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ENERGETIC METHOD OF SOLVING THE STABILITY PROBLEM OF A SEMI-SPHERICAL SHELL LOADED WITH TORQUE[1]

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A thin-walled spherical shell is pivoted at both ends. The upper edge of the shell, loaded with a torque, may rotate around the shell axis. The problem of the loss of stability of the shell is solved with an energetic method. The change in the total energy of the shell while losing stability is determined. This requires the forms of the deflection and force functions to be assumed, according to actual boundary conditions. Coefficients of the force function are determined from the solution to the inseparability equation with the Bubnov-Galerkin method. The stability equation of the shell is formulated as a result of application of the Ritz method to the total energy variation. It is an algebraic equation serving for determination of the critical load. It is equal to the minimal value of the load. The work ends with a numerical example.

Key words: shells, non-linear stability

1. Introduction

A thin-walled spherical shell being a subject of the analysis is shown in Fig. 1. Its bottom edge is fixed and pivoted. The upper edge is also pivoted but may rotate around the vertical axis of the shell. The upper edge is loaded with a torque. A non-linear problem of the loss of stability is considered. The problem is solved with an energetic method. For the assumed modes of the deflection and force function the Bubnov-Galerkin method is used in order to solve the inseparability equation and, afterwards, the total energy is calculated.

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In order to formulate the stability equation the Ritz method is applied. The final goal of the work consists in the determination of the critical load. This, however, is possible only in the case of a numerical example, since the problem is of high complexity.

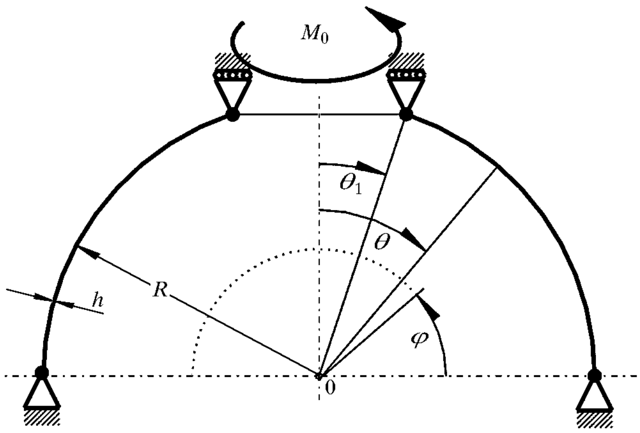

Fig. 1. Scheme of the shell

2. Mathematical description of the problem

2.1. Total energy of the system

The total energy of the system while losing the stability amounts to

$$V = U_1 + U_2 - L \tag{2.1}$$

where U_1 is the energy of the membrane state, U_2 – energy of the bending state, L – work of external forces and

$$U_1 = \frac{1}{Eh} \int\limits_A [T_1^2 + T_2^2 - 2\nu T_1 T_2 + 2(1+)S] \; dA$$

$$U_2 = \frac{Eh^3}{24(1-\nu^2)} \int\limits_A [\kappa_{11}^2 + \kappa_{22}^2 + 2\nu\kappa_{11}\kappa_{22} + 2(1-\mu)\kappa_{12}^2] \; dA$$

$$L = \iint \overline{S} \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial \varphi} \; d\theta d\varphi$$

A is the middle surface of the shell, w – deflection function upon the loss stability, κ_{11}, κ_{22}, κ_{12} are variations in the main curvatures of the spherical shell (Mushtari and Galimov, 1957), T_1, T_2, S are forces of the bending state, Ψ denotes the force function and

$$\kappa_{11} = -\frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2} \qquad \kappa_{22} = -\frac{1}{R^2}\Big(\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} + \cot\theta\frac{\partial w}{\partial\theta}\Big)$$

$$\kappa_{12} = \frac{1}{R^2\sin\theta}\Big(\cot\theta\frac{\partial w}{\partial\varphi} - \frac{\partial^2 w}{\partial\varphi\partial\theta}\Big) \qquad T_1 = \frac{1}{R^2}\Big(\frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} + \cot\theta\frac{\partial\Psi}{\partial\theta}\Big)$$

$$T_2 = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2} \qquad S = \frac{1}{R^2\sin\theta}\Big(\cot\theta\frac{\partial\Psi}{\partial\varphi} - \frac{\partial^2\Psi}{\partial\varphi\partial\theta}\Big)$$

The force of the membrane state in (2.1) has the form

$$\overline{S} = \frac{M_0}{2\pi R^2\sin^2\theta} \tag{2.2}$$

2.2. Inseparability equation

In order to determine the forces of the bending state, the force functions must be known. This requires solving the equation of the inseparability of displacements. The inseparability equation is of the form (Mushtari and Galimov, 1957)

$$\nabla^2\nabla^2\Psi - Eh(\kappa_{12}^2 - \kappa_{11}\kappa_{22} - \kappa_{11}k_{22} - \kappa_{22}k_{11}) = 0 \tag{2.3}$$

where

$$\nabla^2 = \frac{1}{R^2}\Big(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Big)$$

and k_{ii} are main curvatures.

3. Boundary conditions

The following conditions should be met at the shell boundaries

$$\begin{array}{lll} \theta = \theta_1 & w = 0 & M_\theta = 0 \\ S = \dfrac{M_0}{2\pi R^2\sin^2\theta_1} & T_1 = 0 & \end{array} \tag{3.1}$$

and

$$\begin{aligned} &\theta = \frac{\pi}{2} \qquad && w = 0 \qquad && M_\theta = 0 \\ &S = \frac{M_0}{2\pi R^2} \qquad && T_1 = 0 \end{aligned} \tag{3.2}$$

4. Deflection and force functions

The following forms of the force and deflection functions were assumed

$$\begin{aligned} &\Psi = [b\varphi + c\sin(m\varphi)]\sin^2\theta \\ &w = a\sin\frac{2\pi(\theta-\theta_1)}{\pi-2\theta_1}\sin\Big[\frac{2\pi(\theta-\theta_1)}{\pi-2\theta_1} + m\varphi\Big]\sin^2\theta \end{aligned} \tag{4.1}$$

where a, b, c are constants, m – an integer number defining the mode of the loss of stability.

The deflection function explicitly satisfies the first two conditions ((3.1) and (3.2)), while the third one is satisfied in the integral sense. On the other hand, the force function meets the third condition to a constant, without satisfying the condition required for the normal force T_1.

5. Solution to the equation of inseparability

Equation of inseparability (2.3) is solved with the Bubnov-Galerkin method. The ortogonalization conditions are of the following form

$$\begin{aligned} &\int\limits_{\theta_1}^{\frac{\pi}{2}}\int\limits_0^{2\pi} F(\theta,\varphi)\varphi\sin^3\theta\; d\theta d\varphi = 0 \\ &\int\limits_{\theta_1}^{\frac{\pi}{2}}\int\limits_0^{2\pi} F(\theta,\varphi)\sin(m\varphi)\sin^3\theta\; d\theta d\varphi = 0 \end{aligned} \tag{5.1}$$

where $F(\theta,\varphi)$ denotes the left-hand side of equation (2.3), with force function components $(4.1)_1$ applied as ortogonalization factors.

Once equations (5.1) and (5.2) are solved, the following expressions for constants b and c are obtained

$$b = Eh(H_1 a^2 + G_1 aR) \qquad\qquad c = Eh(H_2 a^2 + G_2 aR) \tag{5.2}$$

where H_i, G_i are constants.

The final form of the force function is as follows

$$\Psi = Eha^2\Big[\Big(H_1 + G_1\frac{R}{a}\Big)\varphi + \Big(H_2 + G_2\frac{R}{a}\Big)\sin(m\varphi)\Big]\sin^2\theta \tag{5.3}$$

6. Solution to the problem of the loss of stability

The deflection and force functions should be substituted into the equation of total energy variation in order to calculate the variation itself. The total energy is a function of a. While losing the stability, the variation of the total energy takes the minimal value. According to the Ritz method, the condition for the minimum of the total energy variation has the form

$$\frac{\partial V}{\partial a} = 0 \tag{6.1}$$

The implementation of the above expression leads to the formulation of an equation of the dimensionless torque

$$M = \frac{M_0}{Eh^3} = C_1\Big(\frac{a}{h}\Big)^2 + C_2\frac{R}{h}\frac{a}{h} + C_3\Big(\frac{R}{h}\Big)^2 + C_4 \tag{6.2}$$

where C_i are constants depending on θ_1, the number m and Poisson's ratio ν.

Equation (6.2) is an answer to the problem of the loss of stability. It should serve for calculating the critical load. The critical load corresponds to the minimal value of the dimensionless load parameter M. It is determined for fixed dimensions of the shell, defined by the number m. For solving ortogonalization conditions (5.1) and equation (6.1) the *Derive* software was used. It includes procedures enabling transformations of algebraic expressions, differentiating, and integrating, thus enabling all the operations required by the above equations.

7. Numerical example

The critical load can be found only with a numerical method. This is a consequence of the complex structure of coefficients in equation (6.2). First of all, the dimensions of the shell should be assumed in such a way so that to allow for calculating the constants C_i corresponding to different values of the number m. This was carried out with the *Derive* program. The critical load was sought by means of a graphical method. The plots should be drawn in the $M - a/h$ coordinates for a series of the numbers m. The plots enabled finding the minimal value of the load M, i.e. M_{cr}. The value m corresponding to the minimal level of the load M is considered as the critical one m_{cr}. The plots were drawn with the help of the *Derive for Windows* software. Figure 2 shows plots of the dimensionless torque M as a function of a/h for a shell with the dimensions $\theta_1 = \pi/6$, $R/h = 150$, $\nu = 0.3$ and different values of m. The minimal value of the load $M = M_{cr}$ is found for $m = m_{cr} = 8$. It should be noticed that for $\theta_1 = \pi/10$, and $\theta_1 = 3\pi/14$ the problem remains unsolved. This is certainly a result of the assumed forms of the deflection and force functions.

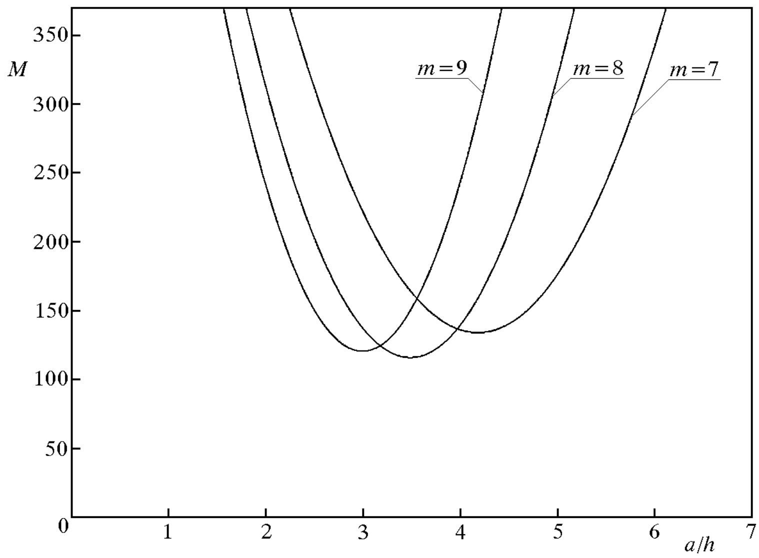

Fig. 2. Diagrams m-a/h

The critical loads of the shells of different dimensions are presented in Table 1. Apart from the M_{cr} values, the numbers m_{cr} are provided in brackets.

The comparison of the critical moments, contained in Table 1, allows one to come to an obvious conclusion that the growing ratio R/h decreases the resistance of the shell to the loss of stability.

Table 1. Values of the critical moments

R/h	M_{cr}	
	$\theta_1 = \pi/12$	$\theta_1 = \pi/6$
100	115.6 (8)	104.4 (8)
150	98.4 (10)	98.1 (10)
200	20.0 (11)	47.8 (11)

The same problem was solved in the paper by Joniak (2003a) with the Bubnov-Galerkin method. The solution, however, was unsatisfactory, being valid only for delimited range of shell dimensions. On the other hand, the solution shown in the present paper is free of this fault.

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Metoda energetyczna rozwiązania problemu stateczności powłoki półkulistej obciążonej momentem obrotowym

Streszczenie

Cienkościenna powłoka półkulista jest podparta przegubowo na obu brzegach. Górny brzeg powłoki ma możliwość obrotu wokół osi powłoki; do tego brzegu przyłożony jest moment obrotowy. Rozpatrywany jest problem utraty stateczności tej powłoki metodą energetyczną. Wyznacza się zmianę energii całkowitej powłoki podczas utraty stateczności. Wymaga to przyjęcia postaci funkcji ugięcia po utracie stateczności i funkcji sił odpowiednich do warunków brzegowych. Współczynniki funkcji sił wyznacza się z rozwiązania równania nierozdzielności metodą Bubnowa-Galerkina. Równanie utraty stateczności powstaje po zastosowaniu do całkowitej zmiany energii metody Ritza. Jest to równanie algebraiczne, z którego wyznacza się obciążenie krytyczne; odpowiada ona minimalnej wartości parametru obciążenia.

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