method
- boundary element method
- method of elementary balances
- finite difference method.

Paper by Ciałkowski and Magnucki (1982) presented a mathematical basis of finite element methods, the examples provided referred to FEM applications to theory of elasticity. Papers by Gdula (1984), Szargut (1992) outlined approximation methods. The authors also showed how those methods could be used to solve stationary and non-stationary issues in heat transfer.

All these approximate methods belong to a group of numerical methods. They have been widely applied due to rapid development of computers. Unlike analytical methods, they can simply allow for a complex shape of a body, nonlinearity of boundary conditions, dependence of material coefficients on temperature. They usually result in simple relations which do not require the use of complicated mathematical calculations to be solved. They are rather solved with the use of elementary calculation methods. A method that combines characteristics of analytical and numerical methods is the so called method of heat polynomials. It makes it possible to achieve a solution that strictly satisfies a differential equation, and approximately satisfies given initial and boundary conditions. It is suitable for solving stationary and nonstationary problems, while taking into consideration any size.

The problem of the identification of the temperature field generated by a moving heat source has been investigated in numerous papers. Particular attention should be paid to work by Rożnowski (1988) as it dealt exclusively with moving heat sources. This paper aimed at the determination of the temperature field and thermal stress tensor in a cylinder and half-plane under dynamic loads of movable heating sources. The investigations focused on nonstationary issues of heat conduction and thermal stresses in the cylinder and half-plane resulting from a discontinuous temperature field moving on the external surface. Apart from the subject matter, it also provided an extensive overview of the literature on the problems under investigation.

2. Formulation of the problem

The aim of the present paper is to provide an approximate solution to a transient two-dimensional problem of heat flow in the case when a part of the boundary of the area under analysis is subject to heating with a moving heat source. The approximate solution is based on the method of finite elements, taking advantage of heat polynomials as base functions. Due to the fact that heat is assumed to spread bidirectionally, the temperature distribution is sought in a flat rectangular area with the length l and the width b. It is assumed that the heat source with the length a is moving along one of the sides of the rectangle with a constant velocity v in a periodic manner. A boundary condition of the second kind is adopted for calculations. This condition may be expressed in the form: $-\lambda\partial T/\partial n = q_n$, where λ denotes thermal conductivity coefficient [W/mK], $\partial T/\partial n$ – a derivative in the direction perpendicular to the rectangle side, $-q_n = \overline{q}_n f(x,t)$ – normal component of the heat flux density $[W/m^2]$, $\overline{q}_n$ its extreme value ($\overline{q}_n > 0$), $f(x,t)$ – polynomial function ($f(x,t) > 0$), Fig. 1.

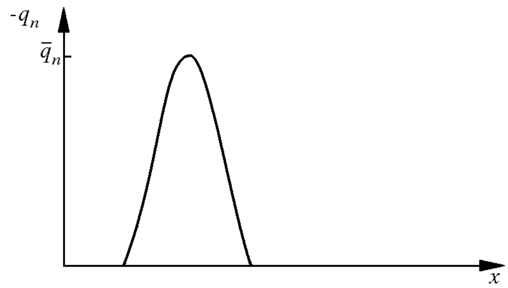

Fig. 1. Distribution of the heat flux for $y = b$ at a fixed time instant

Apart from the contact of the source with the side of the rectangle (also at the opposite side), it is assumed that thermal insulation is present. At the remaining boundaries of the rectangle, the boundary condition of the fourth kind is adopted in order to realise a process of repeated heating of the body with the moving source (heating of a rail by a passing train, heating of a brake drum at breaking, heating of a body during a grinding process).

It is assumed that the body is made of a homogenous and isotropic material, and the thermal conductivity coefficient λ and thermal diffusivity κ of the body do not depend on temperature. It is assumed that, at the initial moment, the temperature of the body and environment is constant and equal to Θ_0. The problem under analysis is formulated mathematically in a dimensionless form. Dimensionless (reduced) temperature is defined as follows

$$T = \frac{\Theta - \Theta_0}{\frac{\overline{q}_n d}{\lambda}}$$

where Θ denotes absolute temperature [K], Θ_0 – absolute temperature at the initial moment [K], $\overline{q}_n$ – extreme value of density of the heat flux originated as a result of action of the moving heat source $[W/m^2]$, d – linear characteristic dimension [m], λ – thermal conductivity coefficient [W/mK]. Because the main direction of thermal conductivity is the direction perpendicular to the acting surface of the source, so the linear characteristic dimension is the height of the body $\overline{b}$ [m]. The dimensionless coordinates are expressed in the following manner: $x = \overline{x}/d$, $y = \overline{y}/d$, $t = \kappa\overline{t}/d^2$ where κ is the thermal diffusivity coefficient (temperature balance coefficient) $[m^2/s]$. We define dimensionless parameters: $b = \overline{b}/d$, $l = \overline{l}/d$, $a = \overline{a}/d$, $v = \overline{v}d/\kappa$ ($\overline{a}$ is width of the source [m], $\overline{b}$ – height of the body, $\overline{l}$ – length of the body [m], $\overline{v}$ – actual velocity of the source [m/s]).

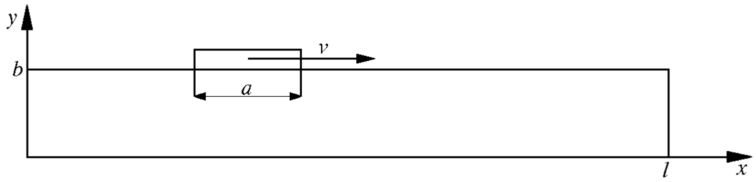

Fig. 2. Model for identification of temperature in a process of heating with a moving heat source

The following dimensionless form of the problem under analysis is achieved

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{\partial T}{\partial t} = 0 \tag{2.1}$$

for $(x,y) \in \Omega$, $t > 0$, $\Omega = \{(x,y) \in R^2 :\ 0 < x < l,\ \ 0 < y < b\}$.

The initial and boundary conditions are

$$\begin{gathered} T(x,y,0) = 0 \qquad\qquad \frac{\partial T}{\partial y}(x,0,t) = 0 \\ \frac{\partial T}{\partial y}(x,b,t) = f(x,t) = \begin{cases} \Big(\dfrac{2}{a}\Big)^4 A^2(-a+A)^2 & \text{for} \quad A \leqslant a \\ 0 & \text{for} \quad a \leqslant A < l \end{cases} \end{gathered} \tag{2.2}$$

where $A = (x - vt) mod.l$ gives the remainder of the division of $(x - vt)$ by l.

Additionally, the consistency conditions are required in the following form

$$T(0,y,t) = T(l,y,t) \qquad\qquad \frac{\partial T}{\partial x}(0,y,t) = \frac{\partial T}{\partial x}(l,y,t) \tag{2.3}$$

Equations (2.1)-(2.3) have the following analytical solution (Maciejewska, 2004)

$$\begin{aligned} T(x,y,t) &= \frac{q_0 t}{bl} + \frac{2}{bl}\sum_{k=1}^{\infty} \frac{(-1)^k q_0 \cos\alpha_k y}{\alpha_k^2}(1 - \mathrm{e}^{-\alpha_k^2 t}) + \\ &+ \frac{2}{bl}\sum_{n=1}^{\infty} \frac{q_{n1}\cos\gamma_{n0}}{\lambda_n^2}[\cos(\lambda_n(x - vt) + \gamma_{n0}) - \mathrm{e}^{-\lambda_n^2 t}\cos(\lambda_n x + \gamma_{n0})] \\ &+ \frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^k q_{n1}\cos\gamma_{nk}\cos\alpha_k y}{\lambda_n^2 + \alpha_k^2} \cdot \\ &\cdot [\cos(\lambda_n(x - vt) + \gamma_{nk}) - \mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}\cos(\lambda_n x + \gamma_{nk})] \end{aligned} \tag{2.4}$$

where $n = 1, 2, ..., k = 0, 1, 2, ...$ and

$$\cos\gamma_{nk} = \frac{\lambda_n^2 + \alpha_k^2}{\sqrt{(v\lambda_n)^2 + (\lambda_n^2 + \alpha_k^2)^2}}$$

$$\lambda_n = \frac{2\pi n}{l} \qquad \alpha_k = \frac{k\pi}{b} \qquad q_0 = \frac{8a}{15}$$

$$q_{n1} = -\frac{8l^3\cos(n\pi)[3aln\pi\cos\frac{an\pi}{l} + ((a\pi n)^2 - 3l^2)\sin\frac{an\pi}{l}]}{a^4(n\pi)^5}$$

3. Approximation of the solution to the thermal conductivity equation on the entire area $\Omega\times <0, t_k>$

The approximate solution of system equations (2.1)-(2.3) has the form

$$T(x,y,t) \approx \sum_{n=1}^{N} A_n v_n(\widehat{x},\widehat{y},\widehat{t}) = \widehat{T}(x,y,t) \tag{3.1}$$

where $(x,y) \in \Omega$, $t \in <0, t_k>$, t_k denotes the length of the time interval, $\widehat{x} = x - x_0$, $\widehat{y} = y - y_0$, $\widehat{t} = t - t_0$ and $v_n(\widehat{x},\widehat{y},\widehat{t})$ denote heat polynomials. Coefficients A_n are sought from the minimization of functional (3.2) of fitting the approximate solution to the given initial and boundary conditions (Ciałkowski and Frąckowiak, 2000; Hożejowski, 1999)

$$\begin{aligned} I = & \int\limits_0^b\int\limits_0^l [\widehat{T}(x,y,0)]^2\,dxdy + \int\limits_0^b\int\limits_0^{t_k}\Big[\frac{\partial\widehat{T}}{\partial x}(0,y,t) - \frac{\partial\widehat{T}}{\partial x}(l,y,t)\Big]^2\,dtdy + \\ & + \int\limits_0^b\int\limits_0^{t_k}[\widehat{T}(0,y,t) - \widehat{T}(l,y,t)]^2\,dtdy + \int\limits_0^l\int\limits_0^{t_k}\Big[\frac{\partial\widehat{T}}{\partial y}(x,0,t)\Big]^2\,dtdx + \\ & + \int\limits_0^l\int\limits_0^{t_k}\Big[\frac{\partial\widehat{T}}{\partial y}(x,b,t) - f(x,t)\Big]^2\,dtdx \end{aligned} \tag{3.2}$$

With $N \to \infty$, the solution expressed by formula (3.1) tends to the exact solution. The application of the method of heat polynomials to numerical calculations requires a finite number of polynomials. A satisfactory approximation of a particular problem can be achieved in two ways

1. By finding global solutions (on $\Omega\times <0,t_k>$) with the use of higher degree polynomials

2. By finding local solutions (on parts of $\Omega\times <0,t_k>$) with the use of lower degree polynomials.

The first approach is presented in the fourth section of the paper and the second one in its fifth section.

4. Numerical example

Let us consider a thermal field in a body made of carbon steel. Characteristics of the body are: $\lambda = 45$ W/mK, $\kappa = 1.19\cdot 10^{-5}$ m^2/s, $\overline{l} = 1$ m (length of the body), $\overline{b} = 0.1$ m (height of the body). We take $\overline{v} = 0.006$ m/s, $\Theta_0 = 0$ K, $\overline{q}_n = 13\cdot 10^3$ W/m^2, $\overline{a} = 0.3$ m. 22 heat polynomials will be used for approximation.

The polynomials have the following form (Ciałkowski and Frąckowiak, 2003): 1, x, y, xy, $\frac{x^2}{2}-\frac{y^2}{2}$, $\frac{x^3}{6}-\frac{xy^2}{2}$, $\frac{x^2y}{2}-\frac{y^3}{6}$, $t+\frac{y^2}{2}$, $tx+\frac{xy^2}{2}$, $ty+\frac{y^3}{6}$, $txy+\frac{xy^3}{6}$, $\frac{x^2y^2}{4}-\frac{y^4}{12}+t\Big(\frac{x^2}{2}-\frac{y^2}{2}\Big)$, $\frac{x^3y^2}{12}-\frac{xy^4}{12}+t\Big(\frac{x^3}{6}-\frac{xy^2}{2}\Big)$, $\frac{t^2}{2}+\frac{ty^2}{2}+\frac{y^4}{24}$, $x\Big(\frac{t^2}{2}+\frac{ty^2}{2}+\frac{y^4}{24}\Big)$, $y\Big(\frac{t^2}{2}+\frac{ty^2}{6}+\frac{y^4}{120}\Big)$, $xy\Big(\frac{t^2}{2}+\frac{ty^2}{6}+\frac{y^4}{120}\Big)$, $\frac{x^2y^4}{48}-\frac{y^6}{240}+\frac{t^2}{2}\Big(\frac{x^2}{2}-\frac{y^2}{2}\Big)+t\Big(\frac{x^2y^2}{4}-\frac{y^4}{12}\Big)$, $\frac{t^3}{6}+\frac{t^2y^2}{4}+\frac{ty^4}{24}+\frac{y^6}{720}$, $x\Big(\frac{t^3}{6}+\frac{t^2y^2}{4}+\frac{ty^4}{24}+\frac{y^6}{720}\Big)$, $y\Big(\frac{t^3}{6}+\frac{t^2y^2}{6}+\frac{ty^4}{120}+\frac{y^6}{5040}\Big)$, $xy\Big(\frac{t^3}{6}+\frac{t^2y^2}{6}+\frac{ty^4}{120}+\frac{y^6}{5040}\Big)$.

In Fig. 3 and Fig. 4, one can see a temperature graph for the instant $\overline{t} = 25.2$ s. Fig. 3 show the approximate temperature distribution established on the basis of formula (3.1), whereas Fig. 4 present the precise temperature distribution determined with formula (2.4). Pairs of Fig. 3 and Fig. 4 are only different presentations of the temperature graph at the instant $\overline{t} = 25.2$ s.

When comparing the temperature diagrams, one can see that the approximate solution is far from the exact one. An apparent effect of inertia caused by the moving heat source active within the system is difficult to model. The given number of polynomials is not sufficient for a satisfactory approximation of the solution, and the use of polynomials of higher degrees results in an adverse numerical conditions of the problem under analysis.

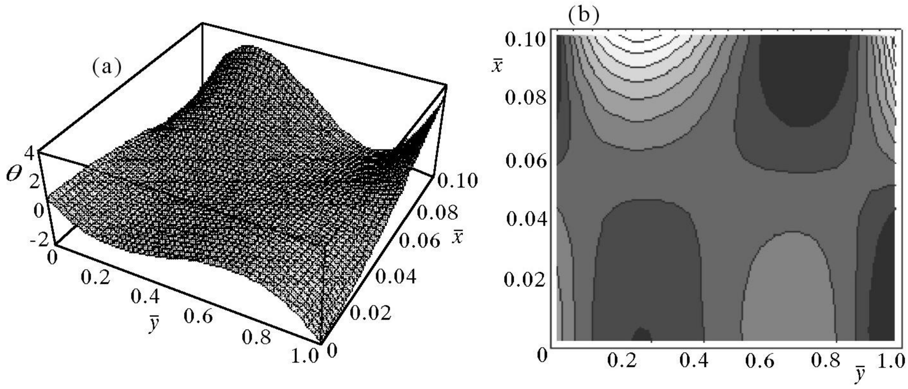

Fig. 3. (a) An approximate temperature distribution achieved by combination of 22 heat polynomials; (b) a contour diagram

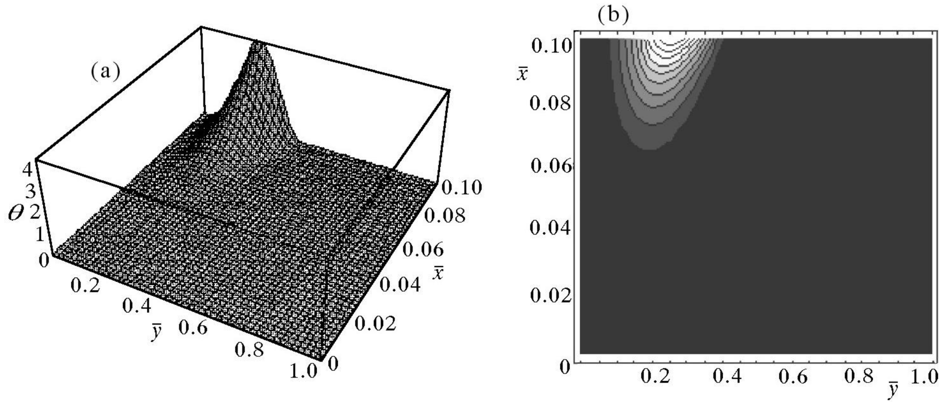

Fig. 4. (a) The exact temperature distribution; (b) a contour diagram

5. Approximation of the solution to the heat equation within an area divided into finite elements

When the number of polynomials is increasing, equation (3.1) better approximates the temperature distribution within a particular area, however, a greater number of polynomials causes numerical problems. Instead of approximating the solution within the entire area $\Omega\times <0, t_k>$ with a great number of polynomials, the area can be divided into smaller subareas and the solution can be approximated in each of them with a combination of a considerably smaller number of polynomials. The area Ω is divided into rectangles Ω_j for $j = 1, 2, ..., J$. The interval $<0, t_k>$ is divided into subintervals

$< r\Delta t, (r+1)\Delta t >$ for $r = 0, 1, 2, ..., R$. The approximate solution to equation (2.1) on $\overline{\Omega}_j = \Omega_j \times < t_0, t_0 + \Delta t >$, $t_0 = r\Delta t$ for fixed r has the form

$$\widehat{T}_j(x, y, t) = \sum_{n=1}^{N} A_{jn} v_n(\widehat{x}, \widehat{y}, \widehat{t}) \tag{5.1}$$

where $\widehat{x} = x - x_{0j}$, $\widehat{y} = y - y_{0j}$, $\widehat{t} = t - t_{0j}$, (x_{0j}, y_{0j}, t_{0j}) is a fixed point belonging to $\overline{\Omega}_j$ (most often the centre of the element $\overline{\Omega}_j$), $v_n(\widehat{x}, \widehat{y}, \widehat{t})$ denotes the nth heat polynomial. The manner of solving equation (2.1) is a generalization of the method presented in the paper by Ciałkowski and Frąckowiak (2000) and amounts to sequential solving of the equation in subsequent time intervals. We establish the number of appropriately situated nodes in the element $\overline{\Omega}_j$. Three coordinates $(\widehat{x}, \widehat{y}, \widehat{t})$ are assigned to each node. In order to determine constants A_{jn} in equation (5.1) on the assumption that temperatures in the nodes of $\overline{\Omega}_j$ are known, the system of equations is solved

$$\widehat{T}_j(x_k, y_k, t_k) = T_{jk} = \sum_{n=1}^{N} A_{jn} v_n(\widehat{x}, \widehat{y}, \widehat{t}) \qquad k = 1, 2, ..., N \tag{5.2}$$

The matrix form of that system has the form

$$\begin{bmatrix} v_1(\widehat{x}_1, \widehat{y}_1, \widehat{t}_1) & v_2(\widehat{x}_1, \widehat{y}_1, \widehat{t}_1) & \dots & v_N(\widehat{x}_1, \widehat{y}_1, \widehat{t}_1) \\ v_1(\widehat{x}_2, \widehat{y}_2, \widehat{t}_2) & v_2(\widehat{x}_2, \widehat{y}_2, \widehat{t}_2) & \dots & v_N(\widehat{x}_2, \widehat{y}_2, \widehat{t}_2) \\ \vdots & \vdots & \vdots & \vdots \\ v_1(\widehat{x}_N, \widehat{y}_N, \widehat{t}_N) & v_2(\widehat{x}_N, \widehat{y}_N, \widehat{t}_N) & \dots & v_N(\widehat{x}_N, \widehat{y}_N, \widehat{t}_N) \end{bmatrix} \begin{bmatrix} A_{j1} \\ A_{j2} \\ \vdots \\ A_{jN} \end{bmatrix} = \begin{bmatrix} T_{j1} \\ T_{j2} \\ \vdots \\ T_{jN} \end{bmatrix} \tag{5.3}$$

An abbreviated form of the system (5.2) is

$$\mathbf{vA} = \mathbf{T} \tag{5.4}$$

Thus, inversion of the matrix $\mathbf{v}$ results in

$$\mathbf{A} = \mathbf{v}^{-1}\mathbf{T} = \mathbf{VT} \qquad A_{jn} = \sum_{k=1}^{N} V_{nk} T_{jk} \tag{5.5}$$

Substitution of (5.5) to (5.1) yields base functions φ_{jk} characteristic of the element $\overline{\Omega}_j$ in the following manner

$$\begin{aligned} \widehat{T}_j(x, y, t) &= \sum_{n=1}^{N} \Bigl(\sum_{k=1}^{N} V_{nk} T_{jk} \Bigr) v_n(\widehat{x}, \widehat{y}, \widehat{t}) = \\ &= \sum_{k=1}^{N} \Bigl(\sum_{n=1}^{N} V_{nk} v_n(\widehat{x}, \widehat{y}, \widehat{t}) \Bigr) T_{jk} = \sum_{k=1}^{N} \varphi_{jk}(x, y, t) T_{jk} \end{aligned} \tag{5.6}$$

thus

$$\varphi_{jk} = \sum_{n=1}^{N} V_{nk} v_n(\widehat{x}, \widehat{y}, \widehat{t}) \tag{5.7}$$

where $\widehat{x} = x - x_{0j}$, $\widehat{y} = y - y_{0j}$, $\widehat{t} = t - t_{0j}$.

The characteristics of the base functions are

1. $\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\Big)\varphi_{jk}(\widehat{x}, \widehat{y}, \widehat{t}) = 0$

2. $\varphi_{jk}(\widehat{x}_m, \widehat{y}_m, \widehat{t}_m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$ $(\widehat{x}_m, \widehat{y}_m, \widehat{t}_m)$ is a node of an element, $k = 1, ..., N$, $m = 1, ..., N$

3. $\sum_{k=1}^{N} \varphi_{jk}(x, y, t) \equiv 1$

In order to achieve an approximate solution to equation (2.1) on $\overline{\Omega} = \Omega \times < t_0, t_0 + \Delta t >$, a square grid parallel to the axis of the system was introduced. The grid originated as a result of dividing the section of the axis OX into $L1$ parts and the section of the axis OY into $L2$ parts. In each cubicoid element $\overline{\Omega}_j$, a system of eight nodes located at vertices of the elements was designated, Fig. 5. Thus, there are $2(L1 + 1)(L2 + 1)$ nodes within the entire area $\overline{\Omega}$. As it results from the form of formulas (5.1), (5.7) and the system of equations (5.3), by introduction of a local coordinate system $\widehat{x}$, $\widehat{y}$, $\widehat{t}$ it is sufficient to find only once the base functions in a local element and then move them in accordance with formulas $\widehat{x} = x - x_{0j}$, $\widehat{y} = y - y_{0j}$, $\widehat{t} = t - t_{0j}$ to obtain the base functions characteristic for each element $\overline{\Omega}_j$. These functions are a combination of the following eight heat polynomials: 1, $\widehat{x}$, $\widehat{y}$, $\widehat{x}\widehat{y}$, $\widehat{t} + \frac{\widehat{y}^2}{2}$, $\widehat{t}\widehat{x} + \frac{\widehat{x}\widehat{y}^2}{2}$, $\widehat{t}\widehat{y} + \frac{\widehat{y}^3}{6}$, $\widehat{t}\widehat{x}\widehat{y} + \frac{\widehat{x}\widehat{y}^3}{6}$.

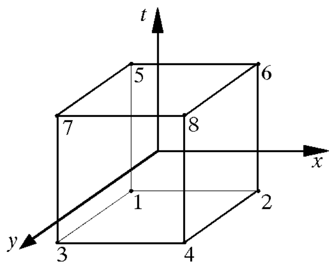

Fig. 5. Location of eight nodes in an element

The temperature in each element is expressed by the relation

$$\widehat{T}_j(x, y, t) = \sum_{k=1}^{8} \varphi_{jk}(x, y, t) T_{jk} \qquad j = 1, 2, ..., L1 \times L2 \tag{5.8}$$

Every node has its global numeration (which comes from numeration of all nodes within $\overline{\Omega}$) and its local numeration (the number of an element and the number of the node in the element is given). It is advantageous to assign in calculations the global numeration to temperatures at the nodes because it guarantees conformity of temperatures at the nodes between elements and reduction of the number of unknowns from $8 \times L1 \times L2$ to $2 \times (L1+1) \times (L2+1)$. An exemplary division of the area $\overline{\Omega}$ into 24 elements $(L1 = 6,\ L2 = 4)$ with eight nodes in corners of each element is presented below. Assigning numbers to nodes from 1 to $2 \times (L1 + 1) \times (L2 + 1)$, like in the example in Fig. 6, and marking the temperature at a node with T^n, where n is the number of the node, the temperature at points of $\overline{\Omega}_{10}$ can be approximated with a relation

$$
\begin{aligned}
\widehat{T}_{10}(x, y, t) = {} & \varphi_{10,1}T^{11} + \varphi_{10,2}T^{12} + \varphi_{10,3}T^{18} + \varphi_{10,4}T^{19} + \varphi_{10,5}T^{46} + \\
& + \varphi_{10,6}T^{47} + \varphi_{10,7}T^{53} + \varphi_{10,8}T^{54}
\end{aligned}
$$

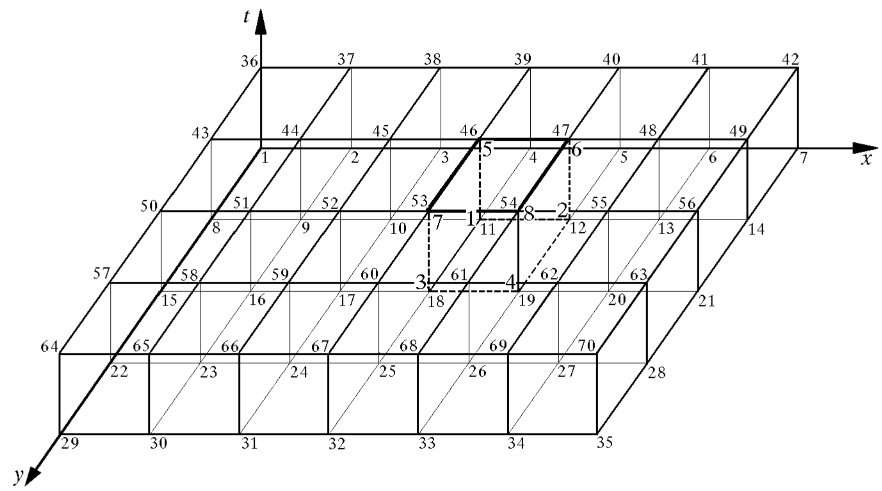

Fig. 6. Exemplary global numeration of nodes in 24 elements with nodes located at corners of these elements

The base functions φ_{jk} (as combinations of heat polynomials) strictly satisfy equation (2.1). Therefore, it is sufficient to fit an approximate solution only for the initial and boundary conditions and minimise defects of heat quantities flowing between the elements. Both, the temperatures at nodes and heat fluxes must be consistent (in the square-mean sense). The boundary-initial problem is solved sequentially in consecutive time intervals $\langle 0, \Delta t\rangle$, $\langle \Delta t, 2\Delta t\rangle$, $\langle 2\Delta t, 3\Delta t\rangle$,... ($\Delta t$ – length of the time subinterval). In the first time interval, the initial condition has the form $(2.2)_1$, while in the rest of time steps the initial condition in the analysed time interval is equal to the temperature at the moment ending the previous time interval. The temperatures at nodes from $n = 1$ to $(L1 + 1)(L2 + 1)$ (with eight-node elements with nodes located in corners of those elements) are known from the initial condition, while the temperatures at the remaining nodes are unknown. They are found by solving a system of linear equations resulting from the minimization of the functional J

$$J = W_P + W_I + W_G + W_{PT} + W_{PS} + W_{SX} + W_{SY} \tag{5.9}$$

Summands of the functional represent the fitting of the approximate solution to: the initial condition, the insulating of the surface $y = 0$, the heating of the surface $y = b$, condition $(2.3)_1$ that indicates periodicity of the temperature, the condition $(2.3)_2$ that indicates periodicity of the heat flux, reduction of the amount of heat flowing between the elements in the direction of the OX axis, reduction of the amount of heat flowing between the elements in the direction of the OY axis

$$\begin{gathered}
W_P = \sum_{i=1}^{L1L2} \iint\limits_{D_1^i} [T_i(x, y, t_0) - T_0]^2 \, dxdy \\
\bigcup_{i=1}^{L1L2} D_1^i = \{(x, y, t_0) \in R^3 : \;\; 0 \leqslant x \leqslant l, \;\; 0 \leqslant y \leqslant b\}
\end{gathered} \tag{5.10}$$

$$\begin{gathered}
W_I = \sum_{i=1}^{L1} \iint\limits_{D_2^i} \Big[\frac{\partial T_{j-1+i}}{\partial y}(x, 0, t)\Big]^2 \, dxdt \\
\bigcup_{i=1}^{L1} D_2^i = \{(x, 0, t) \in R^3 : \;\; 0 \leqslant x \leqslant l, \;\; t_0 \leqslant t \leqslant t_0 + \Delta t\}
\end{gathered} \tag{5.11}$$

$$\begin{gathered}
W_G = \sum_{i=1}^{L1} \iint\limits_{D_3^i} \Big[\frac{\partial T_{k-1+i}}{\partial y}(x, b, t) - f(x, t)\Big]^2 \, dxdt \\
\bigcup_{i=1}^{L1} D_3^i = \{(x, b, t) \in R^3 : \;\; 0 \leqslant x \leqslant l, \;\; t_0 \leqslant t \leqslant t_0 + \Delta t\}
\end{gathered} \tag{5.12}$$

$$W_{PT} = \sum_{i=1}^{L2} \iint\limits_{D_4^i} [T_{p+(i-1)L1}(0,y,t) - T_{p-1+iL1}(l,y,t)]^2 \, dydt$$

$$\bigcup_{i=1}^{L2} D_4^i = \{(0,y,t) \in R^3 : \ \ 0 \leqslant y \leqslant b, \ \ t_0 \leqslant t \leqslant t_0 + \varDelta t\} \tag{5.13}$$

$$W_{PS} = \sum_{i=1}^{L2} \iint\limits_{D_4^i} \Big[\frac{\partial T_{p+(i-1)L1}}{\partial y}(0,y,t) - \frac{\partial T_{p-1+iL1}}{\partial y}(l,y,t)\Big]^2 \, dydt$$

$$\bigcup_{i=1}^{L2} D_4^i = \{(0,y,t) \in R^3 : \ \ 0 \leqslant y \leqslant b, \ \ t_0 \leqslant t \leqslant t_0 + \varDelta t\} \tag{5.14}$$

$$W_{SX} = \sum_{i=1}^{L2} \iint\limits_{D_5^i} \Big[\frac{\partial T_{l_m+(i-1)L1}}{\partial x}(x_m,y,t) - \frac{\partial T_{l_m+1+(i-1)L1}}{\partial x}(x_m,y,t)\Big]^2 \, dydt$$

$$\bigcup_{i=1}^{L2} D_5^i = \{(x_m,y,t) \in R^3 : \ \ 0 \leqslant y \leqslant b, \ \ t_0 \leqslant t \leqslant t_0 + \varDelta t, \quad m = 1,2,...,L1-1\} \tag{5.15}$$

$$W_{SY} = \sum_{i=1}^{L1} \iint\limits_{D_6^i} \Big[\frac{\partial T_{l_n+(i-1)}}{\partial y}(x,y_n,t) - \frac{\partial T_{l_n+(i-1)+L1}}{\partial y}(x,y_n,t)\Big]^2 \, dxdt$$

$$\bigcup_{i=1}^{L1} D_6^i = \{(x,y_n,t) \in R^3 : \ \ 0 \leqslant x \leqslant l, \ \ t_0 \leqslant t \leqslant t_0 + \varDelta t, \quad n = 1,2,...,L2-1\} \tag{5.16}$$

where j, k, p, l_m and l_n are certain constant numbers corresponding to the assumed numeration of elements within the area $\overline{\varOmega}$.

Figures 7a,b show the reduced heat flux distribution assumed in $(2.2)_3$. Taking the height of the body as characteristic dimension we get: $l = 10$ (dimensionless length of the body), $a = 3$ (dimensionless width of the source), $v \approx 50$ (dimensionless velocity of the source).

We consider the problem in time intervals: $\langle 0, \varDelta t\rangle$, $\langle \varDelta t, 2\varDelta t\rangle$, $\langle 2\varDelta t, 3\varDelta t\rangle$, ... where $\varDelta t$ is the time needed for a source of heat to come from the beginning of one element to the beginning of the next element. It is apparent from Fig. 7b that the time step is so small that the initial location of the source only slightly differs from the final location in a given time interval. An exemplary division

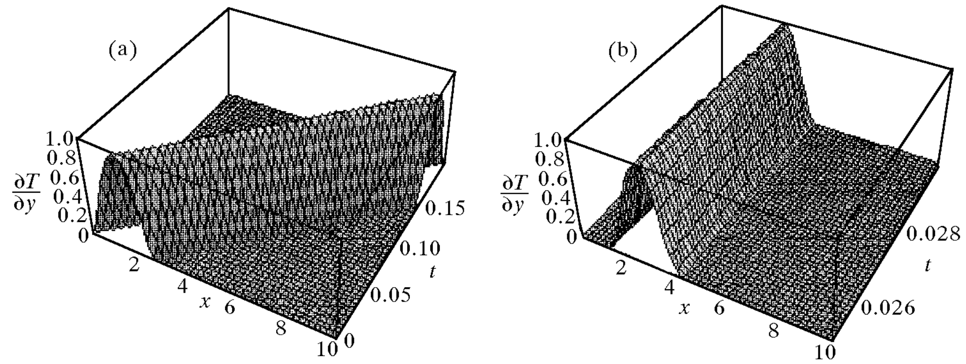

Fig. 7. (a) A graph of the function described in condition $(2.2)_3$ during passing of the source along the length of the entire body; (b) a graph of the function described in condition $(2.2)_3$ during passing along a section equal to 1/40 of the length of the body from t=0.025 do t=0.003 (dimensionless)

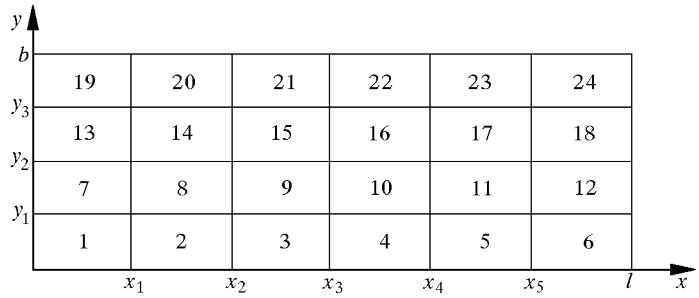

Fig. 8. Numbered bases of space-time elements and edges of the planes: $x = x_1, ..., x = x_5,\ y = y_1, y = y_2, y = y_3$

of the area $\overline{\Omega}$ into 24 elements $(L1 = 6,\ L2 = 4)$ is presented in Fig. 6. To illustrate the assumed symbols, Fig. 8 presents numbered bases of particular space-time elements and edges of the planes: $x = x_m$, for $m = 1, ..., 5,\ y = y_n$, for $n = 1, 2, 3$. Because insulation is given at the boundary $y = 0$, thus the formula (5.11) includes the temperature which is a combination of base functions from six elements of the first line, thus $j = 1$ in the elements numbered as in Fig. 8. The second-kind boundary condition is given at the boundary $y = b$, thus in the formula (5.12) the temperature is a combination of base functions from six elements of the fourth line, so $k = 19$. The condition of periodicity requires conformity of temperatures and heat fluxes at the boundaries: $x = 0,\ x = l$, thus in formulas (5.13), (5.14), for $p = 1$, the conformity of temperatures and heat fluxes (in the square-mean sense) occurs between elements: 1 and 6, 7 and 12, 13 and 18, 19 and 24. Additionally, we assume conformity of heat fluxes q_x between the elements, which is represented by formula (5.15). For example, the assumption that $l_m = 3$ in the plane $x = x_3$ results in conformity of heat fluxes (in the square-mean sense) at the boundary between the following pairs of elements: 3 and 4, 9 and 10, 15 and 16, 21 and 22. In general, in the planes $x = x_m$, we have $l_m = m$, for $m = 1, 2, 3, 4, 5$. By analogy, from (5.16), we obtain that $l_n = n$ $(n = 1, 2, 3)$ for the planes $y = y_n$.

Equation (2.1) with conditions (2.2) and (2.3) was solved with the help of the presented method in the area $\overline{\Omega}$ divided into 400 eight-node elements. The local coordinate system was placed in the centre of an element. The temperatures at nodes numbered from 1 to 451 are given as initial conditions and the temperatures at nodes numbered from 452 to 902 are unknown and wanted. The heat source (with the length of twelve elements) passes a section 1/40 m in a period of time of 4.21 s. Figures 9a,b show an approximate temperature distribution in the sixth time interval; the numerical data are the same as in the fourth section of the paper. The difference between the exact and approximate solution is shown in Fig. 10. The exact and approximate solutions are alike in terms of quantity and quality, i.e. the effect of inertia apparent in Fig. 4 has the same character as observed in Fig. 9.

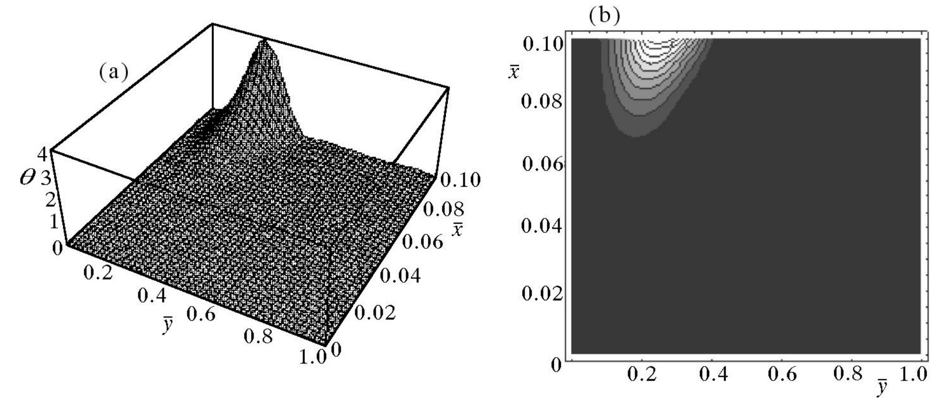

Fig. 9. (a) An approximate temperature distribution in the sixth time interval, i.e. from $\overline{t} = 21$ s to $\overline{t} = 25.21$ s; (b) a contour diagram

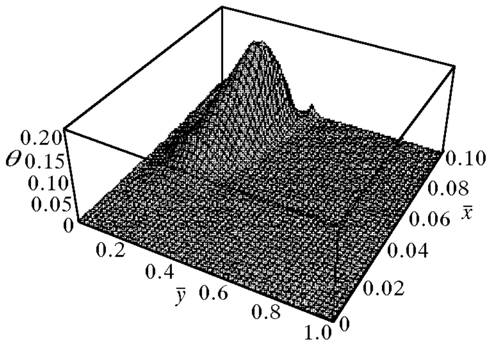

Fig. 10. The difference between the exact and approximate solutions

6. Final remarks and conclusions

The area $\Omega\times <0,t_k>$ under analysis has been divided into simplest eight-node cubicoid elements with nodes at corners of the elements. The base functions have then eight constituents, and we can approximate the exact solution with a polynomial of a low degree, which has a significant effect (positive) on numerical conditions of the analysed problem. However, a different location of the nodes can be assumed as well as a different shape of the elements. While selecting the number and location of the nodes, one should take into consideration the invertibility of the matrix $\mathbf{v}$ expressed by relation (5.4). Any location of the nodes is related to a system of heat polynomials necessary to formulate the base functions on the basis of formulas (5.2)-(5.6).

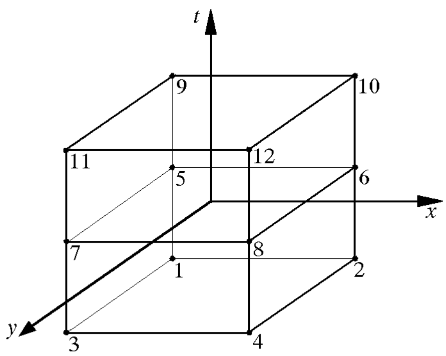

Fig. 11. Location of twelve nodes in an element

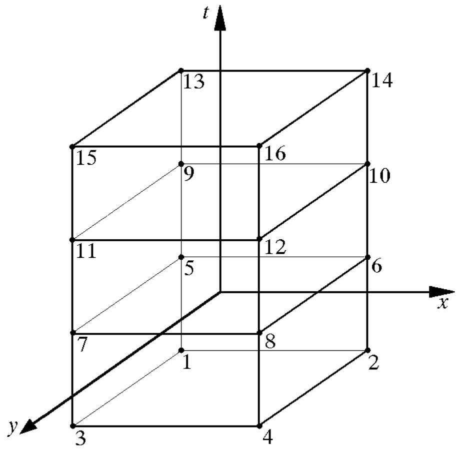

Fig. 12. Location of sixteen nodes in an element

Examplary locations of the nodes are presented in Fig. 11 and Fig. 12. While establishing the number of nodes in the area $\overline{\Omega}$, one should represent the boundary conditions well. The presented example shows that the heat polynomials are suitable for formulating time-space shape functions in the method of finite elements in the case of a body heated with a moving heat source. After performing a great number of numerical experiments consisting in different divisions of the area $\overline{\Omega}$, the division of the area into 400 elements seems to be sufficient to satisfy the initial and boundary conditions with the assumed numerical data. For source moving with a greater velocity, it is necessary to divide the area into a greater number of elements. Similarly, if the moving heat source is smaller, the grid of finite elements must be more dense. Taking advantage of heat polynomials and finite elements, one can solve the problems in areas with more complex shapes and non-stationary boundary conditions.

The system of polynomials for twelve nodes in the element: 1, x, y, xy, $t+\frac{y^2}{2}$, $tx+\frac{xy^2}{2}$, $ty+\frac{y^3}{3!}$, $txy+\frac{xy^3}{3!}$, $\frac{t^2}{2}+\frac{y^2t}{2}+\frac{y^4}{4!}$, $\frac{t^2x}{2}+\frac{y^2xt}{2}+\frac{t^4x}{4!}$, $\frac{t^2y}{2}+\frac{y^3t}{3!}+\frac{y^5}{5!}$, $\frac{t^2xy}{2}+\frac{y^3xt}{3!}+\frac{y^5x}{5!}$.

System of polynomials for sixteen nodes in an element: 1, x, y, xy, $t+\frac{y^2}{2}$, $tx+\frac{xy^2}{2}$, $ty+\frac{y^3}{3!}$, $txy+\frac{xy^3}{3!}$, $\frac{t^2}{2}+\frac{y^2t}{2}+\frac{y^4}{4!}$, $\frac{t^2x}{2}+\frac{y^2xt}{2}+\frac{t^4x}{4!}$, $\frac{t^2y}{2}+\frac{y^3t}{3!}+\frac{y^5}{5!}$, $\frac{t^2xy}{2}+\frac{y^3xt}{3!}+\frac{y^5x}{5!}$, $\frac{t^3}{3!}+\frac{y^2t^2}{2!2!}+\frac{y^4t}{4!}+\frac{y^6}{6!}$, $\frac{t^3x}{3!}+\frac{y^2t^2x}{2!2!}+\frac{y^4tx}{4!}+\frac{y^6x}{6!}$, $\frac{t^3y}{3!}+\frac{y^3t^2}{3!2!}+\frac{y^5t}{5!}+\frac{y^7}{7!}$, $\frac{t^3yx}{3!}+\frac{y^3t^2x}{3!2!}+\frac{y^5tx}{5!}+\frac{y^7x}{7!}$.

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Zastosowanie zmodyfikowanej metody elementów skończonych do identyfikacji temperatury ciała nagrzewanego ruchomym zrodłem ciepła

Streszczenie

W pracy rozważane jest dwuwymiarowe niestacjonarne zagadnienie przepływu ciepła z ruchomym źródłem ciepła na brzegu obszaru. Zakłada się, że źródło ciepła porusza się wzdłuż brzegu prostokąta ze stałą prędkością. Poszukiwany jest rozkład temperatury w obszarze o kształcie prostokąta. Przybliżone rozwiązanie problemu zaprezentowane w pracy opiera się na metodzie elementów skończonych ze zmodyfikowanymi funkcjami bazowymi. Skonstruowane czasoprzestrzenne funkcje bazowe są kombinacją wielomianów cieplnych, tzn. wielomianów ściśle spełniających równanie przewodnictwa ciepła. Zagadnienie rozwiązame zostało we współrzędnych kartezjańskich.

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