# APPLICATION OF THE TOLERANCE AVERAGING METHOD TO ANALYSIS OF DYNAMICAL STABILITY OF THIN PERIODIC PLATES[1]

Jarosław Jędrysiak

*Department of Structural Mechanics, Łódź University of Technology*

*e-mail: jarek@p.lodz.pl*

Some problems of a dynamical stability of thin periodic plates are considered. As a tool to derive the governing equations of an averaged non-asymptotic plate model *the tolerance averaging* is applied, proposed for periodic composites and structures by Woźniak and Wierzbicki (2000). This method applied to the known Kirchhoff-type plate equation leads to averaged models taking into account *the effect of the period lengths* on the overall plate behaviour (Jędrysiak, 2001). Here, a non-asymptotic model describing the problems of a dynamical stability of periodic plates is formulated. Moreover, it is shown that the effect of the period lengths plays a crucial role in some special cases of dynamical stability of such plates, i.e. for higher oscillation frequencies of compressive forces in the plate midplane.

*Key words:* thin periodic plate, dynamical stability, the length-scale effect

## 1. Introduction

Thin periodic plates, which are considered in this paper, are composed of many identical small elements (Fig. 1). These elements are treated as thin plates with spans $l_1$, $l_2$ and called *periodicity cells*. In mechanical problems of these plates the effect of the period lengths, which will be called *the length-scale effect*, on the overall plate behaviour, in particular on dynamics problems, is very interesting.

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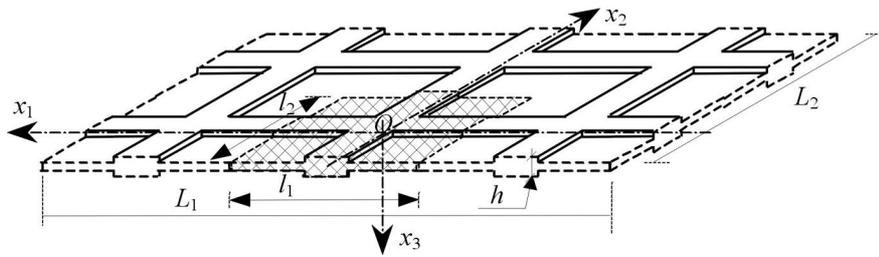
Fig. 1. Portion of a thin periodic plate

The exact equations of the plate theory for periodic plates involve highly oscillating, non-continuous, periodic coefficients and thus, they are too complicated to apply them to investigations of engineering problems. Thus, certain simplified models have been proposed. Two ways of formulation of averaged 2D-models of thin elastic plates having a periodic structure along the midplane can be mentioned. Using the first of them, based on the multiscale asymptotic expansions, the 2D-models of homogenised plates are derived from the 3D-model of the elastic solid. Equations of these models are similar to the known Kirchhoff-type equation of a homogeneous plate (cf. Caillerie, 1984; Kohn and Vogelius, 1984). Using this approach, a homogenised model of a pre-stressed periodic plate can be obtained, cf. Kolpakov (2000). However, in these averaged models the length-scale effect is neglected.

In the second approach – *the tolerance averaging method*, presented in the book (Woźniak and Wierzbicki, 2000) for periodic composites and structures, is applied to the 2D-model equations of periodic plates. Under different assumptions of periodic plates, this approach leads to certain non-asymptotic models of these plates, described by differential equations with constant coefficients, e.g. for the Hencky-Boole-type plates by Baron (2002, dynamic stability), for wavy-type plates by Michalak (1998, stability; 2001, dynamics and stability; 2002, dynamics), for thin plates (with the period lengths and the plate thickness being of the same order) by Mazur-Śniady *et al.* (2003, dynamics), for Kirchhoff-type plates (with the plate thickness being small in comparison to the period lengths) by Jędrysiak (2000, 2003b, stability; 2001, dynamics and stability; 2003a, dynamics). Internal instability of periodic structures was analysed by Wierzbicki and Woźniak (2002). Models of this kind make it possible to investigate the length-scale effect on the overall plate behaviour, in contrast to the aforementioned homogenised models.

The main aim of this paper is to derive a non-asymptotic 2D-model of thin periodic plates with forces acting in the midplane, which takes into account

the length-scale effect on the dynamic stability of such plates. The tolerance averaging method will be applied as a tool for the modelling. As an illustrative example, a dynamic stability of a periodic plate band with span $L$ along the $x_1$-axis will be analysed. It will be shown that in some special cases of a dynamical stability of that plate, the length-scale effect could not be neglected, e.g. for high values of oscillation frequencies of compressive forces in the plate midplane.

## 2. Modelling approach

Let $Ox_1x_2x_3$ be the orthogonal Cartesian co-ordinate system in the physical space, $t$ be the time co-ordinate and indices $\alpha, \beta, ...$ run over $1, 2$; $A, B, ...$ run over $1, ..., N$. Summation convention holds for all aforementioned indices. Let $\Omega \equiv \{(\boldsymbol{x}, z) : -h(\boldsymbol{x})/2 < z < h(\boldsymbol{x})/2, \quad \boldsymbol{x} \in \Pi\}$ be the region of undeformed plate, where $\boldsymbol{x} \equiv (x_1, x_2)$; $z \equiv x_3$; $\Pi$ is the plate midplane and $h(\boldsymbol{x})$ is the plate thickness at the point $\boldsymbol{x} \in \Pi$. The periodicity cell on the $Ox_1x_2$ plane is denoted by $\Delta \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$, where $l_1$, $l_2$ are the cell length dimensions along the $x_1$-, $x_2$-axis. Let $l \equiv \sqrt{l_1^2 + l_2^2}$ be the parameter describing the size of the cell. Since of the parameter $l$ is assumed to be sufficiently small compared to the minimum characteristic length dimension of $\Pi$ and sufficiently large compared to the maximum plate thickness ($h_{max} \ll l \ll L_{\Pi}$), it will be called *the mesostructure parameter*. Assume that $h$ is a $\Delta$-periodic function in $\boldsymbol{x}$ and all material and inertial plate properties, e.g. mass density $\rho = \rho(\boldsymbol{x}, z)$ and elastic moduli $a_{ijkl} = a_{ijkl}(\boldsymbol{x}, z)$, are also $\Delta$-periodic functions in $\boldsymbol{x}$ and even functions in $z$ (cf. Jędrysiak, 2001, 2003a). Periodic plates with the structure will be called *mesoperiodic plates*. Denote by $w$ the plate deflection and by $p^-$, $p^+$ loadings in the $z$-axis direction acting on the upper and lower plate boundaries. The non-zero terms of the elastic moduli tensor are $a_{\alpha\beta\gamma\delta}, a_{\alpha\beta33}, a_{3333}$; denote $c_{\alpha\beta\gamma\delta} \equiv a_{\alpha\beta\gamma\delta} - a_{\gamma\delta33}a_{\alpha\beta33}(a_{3333})^{-1}$. Our considerations are based on the well-known Kirchhoff-type plate theory assumptions (cf. Jędrysiak, 2001). Let us introduce $\Delta$-periodic functions of the mean plate properties – mass density, rotational inertia, bending stiffnesses

$$\mu \equiv \int\limits_{-h/2}^{h/2} \rho \, dz \qquad \vartheta \equiv \int\limits_{-h/2}^{h/2} \rho z^2 \, dz \qquad d_{\alpha\beta\gamma\delta} \equiv \int\limits_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta} \, dz$$

As the starting point of the modelling we assume the well known fourth order differential equation

$$(d_{\alpha\beta\gamma\delta}w_{,\gamma\delta})_{,\alpha\beta} - (n_{\alpha\beta}w_{,\alpha})_{,\beta} + \mu\ddot{w} - (\vartheta\ddot{w}_{,\alpha})_{,\alpha} = p \qquad (2.1)$$

in the form known for homogeneous thin plates from the books Timoshenko and Gere (1961), Volmir (1972), Kaliski (1992); here $p \equiv p^+ + p^-$; $n_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) are forces in the plate midplane, such that $n_{\alpha\beta,\beta} = 0$. However, for mesoperiodic plates the coefficients in the above equation are highly oscillating $\Delta$-periodic and also non-continuous functions.

In order to derive the averaged governing equations of mesoperiodic plates, having constant coefficients and taking into account the length-scale effect, *the tolerance averaging method* developed in the book Woźniak and Wierzbicki (2000) for periodic composites will be applied. In this method some additional concepts such as e.g. an averaging operator, a tolerance system, a slowly varying function, a periodic-like function and an oscillating function explained in detail in the above book, are used.

Let $\Delta(\boldsymbol{x}) = \Delta + \boldsymbol{x}$ be a periodicity cell at $\boldsymbol{x} \in \Pi_\Delta$, $\Pi_\Delta = \{\boldsymbol{x} : \boldsymbol{x} \in \Pi, \Delta(\boldsymbol{x}) \subset \Pi\}$. In the analysis of periodic structures, the known *averaging operator* (cf. Woźniak and Wierzbicki, 2000; Jędrysiak, 2001, 2003a,b) is applied

$$\langle \varphi \rangle(\boldsymbol{x}) \equiv \frac{1}{l_1 l_2} \int\limits_{\Delta(\boldsymbol{x})} \varphi(\boldsymbol{y}) \, d\boldsymbol{y} \qquad \boldsymbol{x} \in \Pi_\Delta \qquad (2.2)$$

defined on the plate midplane $\Pi$ for an arbitrary integrable function $\varphi$. For a periodic function $\varphi$ in $\boldsymbol{x}$, its averaged value from (2.2) is constant. It is tacitly assumed that all functions under consideration satisfy the required regularity conditions.

In order to make the paper self-consistent, following the book Woźniak and Wierzbicki (2000), some mathematical notions and formulae will be used. The tolerance averaging method is based on the concept that to every considered physical quantity $s$, expressed in terms of a certain unit measure, a positive number $\varepsilon_s$ can be assigned. This number is called *a tolerance parameter* and is such that for every two values $s_1$, $s_2$ of this quantity, if $|s_1 - s_2| \leqslant \varepsilon_s$ then $s_1 \cong s_2$, what means that values $s_1$, $s_2$ can be treated as indistinguishable. Denote by $T$ a certain mapping (called a tolerance system), which assigns to every quantity under consideration a tolerance parameter. Now, the following definitions will be recalled (cf. the above book).

The continuous function $\Psi$, defined on $\overline{\Pi}$, will be called *a slowly varying function*, if for every $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \overline{\Pi}$ such that $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leqslant l$ the following

condition holds $|\Psi(\boldsymbol{x}_1) - \Psi(\boldsymbol{x}_2)| \leqslant \varepsilon_\Psi$. We shall write $\Psi \in SV(T)$ if $\Psi$ and all its derivatives are slowly varying functions.

The continuous function $f$ will be called *a periodic-like function* if for every $\boldsymbol{x} \in \overline{\Pi}$ there exists a certain continuous $\Delta$-periodic function $f_{\boldsymbol{x}}$ such that for every $\boldsymbol{y} \in \overline{\Pi}$ and $\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l$, we obtain $|f(\boldsymbol{y}) - f_{\boldsymbol{x}}(\boldsymbol{y})|\varepsilon_f$. If derivatives of the function $f$ satisfy similar conditions, we will write $f \in PL(T)$. It can be shown Woźniak and Wierzbicki (2000) that averaging (2.1) of periodic-like function is a slowly varying function.

A periodic-like function $f$ will be called *an oscillating function* if it satisfies the condition $\langle \mu f \rangle(\boldsymbol{x}) \cong 0$ for every $\boldsymbol{x} \in \Pi_\Delta$, where $\mu$ is a positive-valued $\Delta$-periodic function. The set of oscillating periodic-like functions with the weight $\mu$ will be denoted by $PL^\mu(T)$.

In the modelling procedure, the lemmas and assertions, formulated and proved in the book Woźniak and Wierzbicki (2000) with the above concepts are applied.

In the tolerance averaging method assumptions formulated below are used.

**The Tolerance Averaging Approximation.** It is assumed that for every $\Delta$-periodic integrable function $\varphi$ defined on $\Pi$ and all integrable functions $\Psi \in SV(T)$, $f \in PL(T)$, the following conditions hold

$$\langle \varphi \Psi \rangle(\boldsymbol{x}) \cong \langle \varphi \rangle \Psi(\boldsymbol{x}) \qquad \langle \varphi f \rangle(\boldsymbol{x}) \cong \langle \varphi f_{\boldsymbol{x}} \rangle(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Pi_\Delta \quad (2.3)$$

**The Conformability Assumption.** It is assumed that the deflection $w(\cdot, t)$ of the plate midplane under consideration is a periodic-like function, $w(\cdot, t) \in PL(T)$, i.e. the deflection is conformable to a periodic plate structure. This condition may be violated only near the plate boundary.

**The Midplane Forces Restriction.** It is assumed that forces in the midplane $n_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are also periodic-like functions, $n_{\alpha\beta}(\cdot, t) \in PL(T)$. Hence, they could be decomposed into $n_{\alpha\beta} = n^0_{\alpha\beta} + \tilde{n}_{\alpha\beta}$, where $n^0_{\alpha\beta} \in SV(T)$ will be the averaged part defined by $n^0_{\alpha\beta} \equiv \langle n_{\alpha\beta} \rangle$, and $\tilde{n}_{\alpha\beta} \in PL^1(T)$ will be the fluctuating part, such that $\langle \tilde{n}_{\alpha\beta} \rangle = 0$.

The modelling procedure of the tolerance averaging can be divided into four steps.

1) Define the averaging part of deflection by setting $W \equiv \langle \mu \rangle^{-1} \langle \mu w \rangle$, where $\mu$ is the plate mass density. Because of $w \in PL(T)$ it is $W \in SV(T)$. Thus, the decomposition can be introduced $w = W + v$, where $v \in PL^\mu(T)$ is *the fluctuating part* satisfying the condition $\langle \mu v \rangle = 0$. The averaged part of deflection $W$ will be called *a macrodeflection*.

2) Formulate *the periodic problem* (cf. Woźniak and Wierzbicki, 2000; Jędrysiak, 2001) on $\Delta(\boldsymbol{x})$ for $v_{\boldsymbol{x}}$ being a $\Delta$-periodic approximation of $v$ on $\Delta(\boldsymbol{x})$ at $\boldsymbol{x} \in \Pi_\Delta$. Function $v_{\boldsymbol{x}}$ satisfies the condition $\langle \mu v_{\boldsymbol{x}} \rangle = 0$.

3) Formulate the Galerkin approximation of the above periodic problem by introducing the system of $N$ linear-independent $\Delta$-periodic functions $g^A$, $A = 1, ..., N$, such that $\langle \mu g^A \rangle = 0$, and by setting $v_{\boldsymbol{x}}(\boldsymbol{y}, t) = g^A(\boldsymbol{y})Q^A(\boldsymbol{x}, t)$, where $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Delta$; $Q^A \in SV(T)$ are new kinematic unknowns. Functions $g^A$ are called *mode-shape functions* and they have to approximate the expected form of the oscillating part of free vibration modes of the periodicity cell. Moreover, values of these functions have to satisfy conditions $l^{-1}g^A(\cdot), g^A_{,\alpha}(\cdot), lg^A_{,\alpha\beta}(\cdot) \in \mathcal{O}(l)$.

4) After some manipulations, the equation for the macrodeflection $W$ and equations for kinematic unknowns $Q^A$ are obtained

$$
\begin{aligned}
&(\langle d_{\alpha\beta\gamma\delta} \rangle W_{,\gamma\delta} + \langle d_{\alpha\beta\gamma\delta} g^B_{,\gamma\delta} \rangle Q^B)_{,\alpha\beta} - n^0_{\alpha\beta} W_{,\alpha\beta} + \langle \mu \rangle \ddot{W} - \langle \vartheta \rangle \ddot{W}_{,\alpha\alpha} - \\
&-\langle \widetilde{n}_{\alpha\beta} g^B_{,\alpha} \rangle Q^B_{,\beta} - \langle \vartheta g^B_{,\alpha} \rangle \ddot{Q}^B_{,\alpha} = \langle p \rangle \\
\\
&(\langle \mu g^A g^B \rangle + \langle \vartheta g^A_{,\alpha} g^B_{,\alpha} \rangle) \ddot{Q}^B + \langle \vartheta g^A_{,\alpha} \rangle \ddot{W}_{,\alpha} + \langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} \rangle W_{,\alpha\beta} + \\
&+\langle \widetilde{n}_{\alpha\beta} g^A_{,\beta} \rangle W_{,\alpha} + \langle d_{\alpha\beta\gamma\delta} g^A_{,\alpha\beta} g^B_{,\gamma\delta} \rangle Q^B + n^0_{\alpha\beta} \langle g^A_{,\alpha} g^B_{,\beta} \rangle Q^B + \\
&+\underline{\langle \widetilde{n}_{\alpha\beta} g^A_{,\alpha} g^B_{,\beta} \rangle} Q^B = \langle p g^A \rangle
\end{aligned}
\tag{2.4}
$$

where $n^0_{\alpha\beta}$ and $\widetilde{n}_{\alpha\beta}$ are the averaged and the fluctuating part of in-plane forces, respectively.

It can be shown that for plates with symmetric cells (cf. Fig. 1) and symmetric mode-shape functions $g^A$, the following terms are equal to zero: $\langle \widetilde{n}_{\alpha\beta} g^A_{,\alpha} \rangle = \langle \vartheta g^0_{,\alpha} \rangle = 0$. In the subsequent considerations it will be assumed that in the above equations the underlined terms $\langle \widetilde{n}_{\alpha\beta} g^A_{,\alpha} g^B_{,\beta} \rangle$ with the fluctuating part of in-plane forces are small in comparison to terms $n^0_{\alpha\beta} \langle g^A_{,\alpha} g^B_{,\beta} \rangle$ with the averaged part of the forces and they will be neglected. Moreover, introducing the in-plane constant forces $N_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ applied on the edges of thin periodic plate, it can be assumed that the averaged part $n^0_{\alpha\beta}$ of in-plane forces can be replaced by $N_{\alpha\beta}$.

It should be emphasized that equations (2.4) are derived without the assumption introduced in Jędrysiak (2000, 2001, 2003b) that in terms $(n_{\alpha\beta} w_{,\alpha})_{,\beta}$ of Eq. (2.1) the deflection $w$ can be replaced by the macrodeflection $W$. Thus, the obtained equations are more general than those presented in the above papers.

### 3. Governing equations

Introducing the following notations

$$D_{\alpha\beta\gamma\delta} \equiv \langle d_{\alpha\beta\gamma\delta} \rangle \qquad D_{\alpha\beta}^{A} \equiv \langle d_{\alpha\beta\gamma\delta} g_{,\gamma\delta}^{A} \rangle \qquad D^{AB} \equiv \langle d_{\alpha\beta\gamma\delta} g_{,\alpha\beta}^{A} g_{,\gamma\delta}^{B} \rangle$$

$$G_{\alpha\beta}^{AB} \equiv l^{-2} \langle g_{,\alpha}^{A} g_{,\beta}^{B} \rangle \qquad m \equiv \langle \mu \rangle \qquad m^{AB} \equiv l^{-4} \langle \mu g^{A} g^{B} \rangle$$

$$j \equiv \langle \vartheta \rangle \qquad j^{AB} \equiv l^{-2} \langle \vartheta g_{,\alpha}^{A} g_{,\alpha}^{B} \rangle \qquad P \equiv \langle p \rangle$$

$$P^{A} \equiv l^{-2} \langle pg^{A} \rangle$$

and neglecting in (2.4) the underlined terms, we arrive at the governing equations of *the non-asymptotic model*

$$(D_{\alpha\beta\gamma\delta} W_{,\gamma\delta} + D_{\alpha\beta}^{B} Q^{B})_{,\alpha\beta} - N_{\alpha\beta} W_{,\alpha\beta} + m\ddot{W} - j\ddot{W}_{,\alpha\alpha} = P$$

$$l^{2}(l^{2} m^{AB} + j^{AB})\ddot{Q}^{B} + D_{\alpha\beta}^{A} W_{,\alpha\beta} + D^{AB} Q^{B} + N_{\alpha\beta} l^{2} G_{\alpha\beta}^{AB} Q^{B} = l^{2} P^{A} \tag{3.1}$$

where some terms depend explicitly on the mesostructure parameter $l$; $N_{\alpha\beta}$ are in-plane forces.

The above equations are a certain generalization of the governing equations obtained by Jędrysiak (2001) for plates with periodic structure along both the axes in the midplane, because they involve additional terms $N_{\alpha\beta} l^{2} G_{\alpha\beta}^{AB} Q^{B}$. These equations, having averaged constant coefficients, make it possible to analyse *the length-scale effect* in dynamic processes and also in stability of periodic plates. The basic unknowns $W$, $Q^{A}$, $A = 1, ..., N$, are slowly varying functions. For a rectangular plate with midplane $\Pi = (0, L_1) \times (0, L_2)$, two boundary conditions should be defined on the edges $x_1 = 0$, $L_1$ and $x_2 = 0$, $L_2$ only for *the macrodeflection* $W$. Hence, functions $Q^{A}$ are called *internal variables*. To derive equations (3.1), the mode-shape functions $g^{A}$, $A = 1, ..., N$, for every periodic plate under consideration have to be previously obtained. In the most cases only one $(N = 1)$ mode-shape function $g = g^{1}$, assumed as an approximate solution to the eigenvalue problem on the cell, is sufficient from the calculative point of view (Jędrysiak, 2001).

At the end of this section it is shown that a model without the length-scale effect is a special case of the non-asymptotic model. Neglecting the terms with parameter $l$ in equations (3.1) and substituting $(3.1)_2$ into $(3.1)_1$ we arrive at

$$[D_{\alpha\beta\gamma\delta} - D_{\gamma\delta}^{A} D_{\alpha\beta}^{B} (D^{AB})^{-1}] W_{,\alpha\beta\gamma\delta} - N_{\alpha\beta} W_{,\alpha\beta} + m\ddot{W} - j\ddot{W}_{,\alpha\alpha} = P \tag{3.2}$$

The above equation of the averaged model without the length-scale effect has form similar to the equations of homogeneous plates. This model will be called *homogenised model.*

## 4.  The problem of dynamic stability

Now, the governing equations of both the models presented in the previous section will be applied to analyse the problem of dynamic stability of a rectangular plate. It is assumed that the plate is made of an isotropic piece-wise periodically homogeneous material along the $x_1$- and $x_2$-axis and has a periodic thickness $h$ along both the axes. Moreover, assume that Poisson's ratio $\nu$ is constant, but the plate mass density $\rho$ and Young's modulus $E$ are periodically variable; loadings $p$ are neglected and the plate is uniformly compressed along the $x_1$- and $x_2$-axis in the midplane, hence $N_{12} = N_{21} = 0$. Let us consider a case with only one mode-shape function $g$ (i.e. $A = N = 1$) assumed as the approximate solution to a certain eigenvalue problem with periodic boundary conditions imposed on the cell in the form: $g = g^1 = l^2[\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2) + c]$, where the constant $c$ is calculated from the condition $\langle \mu g \rangle = 0$. For the assumed symmetric cell and symmetric form of mode-shape function it can be shown that $D_{12}^1 = D_{21}^1 = 0$. Denote $B = \langle Eh^3/[12(1-\nu^2)]\rangle$ and $Q = Q^1$, $x = x_1$, $G_1 \equiv G_{11}^{11}$, $G_2 \equiv G_{22}^{11}$, $D_1 \equiv D_{11}^1$, $D_2 \equiv D_{22}^1$, $D \equiv D^{11}$, and also $N_1 \equiv -N_{11}$, $N_2 \equiv -N_{22}$. Separating variables $\boldsymbol{x} = (x_1, x_2)$ and $t$, the macrodeflection $W$ and the internal variable $Q$ can be assumed in the form

$$W(x_1, x_2, t) = X_{nk}(x_1, x_2)T_{nk}(t)$$

$$Q(x_1, x_2, t) = X_{nk}(x_1, x_2)T_{nk}^Q(t) \qquad n, k = 1, 2, \ldots \tag{4.1}$$

where functions $X_{nk}(\cdot)$ satisfy proper boundary conditions on the opposite plate edges.

For the plate under consideration, substituting (4.1) into equations (3.1) and after some manipulations, the equation for functions $T_{nk}$ in *the non-asymptotic model* is obtained

$$l^2(l^2 m^{11} + j^{11})\{[mX_{nk} - j(X_{nk,11} + X_{nk,22})]\frac{d^4}{dt^4}T_{nk} +$$

$$+\frac{d^2}{dt^2}[(BX_{nk,\alpha\alpha\beta\beta} + N_1 X_{nk,11} + N_2 X_{nk,22})T_{nk}]\} +$$

$$+(D - l^2 N_1 G_1 - l^2 N_2 G_2)\{[mX_{nk} - j(X_{nk,11} + X_{nk,22})]\frac{d^2}{dt^2}T_{nk} + \tag{4.2}$$

$$+(BX_{nk,\alpha\alpha\beta\beta} + N_1 X_{nk,11} + N_2 X_{nk,22})T_{nk}\} -$$

$$-(D_1^2 X_{nk,1111} + 2D_1 D_2 X_{nk,1122} + D_2^2 X_{nk,2222})T_{nk} = 0$$

Substituting the proper functions $X_{nk}(\cdot)$ satisfying the boundary conditions into (4.2) we obtain the frequency equation.

In *the homogenised model* of the considered plate, after similar manipulations, from equation (3.2), we obtain

$$\frac{d^2}{dt^2}[(BX_{nk,\alpha\alpha\beta\beta} + N_1 X_{nk,11} + N_2 X_{nk,22})T_{nk}] +$$

$$+D\{[mX_{nk} - j(X_{nk,11} + X_{nk,22})]\frac{d^2}{dt^2}T_{nk} +$$

$$+(BX_{nk,\alpha\alpha\beta\beta} + N_1 X_{nk,11} + N_2 X_{nk,22})T_{nk}\} -$$

$$-(D_1^2 X_{nk,1111} + 2D_1 D_2 X_{nk,1122} + D_2^2 X_{nk,2222})T_{nk} = 0$$

(4.3)

where the length-scale effect described by terms with parameter $l$ is neglected.

In the subsequent section an example of application of the above equations will be shown.

## 5. Example – the dynamic stability of simply supported plate strip

### 5.1. General analysis

As an example, let us consider a simply supported plate strip with span $L_1 = L$ along the $x$-axis ($x = x_1$), having periodic structure along both the axes in the midplane. The plate periodicity is caused by the periodic thickness. However, mass density $\rho$ and Young's modulus $E$ are constant. It is assumed that the periodicity cell is square, i.e. $\Delta \equiv (-l/2, l/2) \times (-l/2, l/2)$, and hence the mode-shape function is $g = l^2[\cos(2\pi x_1/l)\cos(2\pi x_2/l)+c]$. It can be shown that for such a plate $D_1 = D_2$ and $G_1 = G_2$. Functions $X_{nk}$ satisfying the boundary conditions for the simply supported plate strip on the edges $x = 0, L$ have the form ($k = 1$)

$$X_n(x) = X_{nk}(x_1, x_2) = \sin(\alpha_n x) \tag{5.1}$$

where $\alpha_n = n\pi/L$, $n = 1, 2, ....$ Denote $T = T_n = T_{nk}$. Because the wavelengths of $X_n$ are sufficiently large compared to $l$ and hence $\alpha_n l \ll 1$ and also $h/l \ll 1$, in the sequel the simplified form of Eq. (4.2) will be applied, in which terms $l^2(l^2 m^{11} + j^{11})d^2[(BX_{n,1111} + N_1 X_{n,11})T]/dt^2$ can be neglected

as small compared to $[D - l^2 G_1(N_1 + N_2)](mX_n - jX_{n,11})d^2T/dt^2$. Substituting (5.1) into Eq. (4.2), bearing in mind that $l/L \ll 1$, from (4.2) the explicit asymptotic formulae can be derived

$$N_- \equiv \alpha_n^2(B - D_1^2 D^{-1}) \qquad\qquad N_+ \equiv D(G_1 l^2)^{-1} - N_2 + \alpha_n^2 D_1^2 D^{-1}$$

$$\widetilde{N}_+ \equiv D(G_1 l^2)^{-1} \qquad\qquad \omega_-^2 \equiv \alpha_n^4(B - D_1^2 D^{-1})(m + j\alpha_n^2)^{-1}$$

$$\omega_+^2 \equiv Dl^{-2}(l^2 m^{11} + j^{11})^{-1}$$

$$(5.2)$$

for the "fundamental" lower critical force $N_-$, for the "additional" higher critical force $N_+$ and its approximation $\widetilde{N}_+$, for the lower free vibration frequency $\omega_-$, for the higher free vibration frequency $\omega_+$. Using these asymptotic formulae, the frequency equation within *the non-asymptotic model* can be written in the approximate form

$$\frac{d^4}{dt^4}T + \omega_+^2[1 - (N_1 + N_2)\widetilde{N}_+^{-1}]\frac{d^2}{dt^2}T +$$

$$(5.3)$$

$$+\omega_+^2\omega_-^2(1 - N_1 N_-^{-1})(1 - N_1 N_+^{-1})N_+\widetilde{N}_+^{-1}T = 0$$

Assuming that only $N_1$ is a time-dependent function by $N_1 = N_a + N_b \cos pt$, where $p$ is the oscillation frequency of force $N_1$; however, $N_2$ is independent of time; introducing a dimensionless time co-ordinate $z = pt$ and denoting $T' = dT/dz$ and also

$$\eta_- = \omega_-^2 p^{-2} \qquad\qquad \eta_+ = \omega_+^2 p^{-2} \qquad\qquad \zeta = N_+\widetilde{N}_+^{-1}$$

$$\chi_- = N_a N_-^{-1} \qquad\qquad \delta_- = N_b N_-^{-1} \qquad\qquad \chi_+ = N_a N_+^{-1}$$

$$\delta_+ = N_b N_+^{-1} \qquad\qquad \widetilde{\chi}_+ = N_a \widetilde{N}_+^{-1} \qquad\qquad \widetilde{\delta}_+ = N_b \widetilde{N}_+^{-1}$$

$$\widetilde{\chi} = N_2 \widetilde{N}_+^{-1} \qquad\qquad \xi = \eta_+(1 - \widetilde{\chi}_+ - \widetilde{\chi}) \qquad\quad \xi_- = \eta_-(1 - \chi_-)$$

$$\xi_+ = \eta_+(1 - \chi_+) \qquad \varphi = \widetilde{\delta}_+(1 - \widetilde{\chi}_+ - \widetilde{\chi})^{-1} \qquad \varphi_- = \delta_-(1 - \chi_-)^{-1}$$

$$\varphi_+ = \delta_+(1 - \chi_+)^{-1}$$

$$(5.4)$$

from (5.3) the following equation is obtained

$$T'''' + \xi(1 - \varphi \cos z)T'' + \xi_-\xi_+(1 - \varphi_- \cos z)(1 - \varphi_+ \cos z)\zeta T = 0 \qquad (5.5)$$

The above equation is a starting point of the analysis of dynamic stability of the considered plate strip in the framework of the non-asymptotic model.

It should be emphasized that, in contrast to the models presented in Jędrysiak (2001), we obtain here not only the additional higher frequency $\omega_+$,

Eq. $(5.2)_5$, but also the additional higher critical force $N_+$, Eq. $(5.2)_2$ in the framework of the proposed non-asymptotic model.

In order to evaluate the obtained results let us consider the above problem within *the homogenised model*. Substituting solutions (5.1) into Eq. (4.3) and using formulae $(5.2)_{1,4}$ we arrive at the frequency equation in the form

$$\frac{d^2}{dt^2}T + \omega_-^2(1 - N_1 N_-^{-1})T = 0$$

Assuming $N_1 = N_a + N_b \cos pt$, introducing a dimensionless time co-ordinate $z = pt$ and using (5.4), the above equation takes the form

$$T'' + \xi_-(1 - \varphi_- \cos z)T = 0 \tag{5.6}$$

It can be observed that equation (5.6) for *the homogenised model* of periodic plates has a form of the known Mathieu equation, which describes dynamic stability or parametric vibrations of different structures (e.g. bars, plates; cf. Timoshenko (1961), Kaliski (1992)). Using this equation, regions of dynamic instability for parameters $\varphi_-$, $\xi_-$ can be determined.

However, in the framework of the non-asymptotic model, the fourth order equation (5.5) is derived, which can be treated as a certain generalization of the Mathieu equation. This equation makes it possible to investigate the length-scale effect on the dynamical stability of periodic plates. Moreover, it should be emphasized that for this equation additional initial conditions imposed on higher-order derivatives of function $T$, i.e. $T''$, $T'''$, have to be formulated. The function $T$ and its derivatives $T'$, $T''$, $T'''$ can be treated as the dimensionless: macrodeflection, velocity, acceleration, higher-order acceleration.

## 5.2. Approximate solutions

An analytical solution of the Mathieu equation (5.6) is known in the literature (cf. Timoshenko, 1961; Volmir, 1972; Kaliski, 1992). This solution can be assumed in the following form

$$T(z) = \sum_{k=0}^{\infty}\left(A_k \sin\frac{kz}{2} + B_k \cos\frac{kz}{2}\right) \tag{5.7}$$

which is restricted in time (time $t$ is hidden in variable $z = pt$).

However, in order to apply (5.7) to the generalized Mathieu equation (5.5) some simplifications in this equation must be introduced. Thus, neglect the

constant part of the force $N_1$ and also the force $N_2$, i.e. $N_a = N_2 = 0$. In this way, $\chi_- = \chi_+ = \widetilde{\chi}_+ = \widetilde{\chi} = 0$ and $\xi = \xi_+$. Moreover, in most cases the following conditions hold: $N_b \ll N_+$, $N_b \ll \widetilde{N}_+$, and hence it can be shown that $\varphi, \varphi_+ \ll 1$. Introduce the notations: $a \equiv \xi_-$, $2b \equiv \xi_- \varphi_-$ and $\varepsilon \equiv \omega_-^2/\omega_+^2$, hence $\xi_+ = a/\varepsilon$. The frequency equations for both the models can be written in the forms:

— for *the non-asymptotic model* (5.5)

$$T'''' + \frac{a}{\varepsilon}T'' + \frac{a}{\varepsilon}\zeta(a - 2b\cos z)T = 0 \qquad (5.8)$$

— for *the homogenised model* (5.6)

$$T'' + (a - 2b\cos z)T = 0 \qquad (5.9)$$

Substituting the solution (5.7) into the above equations, after some manipulations, characteristic equations of the relation between coefficients $a$ and $b$ are derived in the form of continued fractions.

For *the non-asymptotic model* (5.8) the characteristic equations take the form

$$\frac{\varepsilon}{16} - \frac{a}{4} + \zeta a(a - b) = \cfrac{(ab\zeta)^2}{\frac{81\varepsilon}{16} - \frac{9a}{4} + a^2\zeta - \cfrac{(ab\zeta)^2}{\frac{125\varepsilon}{16} - \frac{25a}{4} + a^2\zeta - \dots}} \qquad r = 1, 3, \dots$$

$$\frac{\varepsilon}{16} - \frac{a}{4} + \zeta a(a + b) = \cfrac{(ab\zeta)^2}{\frac{81\varepsilon}{16} - \frac{9a}{4} + a^2\zeta - \cfrac{(ab\zeta)^2}{\frac{125\varepsilon}{16} - \frac{25a}{4} + a^2\zeta - \dots}} \qquad r = 1, 3, \dots$$

$$\frac{a}{2b} = \cfrac{ab\zeta}{\frac{16\varepsilon}{16} - \frac{4a}{4} + a^2\zeta - \cfrac{(ab\zeta)^2}{\frac{256\varepsilon}{16} - \frac{16a}{4} + a^2\zeta - \dots}} \qquad r = 0, 2, \dots$$

$$\frac{16\varepsilon}{16} - \frac{4a}{4} + \zeta a^2 = \cfrac{(ab\zeta)^2}{\frac{256\varepsilon}{16} - \frac{16a}{4} + a^2\zeta - \cfrac{(ab\zeta)^2}{\frac{1296\varepsilon}{16} - \frac{36a}{4} + a^2\zeta - \dots}} \qquad r = 0, 2, \dots$$

$$(5.10)$$

Equations $(5.10)_1$ and $(5.10)_3$ are related to the part of solution (5.7) in the form $\cos(rx/2)$, however equations $(5.10)_2$ and $(5.10)_4$ are referred to the part $\sin(rx/2)$. These equations determine boundaries between the regions of stable and unstable vibrations within *the non-asymptotic model* of periodic plate.

For *the homogenised model* (5.9), the characteristic equations have the form

$$
\begin{aligned}
a - b - \frac{1}{4} &= \cfrac{b^2}{a - \frac{9}{4} - \cfrac{b^2}{a - \frac{25}{4} - ... - \cfrac{b^2}{a - \frac{1}{4}r^2 - ...}}} & r &= 1, 3, \ldots \\[2ex]
a + b - \frac{1}{4} &= \cfrac{b^2}{a - \frac{9}{4} - \cfrac{b^2}{a - \frac{25}{4} - ... - \cfrac{b^2}{a - \frac{1}{4}r^2 - ...}}} & r &= 1, 3, \ldots \\[2ex]
\frac{1}{2}a &= \cfrac{b^2}{a - 4 - \cfrac{b^2}{a - 16 - ... - \cfrac{b^2}{a - r^2 - ...}}} & r &= 0, 2, \ldots \\[2ex]
a - 1 &= \cfrac{b^2}{a - 4 - \cfrac{b^2}{a - 9 - ... - \cfrac{b^2}{a - r^2 - ...}}} & r &= 0, 2, \ldots
\end{aligned}
\tag{5.11}
$$

Similar to the non-asymptotic model, equations $(5.11)_1$ and $(5.11)_3$ are related to the part of solution (5.7) in the form $\cos(rx/2)$, however equations $(5.11)_2$ and $(5.11)_4$ are referred to $\sin(rx/2)$. The above equations determine the boundaries between regions of stable and unstable vibrations within *the homogenised model* of a periodic plate.

It can be observed that the continued fractions (5.11) for the homogenised model have the form known in literature, being obtained from the known Mathieu equation (5.9). Unfortunately, these relations do not describe the length-scale effect. However, equations (5.10) for the non-asymptotic model, derived from equation (5.8), take into account the length-scale effect. This effect is described by terms with the parameter $\varepsilon$.

Diagrams of curves determining boundaries between regions of stable and unstable vibrations by both the models are shown in the subsequent section in Fig. 3.

### 5.3. Numerical solutions

In order to find more exact solutions to equation (5.8) and in particular to equation (5.5) without simplifications, it is necessary to apply numerical methods. For this purpose, the known commercial programs for symbolic and numerical calculations such as *Mathematica* or *MathCad* can be used. Some diagrams of these solutions are shown and discussed in the subsequent section.

## 6.   Numerical results

Numerical examples are calculated for the plate strip of periodically variable thickness along both the axes in the midplane. A square periodicity cell is assumed, i.e. $\Delta \equiv (-l/2, l/2) \times (-l/2, l/2)$, shown in Fig. 2. The plate thickness is defined as

$$
h(\boldsymbol{x}) = \begin{cases}
h_0 & \text{if} \quad \boldsymbol{x} \in \left[-\tfrac{1}{2}l, -\tfrac{1}{2}\gamma l\right) \times \left[-\tfrac{1}{2}\gamma l, \tfrac{1}{2}\gamma l\right] \cup \\
& \quad \cup \left[-\tfrac{1}{2}\gamma l, \tfrac{1}{2}\gamma l\right] \times \left[-\tfrac{1}{2}l, \tfrac{1}{2}l\right] \cup \left(\tfrac{1}{2}\gamma l, \tfrac{1}{2}l\right] \times \left[-\tfrac{1}{2}\gamma l, \tfrac{1}{2}\gamma l\right] \\
h_1 = \eta h_0 & \text{if} \quad \boldsymbol{x} \in \left[-\tfrac{1}{2}l, -\tfrac{1}{2}\gamma l\right) \times \left[-\tfrac{1}{2}l, -\tfrac{1}{2}\gamma l\right) \cup \\
& \quad \cup \left[-\tfrac{1}{2}l, -\tfrac{1}{2}\gamma l\right) \times \left(\tfrac{1}{2}\gamma l, \tfrac{1}{2}l\right] \cup \\
& \quad \cup \left(\tfrac{1}{2}\gamma l, \tfrac{1}{2}l\right] \times \left[-\tfrac{1}{2}l, -\tfrac{1}{2}\gamma l\right) \cup \left(\tfrac{1}{2}\gamma l, \tfrac{1}{2}l\right] \times \left(\tfrac{1}{2}\gamma l, \tfrac{1}{2}l\right]
\end{cases}
\tag{6.1}
$$

where $\gamma, \eta \in [0,1]$, $\boldsymbol{x} = (x_1, x_2)$. Introduce the dimensinless parameters: $\lambda = l/L$, $\eta_0 = h_0/l$, $\phi = N_2/N_-$.
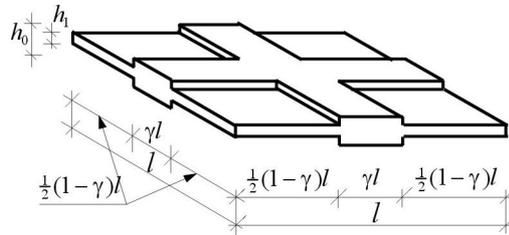


Fig. 2. Periodicity cell of the plate strip under consideration

### 6.1.   Boundaries of regions of stable and unstable vibrations for an approximate solution

Some results for the approximate solution (5.7) to frequency equations (5.8) and (5.9) are shown in Fig. 3. Here, the curves of boundaries of regions of stable and unstable vibrations are presented. These diagrams are made by using the characteristic equations (5.10) and (5.11), which describe the relation between coefficients $a - b$ in equations (5.8) and (5.9) in the framework of the non-asymptotic and the homogenised model, respectively. Plots shown in this figure are made for parameters: $\lambda = 0.1$, $\eta_0 = 0.1$, $\eta = 0.7$, $\gamma = 0.5$, and $n = 1$.
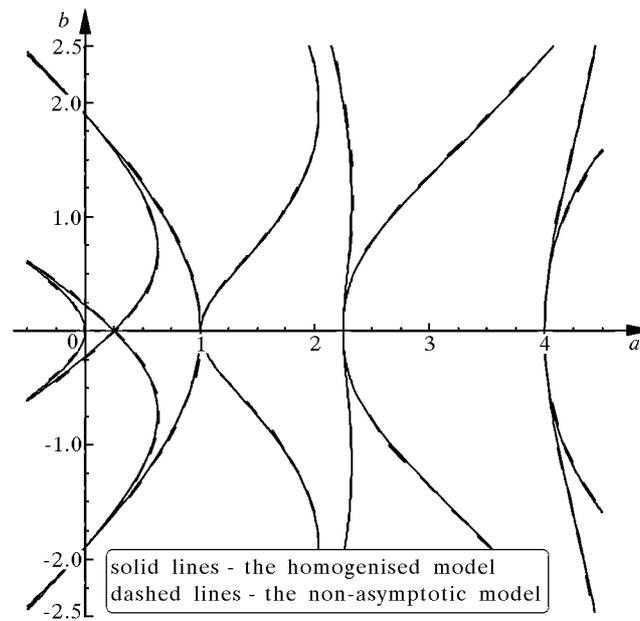
Fig. 3. Curves of relation $a - b$ (boundaries of regions of stable and unstable vibrations)

## 6.2. Solutions to the Mathieu and the generalized Mathieu equations
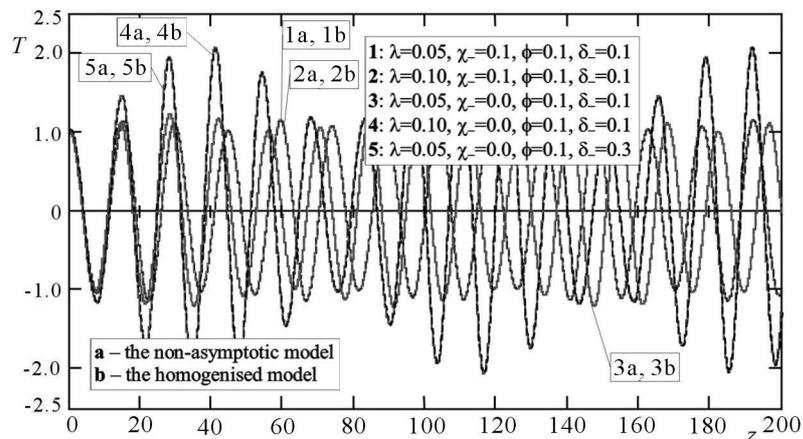


Fig. 4. Solutions to frequency equations for initial conditions at $z = 0$: $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 1$, and parameter $\eta_- = 0.2$
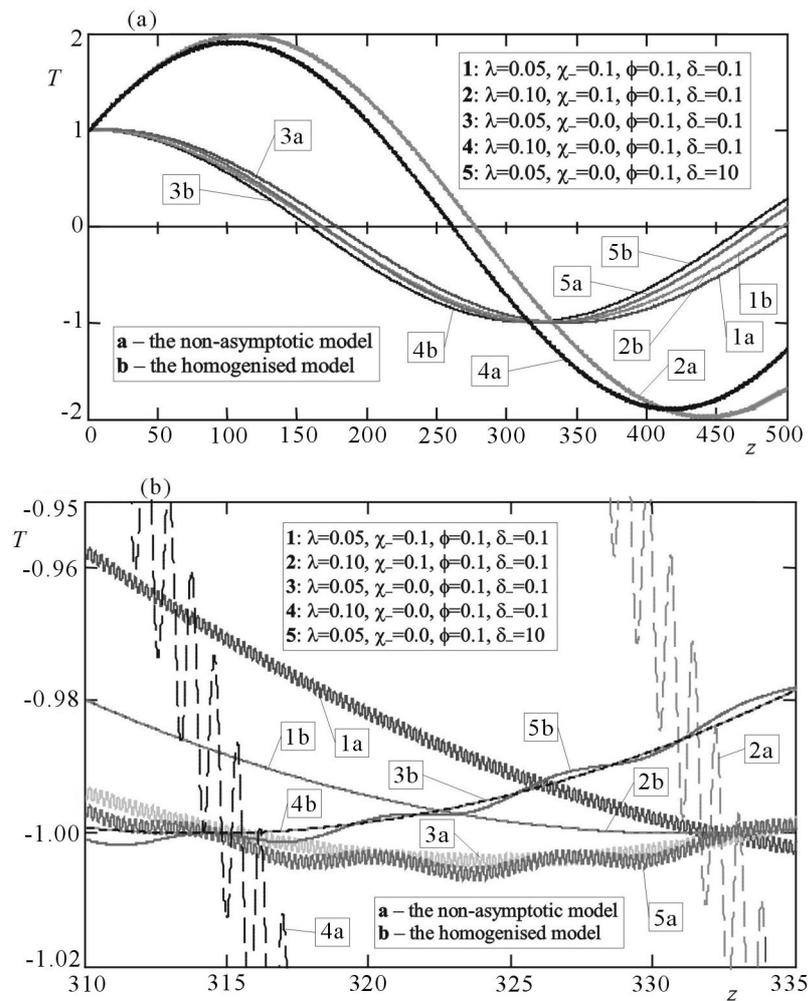
Fig. 5. Solutions to frequency equations for initial conditions at $z = 0$: $T = 1$,
$T' = 0$, $T'' = -1$, $T''' = 1$, and parameter $\eta_- = 10^{-4}$

In Fig. 4-Fig. 8 diagrams of numerical solutions to equations (5.5) and (5.6) are shown. These diagrams are made for different values of parameters: $\eta_0 = 0.1$, $\eta = 0.7$, $\gamma = 0.5$ and $n = 1$; $\lambda = 0.05$, $0.1$; $\phi = 0.1$; $\chi_- = 0.0$, $0.1$; $\delta_- = 0.1$, $0.3$, $10$. In these figures we have curves of solutions for different values of parameter $\eta_-$, Eq. $(5.4)_1$, describing the oscillation frequency of force $N_1$, and different initial conditions at $z = 0$, i.e. in Fig. 4: $\eta_- = 0.2$, $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 1$ for Eq. (5.5) and $T = 1$, $T' = 0$ for Eq. (5.6); in Fig. 5: $\eta_- = 10^{-4}$, $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 1$ for Eq. (5.5) and $T = 1$,
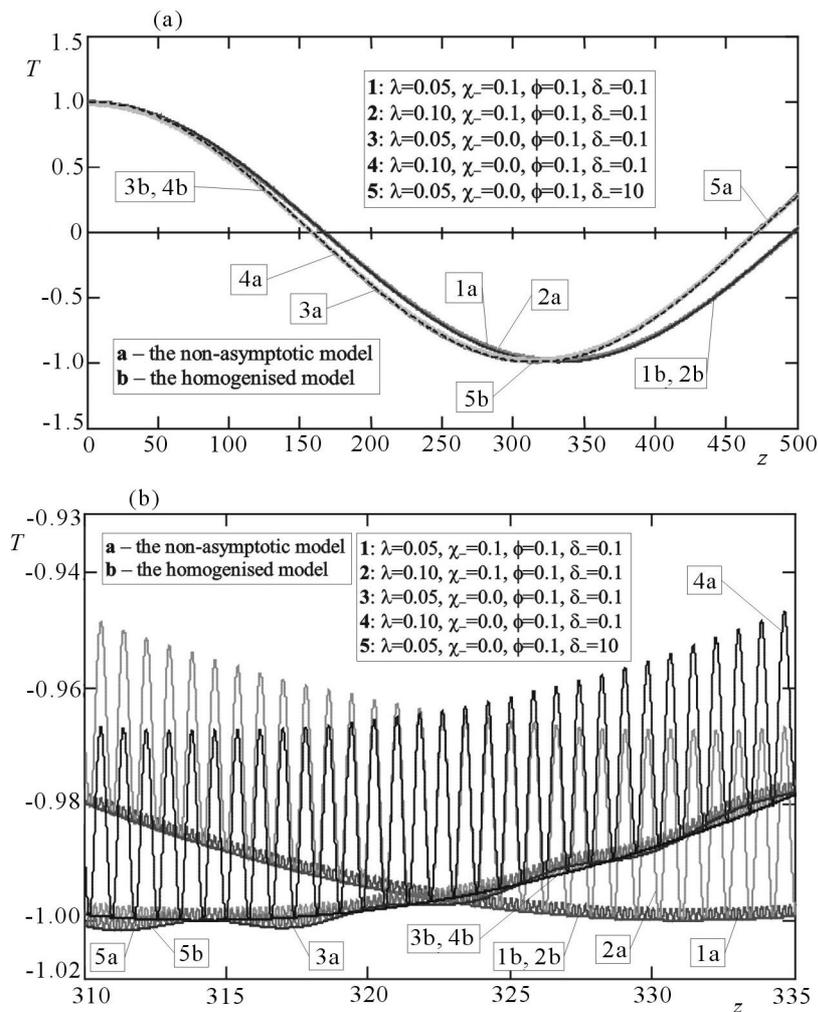
Fig. 6. Solutions to frequency equations for initial conditions at $z = 0$: $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 0$, and parameter $\eta_- = 10^{-4}$

$T' = 0$ for Eq. (5.6); in Fig. 6: $\eta_- = 10^{-4}$, $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 0$ for Eq. (5.5) and $T = 1$, $T' = 0$ for Eq. (5.6); in Fig. 7: $\eta_- = 1.6 \cdot 10^{-6}$, $T = 0$, $T' = 1$, $T'' = 0$, $T''' = 1$ for Eq. (5.5) and $T = 0$, $T' = 1$ for Eq. (5.6); in Fig. 8: $\eta_- = 1.6 \cdot 10^{-6}$, $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 0$ for Eq. (5.5) and $T = 1$, $T' = 0$ for Eq. (5.6). In Fig. 5b and Fig. 6b are shown enlarged fragments of Fig. 5a and Fig. 6a, respectively; however, in Fig. 8c we have an enlarged fragment of Fig. 8a,8b. Diagrams in Fig. 4 are made for $z \in [0, 200]$, in Fig. 5a and Fig. 6a for $z \in [0, 500]$, and in Fig. 7 and Fig 8a,b for $z \in [0, 5000]$.
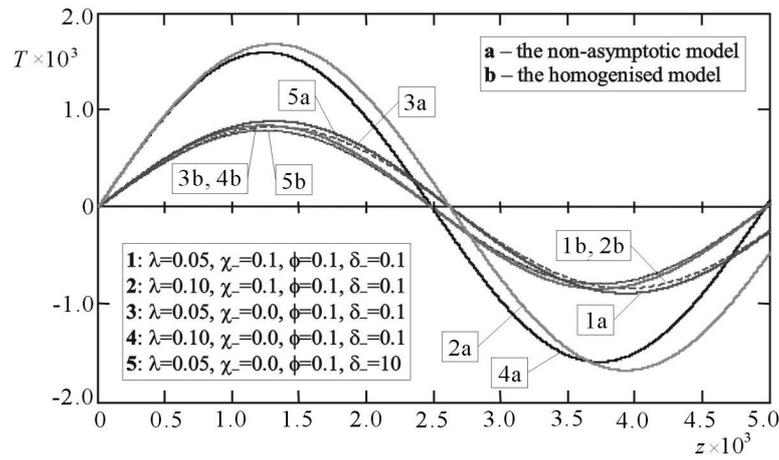
Fig. 7. Solutions to frequency equations for initial conditions at $z = 0$: $T = 0$, $T' = 1$, $T'' = 0$, $T''' = 1$, and parameter $\eta_- = 1.6 \cdot 10^{-6}$

### 6.3.  Discussion of obtained numerical results

From the obtained results of numerical examples some conclusions can be drawn.

- The length-scale effect is negligibly small in the problem of determining the boundaries of regions of stable and unstable vibrations and hence, the homogenised model is sufficient from the point of view of calculation for this problem (Fig. 3).

- Analysing the diagrams of numerical solutions to Eqs (5.5) and (5.6) shown in Fig. 4-Fig. 8 it can be observed:

    – differences between solutions from the non-asymptotic and the homogenised models are negligibly small for additional homogeneous initial conditions of Eq. (5.5) for higher-order derivatives of function $T$, i.e. $T'' = T''' = 0$ (Fig. 4);

    – assuming for Eq. (5.5) additional non-homogeneous initial conditions for higher-order derivatives of solution $T$, i.e. $T''$, $T'''$, and high values of the oscillation frequency $p$ of compressive force $N_1$ (described by small values of the parameter $\eta_-$), differences between solutions from both the models are significant:

        ∗ there are differences between amplitudes of these solutions, in particular for $T''' = 1$ (Fig. 5 and Fig. 7);
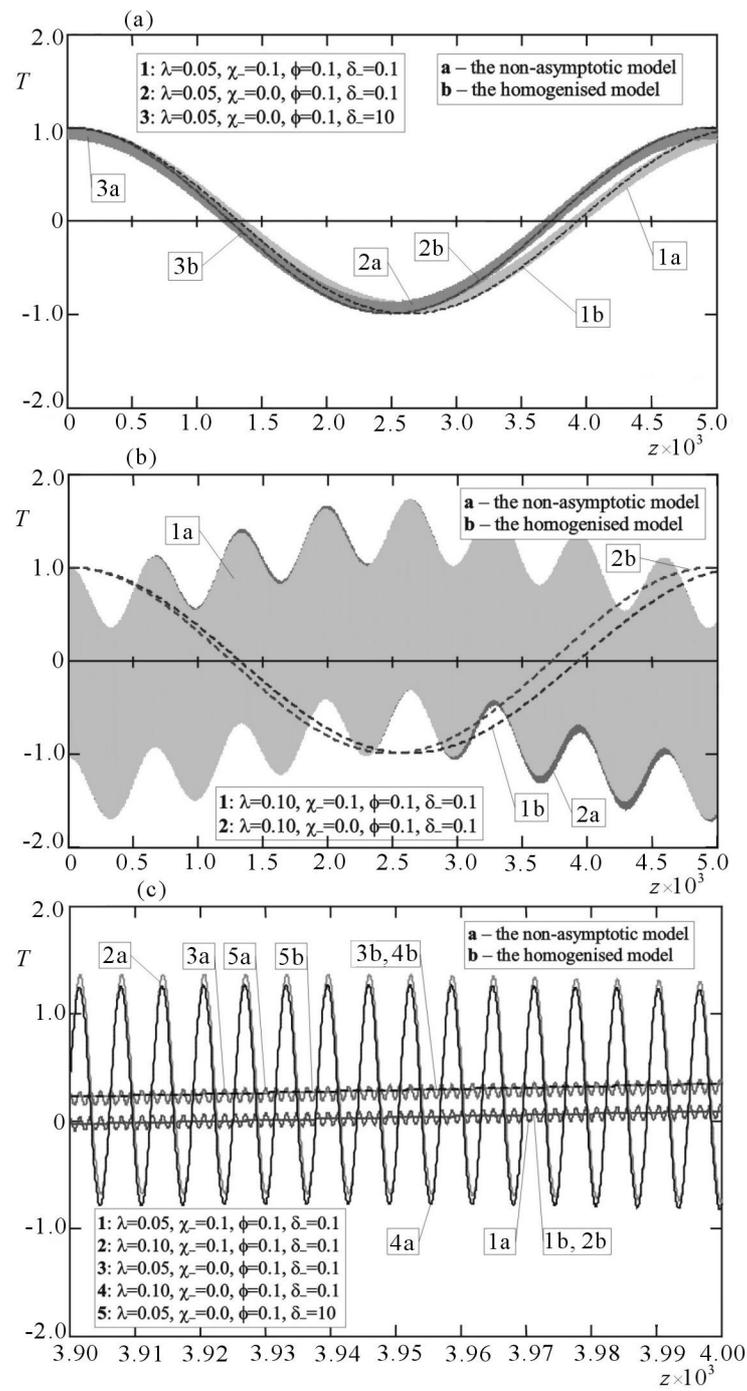
Fig. 8. Solutions to frequency equations for initial conditions at $z = 0$: $T = 1$, $T' = 0$, $T'' = -1$, $T''' = 0$, and parameter $\eta_- = 1.6 \cdot 10^{-6}$

* there are differences of fundamental periods of these solutions (Fig. 5-Fig. 8), in particular for $T'' = -1$, $T''' = 1$ (Fig. 5);

* additional oscillations with small periods appear in solutions $T$ from the non-asymptotic model (Fig. 5b, Fig. 6b and Fig 8c);

* solutions for the non-asymptotic model describe *the phenomenon of beating* for values of the oscillation frequency $p$ of compressive force $N_1$ closer to the higher free vibration frequency $\omega_+$ of this model (Fig. 8b);

* the aforementioned differences increase with increasing oscillation frequency $p$ of compressive force $N_1$ (i.e. for decreasing parameter $\eta_-$) and with increasing parameter $\lambda$, describing the size of the cell;

– large values of amplitude $N_b$ of the compressive force $N_1$ (parameter $\delta_-$) cause additional oscillations in both of the solutions (Fig. 5b and Fig. 6b – curves 5a and 5b).

## 7.  Remarks

It has to be emphasized that the applied modelling approach, i.e. the tolerance averaging technique (Woźniak and Wierzbicki, 2000), different from the known homogenisation methods used for periodic plates, leads to the non-asymptotic models, which make it possible to investigate the effect of the period lengths on the overall plate behaviour (cf. Jędrysiak, 2001, 2003a,b). The main advantage of these models is that the analysed problems are described by relatively simple differential equations with constant coefficients. Thus, the non-asymptotic models can be used to analyse many engineering problems. Moreover, for the proposed non-asymptotic model of thin periodic plates, the conditions of the physical correctness of solutions $W$, $Q^A$ are determined, i.e. *the macrodeflection* $W$ and *the internal variables* $Q^A$, $A = 1, ..., N$, are slowly varying functions.

In this paper, using this model, the effect of the period lengths on dynamic stability problems for Kirchhoff-type plates with periodic structure is taken into account. Applying this model, parametric vibrations of such plates can be considered. This problem can be also extended to problems of loads moving on periodic plates, which can be the equivalent dynamical compressive forces acting in the plate midplane (cf. Szcześniak, 1992).

Summarizing our considerations, the following conclusions can be formulated:

- Taking into account *the effect of the period lengths* on dynamic stability for thin periodic plates leads to *the fourth-order differential equation* for the unknown function of time co-ordinate, which can be treated as *a certain generalization of the Mathieu equation.* On the contrary, within *the homogenised model the known Mathieu equation* is obtained.

- In the framework of the non-asymptotic model proposed in this contribution, the additional higher critical forces can be analysed.

- From numerical solutions to the generalized Mathieu equation it can be observed that the effect of the period lengths plays a crucial role for high values of the oscillation frequency $p$ of compressive force $N_1$, which is manifested in:

  - different fundamental periods of the solution $T$ in the dimensionless time co-ordinate $z$, for additional non-homogeneous initial conditions imposed on the higher-order derivatives of $T$, i.e. on $T''$, $T'''$;

  - large differences between amplitudes of the function $T$, for non-homogeneous additional initial conditions imposed on $T''$, $T'''$;

  - additional oscillations of the solution $T$ with very small periods in $z$, for non-homogeneous additional initial conditions imposed on $T''$, $T'''$;

  - the phenomenon called *the beating* described by the function $T$ for some non-homogeneous additional initial conditions imposed on $T''$, $T'''$ and values of the oscillation frequency $p$ closed to higher free vibration frequency $\omega_+$ by the non-asymptotic model.

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## Zastosowanie metody tolerancyjnego uśredniania do analizy stateczności dynamicznej cienkich płyt periodycznych

Streszczenie

W pracy rozpatrzono pewne zagadnienia dotyczące stateczności dynamicznej cienkich płyt o periodycznej budowie. Przy wyprowadzeniu równań uśrednionego modelu nieasymptotycznego wykorzystano tolerancyjne uśrednianie, zaproponowane przez Woźniaka i Wierzbickiego (2000) do modelowania kompozytów i struktur periodycznych. Zastosowanie tej metody do znanego równania teorii płyt Kirchhoffa prowadzi do modeli uśrednionych, w których uwzględniony jest wpływ długości okresów periodyczności na pracę płyty (Jędrysiak, 2001). W pracy sformułowano nieasymptotyczny model opisujący zagadnienia stateczności dynamicznej płyt periodycznych. Pokazano również, że wpływ długości okresów periodyczności odgrywa znaczącą rolę w pewnych szczególnych przypadkach stateczności dynamicznej takich płyt, np. dla wysokich częstości oscylacji sił ściskających, przyłożonych w płaszczyźnie środkowej płyty.