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ON AUTOFRETTAGE OF CYLINDERS BY LIMITING CIRCUMFERENTIAL RESIDUAL STRESS BASED ON MISES YIELD CRITERION

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The autofrettage technique is an effective and important measure to improve load-bearing capacity and safety and to even distributions of stresses for pressure vessels. Based on the classical fundamental theory on autofrettage and by theoretical analysis of residual stresses, the total stresses, the overstrain and the load-bearing capacity, etc., in view of the cylindrical pressure vessels with outside-to-inside radius ratio larger than the critical ratio, the laws contained in the autofrettage theory are revealed, the essential cause and reason for the obtained laws are analyzed, the inherent and meaning relations between various parameters in the autofrettage theory are brought to light, and the safe depth of the plastic zone (overstrain) as well as the conditions of loading or optimum operation conditions are found out. It is shown that under the optimum operation conditions, pressure vessels are not only safe but also economic, and the equations of the autofrettage theory are simplified greatly and are quite terse, as a result, the essential relations between various parameters in the autofrettage theory are distinct, and these equations are convenient for application in engineering practice.

Key words: pressure vessel, autofrettage, load-bearing capacity, strength theory

Nomenclature

r_i, r_j, r_o	–	Inside radius, radius of elastic-plastic juncture, outside radius, respectively
k	–	outside to inside radius ratio $(k = r_o/r_i)$
k_j	–	depth of plastic zone $(k_j = r_j/r_i)$
k_c	–	critical radius ratio
p, p_a, p_y	–	internal, autofrettage and entire yield pressure, respectively
p_e	–	maximum elastic load-bearing capability of unautofrettaged pressure vessel or initial yield pressure
$\sigma_r, \sigma_\theta, \sigma_z, \sigma_e$	–	radial, circumferential, axial and equivalent stress, respectively
σ_y	–	yield strength

Superscripts: $', T, p$ – residual, total and stress caused by p, respectively.

1. Introduction

Cylindrical pressure vessels are important mechanical equipment and are widely applied in the industry. To raise load-bearing capacity of a pressure vessel and even distribution of stresses in the wall of a pressure vessel, the "autofrettage" technique for thick pressure vessels was created, which is an interesting and absorbing subject. Parker (2001a,b) extended a procedure to model the addition of pressure or material to a tube, providing an associated residual stress profile. Hameed *et al.* (2004) studied the residual stress distribution in an autofrettaged thick--walled cylinder. Maleki *et al.* (2010) evaluated residual stress distributions in autofrettaged homogenous spherical pressure vessels subjected to different autofrettage pressures. Park *et al.* (2008) presented an analytical and numerical study of a machined compound cylinder by using the finite element code ANSYS10.0. Lee *et al.* (2008), conducted residual stress analysis of an autofrettaged compound cylinder based on elastic-perfectly plastic and strain hardening models. Farrahi *et al.* (2009) presented an analytical method for reconstructing residual stresses within axially symmetric cylinders. The optimum autofrettage pressure of a thick-walled cylinder under thermo-mechanical loadings was analyzed by Zheng and Xuan (2010).

The autofrettage damage mechanics model was setup from the ultra-high pressure vessel autofrettage damage mechanism (Lin *et al.*, 2009) and a brief introduction of the autofrettage method, such as mechanical extrusion, direct pressure, expansion of the explosion pressure and solid autofrettage was given by Gao *et al.* (2008). In terms of the point of view of avoiding compressive yield for cylinders experiencing autofrettage handling and raising load-bearing capacity as far as possible simultaneously, Zhu (2008a,b) studied autofrettage of cylinders based on the 3rd and 4th strength theory, respectively. By studying the results published by Zhu (2008a,b), it is found that there are still some problems needed to be studied further, especially when the strength theory (Mises yield criterion) is applied to study the autofrettage for pressure vessels. It is also necessary to pay attention to some problems, such as overstrain to avoid compressive yield, load-bearing capacity, safe conditions for operation, and so on. Based on the work by Zhu (2008a,b), this paper discusses the stresses and depth of the plastic zone (overstrain), which is done in Section 2 and followed by conclusions in Section 3.

2. Discussion about stresses and safe conditions for operation

Figure 1 shows a cross section of a cylinder wall with partially plastic and partially elasticity.

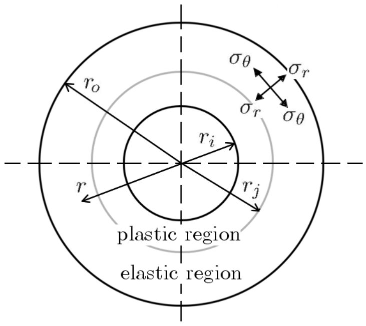

Fig. 1. Cross section of a cylinder wall with marked regions of partial plasticity and partial elasticity

The residual stress at a general radial location within the plastic zone (Zhu, 2008a,b)

$$
\begin{aligned}
\frac{\sigma_z'}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big] \\
\frac{\sigma_r'}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} - 1 + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\Big] \\
\frac{\sigma_\theta'}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} + 1 + \ln\frac{(r/r_i)^2}{k_j^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\Big]
\end{aligned}
\tag{2.1}
$$

Therefore, the equivalent residual stress at a general radial location within the plastic zone is (Zhu, 2008a,b)

$$\frac{\sigma'_e}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta}{\sigma_y} - \frac{\sigma'_r}{\sigma_y}\Big) = 1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2-1)(r/r_i)^2} \tag{2.2}$$

which is the same as that based on the 3rd strength theory (Tresca yield criterion). The reason is as follows.

For a cylinder with closed ends $\sigma_z = (\sigma_r + \sigma_\theta)/2$ (Yu, 1990), then, under the identical stress state and according to the 3rd and 4th strength theories, there is: $\sigma_e^{\rm IV} = \sqrt{3}\sigma_e^{\rm III}/2$, while each component of the residual stresses based on the 4th strength theory is $2/\sqrt{3}$ times of that based on the 3rd strength theory (Zhu, 2008a,b) for the same k and k_j. This is the reason why the equivalent residual stress based on the 3rd strength theory equals that based on the 4th strength theory.

The residual stress at a general radial location within the elastic zone is (Zhu, 2008a,b)

$$\begin{aligned}
\frac{\sigma'_z}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big]\\
\frac{\sigma'_r}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\Big[\frac{k_j^2}{k^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big] = \Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y}\\
\frac{\sigma'_\theta}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\Big[\frac{k_j^2}{k^2} - \Big(1 - \frac{k_j^2}{k^2} + \ln k_j^2\Big)\frac{1}{k^2-1}\Big] = \Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y}
\end{aligned} \tag{2.3}$$

Therefore, the equivalent residual stress at a general radial location within the elastic zone is

$$\frac{\sigma'_e}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta}{\sigma_y} - \frac{\sigma'_r}{\sigma_y}\Big) = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)(r/r_i)^2} \tag{2.4}$$

which is also the same as that based on the 3rd strength theory. The reason is as above mentioned.

In the whole elastic zone $\sigma'_e/\sigma_y > 0$ for $k_j^2 - 1 > \ln k_j^2$ [14], but in the plastic zone σ'_e/σ_y may be positive or negative. For example, on the inside surface of the cylinder where $r/r_i = 1$, from Eq. (2.2), the equivalent residual stress is

$$\frac{\sigma'_{ei}}{\sigma_y} = \frac{k_j^2 - 1 - k^2 \ln k_j^2}{k^2 - 1} \tag{2.5}$$

Since $\ln k_j^2 = 2(k_j^2 - 1)/(k_j^2 + 1) + \ldots$ (Zhu, 2008a,b), $\ln k_j^2 > 2(k_j^2 - 1)/(k_j^2 + 1)$, hence, $k^2 \ln k_j^2 > 2k^2(k_j^2 - 1)/(k_j^2 + 1) > k_j^2 - 1$.

This means that on the inside surface of the cylinder σ'_e/σ_y is always negative. It is not difficult to know that $d\sigma'_e/dr > 0$ within the plastic zone and $d\sigma'_e/dr < 0$ within the elastic zone, and, at the elastic-plastic juncture where $r/r_i = k_j$, both Eq. (2.2) and (2.4) become

$$0 < \frac{\sigma'_e}{\sigma_y} = \frac{\sigma'_{ej}}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)k_j^2} < 1 \tag{2.6}$$

Therefore, within the plastic zone σ'_e/σ_y goes up from the negative value on the inside surface till a positive value at the elastic-plastic juncture, and goes down within the elastic zone from the positive value at the elastic-plastic juncture to a smaller positive value on the outside surface.

In Eq. (2.2), letting

$$1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2-1)(r/r_i)^2} = 0$$

one obtains

$$\frac{r}{r_i} = \sqrt{\frac{k^2 - k_j^2 + k^2 \ln k_j^2}{k^2 - 1}} = \sqrt{\frac{\sigma'_{ei}}{\sigma_y} + 1} < k_j \tag{2.7}$$

On the other hand, the solution for $\sigma'_z = \sigma'_r$, $\sigma'_r = \sigma'_\theta$ and $\sigma'_\theta = \sigma'_z$ within the plastic zone (see Eqs. (2.1)) is also Eq. (2.7). This means that the three curves for the residual stress at a general radial location collect at one point within the plastic zone, and the abscissa of the intersection is just Eq. (2.7), where $\sigma'_e/\sigma_y = 0$.

Since the depth of the plastic zone (k_j) affects the residual stresses and the greater the k_j, the greater the equivalent residual stress, so it is inadvisable to raise k_j, otherwise, compressive yield may occur after removing the autofrettage pressure p_a. However, raising k_j can raise load--bearing capacity of the cylinder thereupon, Zhu (2008a,b) put forward an equation for k_{j^*}, the maximum and optimum plastic depth k_j for a certain k to avoid the compressive yield

$$k^2 \ln k_{j^*}^2 - k^2 - k_{j^*}^2 + 2 = 0 \qquad \text{or} \qquad k = \sqrt{\frac{k_{j^*}^2 - 2}{\ln k_{j^*}^2 - 1}} \qquad (k_{j^*} \geqslant \sqrt{e}) \tag{2.8}$$

where $\sqrt{e} \leqslant k_{j^*} \leqslant k_c = 2.2184574899167\ldots$ and $k \geqslant 2.2184574899167\ldots$, when $k \leqslant 2.2184574899167\ldots = k_c$, $|\sigma'_{ei}/\sigma_y| \geqslant 1$ never occurs irrespective of k_j.

When k_j is determined by Eq. (2.8) – $k_j = k_{j^*}$, Eqs. (2.1)-(2.4) can be rearranged as follows within the plastic zone

$$\begin{aligned} &\frac{\sigma'_z}{\sigma_y} = \frac{\ln(r/r_i)^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{\ln x^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} && \frac{\sigma'_r}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} + \frac{2}{\sqrt{3}x^2} - \frac{2}{\sqrt{3}} \\ &\frac{\sigma'_\theta}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} - \frac{2}{\sqrt{3}x^2} && \frac{\sigma'_e}{\sigma_y} = 1 - \frac{2}{(r/r_i)^2} \end{aligned} \tag{2.9}$$

within the elastic zone

$$\begin{aligned} &\frac{\sigma'_z}{\sigma_y} = \frac{k_j^2 - 2}{\sqrt{3}k^2} && \frac{\sigma'_r}{\sigma_y} = \Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y} = \Big(1 - \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} \\ &\frac{\sigma'_e}{\sigma_y} = \frac{k_j^2 - 2}{(r/r_i)^2} && \frac{\sigma'_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y} = \Big(1 + \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} \end{aligned} \tag{2.10}$$

where x represents r/r_i, the same below.

By taking $k = 3$, $k_j = 1.5, 1.748442$ and 2, respectively, the residual stresses are illustrated as shown in Fig. 2.

In Fig. 2b, $k_j = 1.748442 = k_{j^*}$, as is just determined by Eq. (2.8). $k_j = 1.5 \neq k_{j^*}$ in Fig. 2a, and $k_j = 2 \neq k_{j^*}$ in Fig. 2c or none of them is determined by Eq. (2.8). The former residual stresses are less than those when $k_j = 1.748442 = k_{j^*}$, and the latter residual stresses are greater than those when $k_j = 1.748442 = k_{j^*}$.

Substituting Eq. (2.7) into Eq. $(2.9)_{1-3}$ one obtains

$$\frac{\sigma'_z}{\sigma_y} = \frac{\sigma'_r}{\sigma_y} = \frac{\sigma'_\theta}{\sigma_y} = \frac{\ln 2 - 1}{\sqrt{3}}$$

When $k_j = k_{j^*}$, Eq. (2.7) becomes $r/r_i = \sqrt{2}$. This means that when $k_j = k_{j^*}$, for any k, the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is

$$\Big(\sqrt{2}, \frac{\ln 2 - 1}{\sqrt{3}}\Big) = (1.414214\ldots, -0.17716\ldots)$$

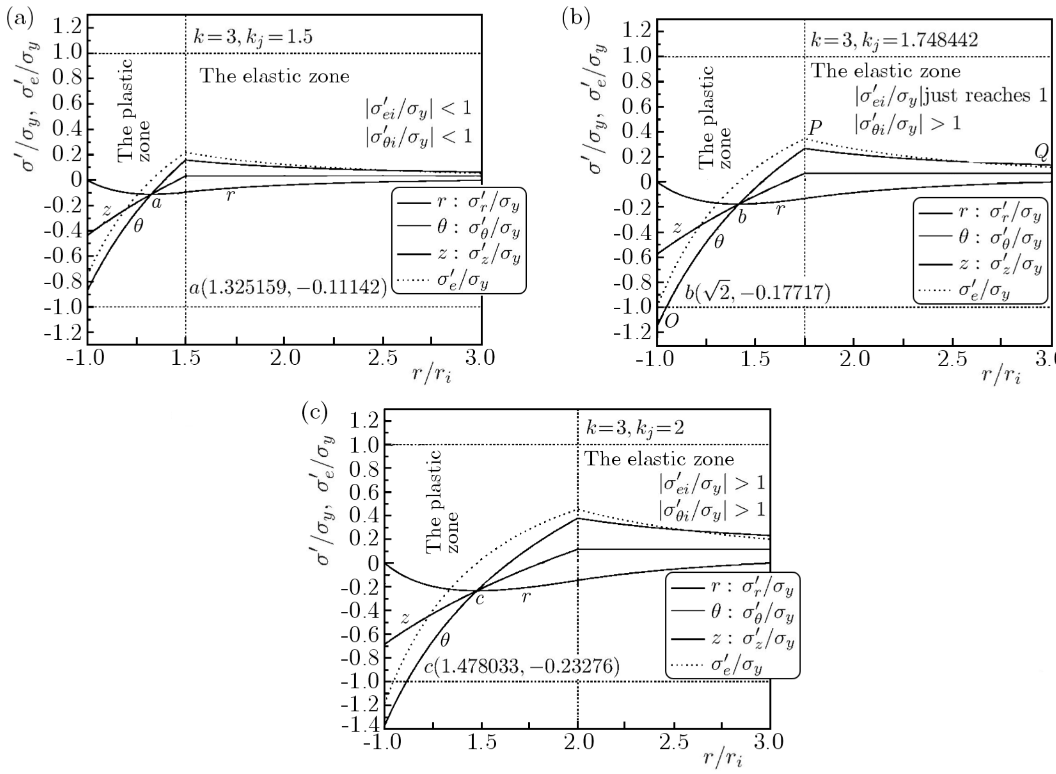

Fig. 2. Distributions of the residual stresses

The residual stresses at the inside surface $(r = r_i)$ can be obtained by letting $r = r_i$ in Eqs. (2.1) or (2.3)

$$
\begin{aligned}
&\frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{\sqrt{3}(k^2-1)}(1 - k_j^2 + k^2 \ln k_j^2) < 0 \qquad && \frac{\sigma'_{ri}}{\sigma_y} = 0 \\
&\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{2}{\sqrt{3}(k^2-1)}(1 - k_j^2 + k^2 \ln k_j^2) = 2\frac{\sigma_{zi}}{\sigma_y}
\end{aligned}
\tag{2.11}
$$

The equivalent residual stress at r_i is

$$
\frac{\sigma'_{ei}}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma'_{\theta i}}{\sigma_y} - \frac{\sigma'_{ri}}{\sigma_y}\Big) = \sqrt{3}\frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{k^2-1}(1 - k_j^2 + 2k^2 \ln k_j)
\tag{2.12}
$$

If $k_j = k_{j^*}$ then, Eqs. (2.11) and (2.12) become

$$
\frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{\sqrt{3}} \qquad \frac{\sigma'_{ri}}{\sigma_y} = 0 \qquad \frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{2}{\sqrt{3}} < -1 \qquad \frac{\sigma'_{ei}}{\sigma_y} = \frac{\sigma'_{ej}}{\sigma_y} = -1
\tag{2.13}
$$

Therefore, if $k_j = k_{j^*}$, the residual stresses and their equivalent stress at the inside surface are always compressive and constant irrespective of k and k_j or the thickness and overstrain of the cylinder. The equivalent residual stress at r_i just reaches $-\sigma_y$. However, $|\sigma'_{\theta i}|$ exceeds the compressive yield strength σ_y. This is an inexorable law, for $\sigma'_{ei} = \sqrt{3}(\sigma'_{\theta i} - \sigma'_{ri})/2$, at the inside surface $\sigma'_{ri} = 0$, then $\sigma'_{ei} = \sqrt{3}\sigma'_{\theta i}/2$, when $\sigma'_{ei}/\sigma_y = -1$, there must be $\sigma'_{\theta i} = -(2/\sqrt{3})\sigma_y < -\sigma_y$.

The residual stresses at elastic-plastic juncture $(r = r_j)$ can be obtained by letting $r = r_j$ in Eqs. (2.1) or (2.3)

$$
\begin{aligned}
\frac{\sigma'_{zj}}{\sigma_y} &= \frac{1}{\sqrt{3}}\Big[\frac{r_j^2}{r_0^2} - \Big(1 - \frac{r_j^2}{r_0^2} + 2\ln\frac{r_j}{r_i}\Big)\frac{1}{k^2-1}\Big] = \frac{1}{\sqrt{3}}\Big[\frac{k_j^2}{k^2} - \Big(1 - \frac{k_j^2}{k^2} + 2\ln k_j\Big)\frac{1}{k^2-1}\Big] \\
&= \frac{k_j^2 - 1 - \ln k_j^2}{\sqrt{3}(k^2-1)} \\
\frac{\sigma'_{rj}}{\sigma_y} &= \Big(1 - \frac{k^2}{k_j^2}\Big)\frac{\sigma'_{zj}}{\sigma_y} \qquad \frac{\sigma'_{\theta j}}{\sigma_y} = \Big(1 + \frac{k^2}{k_j^2}\Big)\frac{\sigma'_{zj}}{\sigma_y}
\end{aligned}
\tag{2.14}
$$

The equivalent residual stress at r_j (σ'_{ej}/σ_y) is

$$
\frac{\sigma'_{ej}}{\sigma_y} = \frac{2k^2}{k_j^2}\frac{\sigma'_{zj}}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{k_j^2(k^2-1)} = 1 - \frac{1}{k^2-1}\Big(\frac{k^2}{k_j^2} - 1 + \frac{2k^2}{k_j^2}\ln k_j\Big) = \frac{k_j^2-1}{k_j^2} - \frac{\sigma'_{ei}/\sigma_y}{k_j^2}
\tag{2.15}
$$

In Eq. $(2.14)_1$, when $k_j = 1$, the numerator $= 0$, and $d(k_j^2 - 1 - \ln k_j^2)/dk_j = 2(k_j^2-1)/k_j > 0$, so, $\sigma'_{zj} > 0$ or tension. Then, $\sigma'_{rj} < 0$, $\sigma'_{\theta j} > 0$.

If $k_j = k_{j^*}$, then Eqs. (2.14) and (2.15) become

$$
\begin{aligned}
\frac{\sigma'_{zj}}{\sigma_y} &= \frac{\ln k_j^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \qquad & \frac{\sigma'_{rj}}{\sigma_y} &= \Big(1 - \frac{k^2}{k_j^2}\Big)\frac{\sigma'_{zj}}{\sigma_y} \\
\frac{\sigma'_{\theta j}}{\sigma_y} &= \Big(1 + \frac{k^2}{k_j^2}\Big)\frac{\sigma'_{zj}}{\sigma_y} \qquad & \frac{\sigma'_{ej}}{\sigma_y} &= \frac{k^2}{k_j^2}(\ln k_j^2 - 1) = \frac{k_j^2 - 2}{k_j^2} > 0
\end{aligned}
\tag{2.16}
$$

It is clear from Eq. (2.15) that $\sigma'_{ej}/\sigma_y < 1$, as a result $\sigma'_{ei}/\sigma_y < 1$ in Eq. (2.15) or no matter whether $k_j = k_{j^*}$, the equivalent residual stress at the elastic-plastic juncture is certainly tension. So, it is quite necessary to pay close attention to the residual stresses at the inside surface instead of those at the elastic-plastic juncture.

When $k = 3$, $k_{j^*} = 1.748442$, at the inside surface $|\sigma'_{ei}|$ and $|\sigma'_{\theta i}|$ just reach σ_y according to the 3rd strength theory, but $|\sigma'_{\theta i}|$ exceeds σ_y according to the 4th strength theory. This may lower the safety of a pressure vessel. So, it is necessary to limit $|\sigma'_{\theta i}|$ to find the maximum k_j for $|\sigma'_{\theta i}|$ to be below σ_y, written as $k_{j\theta}$. Then, letting

$$
\frac{\sigma'_{\theta i}}{\sigma_y} = -\frac{2}{\sqrt{3}(k^2-1)}(1 - k_j^2 + k^2\ln k_j^2) = -1
$$

obtains

$$
2k^2\ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + 2 + \sqrt{3} = 0 \qquad \text{or} \qquad k = \sqrt{\frac{2(k_{j\theta}^2 - 1) - \sqrt{3}}{2\ln k_{j\theta}^2 - \sqrt{3}}}
\tag{2.17}
$$

From Eq. (2.17), $k \to \infty$, $k_{j\theta} = e^{\sqrt{3}/4} = 1.541896\ldots$. Letting $k_{j\theta} = k$ in Eq. (2.17), one obtains

$$
\frac{k^2\ln k^2}{k^2-1} = \frac{2+\sqrt{3}}{2}
\tag{2.18}
$$

The solution to Eq. (2.18) is $k = 2.024678965\ldots = k_{c\theta}$, which means that when $k \leqslant k_{c\theta}$, no matter how deep is the depth of the plastic zone, even when the whole cylinder is yielded $k_j = k$, $|\sigma'_{\theta i}|$ can not exceed σ_y after removing p_a from the cylinder. When $k > k_{c\theta}$, if k_j is greater than what is determined by Eq. (2.17) $(k_j > k_{j\theta})$, $|\sigma'_{\theta i}|$ will exceed σ_y after removing p_a. This paper is in the light of the cylindrical pressure vessels with $k \geqslant k_{c\theta}$. The relation between $k_{j\theta}$ and k is shown in Fig. 3, the solution to Eq. (2.8) is shown in this figure too.

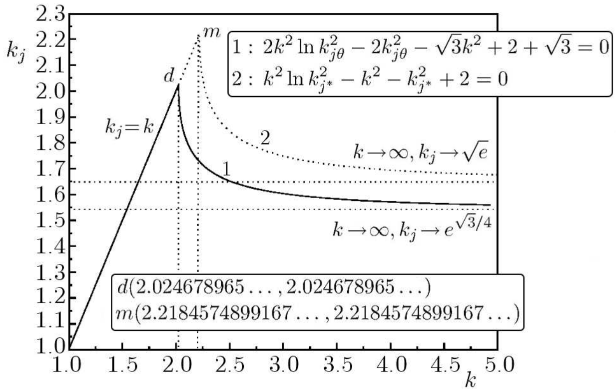

Fig. 3. The depth of plastic zone

The pressure which a cylinder with k_j can contain is (Yu, 1990)

$$\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k_j + \frac{k^2 - k_j^2}{\sqrt{3}k^2} \tag{2.19}$$

Combining Eq. (2.17) with Eq. (2.19), one obtains the ultimate allowable loading of the cylinder with k_j determined by Eq. (2.17) ($k_j = k_{j\theta}$) or by Fig. 3 under the condition $\sigma'_{\theta i} = -\sigma_y$ ($|\sigma'_{ei}| < \sigma_y$) based on the 4th strength theory

$$\frac{p}{\sigma_y} = \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2 - 1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_e}{\sigma_y} \tag{2.20}$$

Therefore, when $k > k_{c\theta}$ and under the condition $\sigma'_{\theta i} = -\sigma_y$ ($|\sigma'_{ei}| < \sigma_y$), the load-bearing capacity of the autofrettaged cylinder should be determined by Eq. (2.20). The value determined by Eq. (2.20) is less than that determined by another equation proposed by Zhu (2008b) based on the 4th strength theory and when $k_j = k_{j*} : p_4/\sigma_y = (2/\sqrt{3})(k^2 - 1)/k^2)$ but greater than that proposed by Zhu (2008a) based on the 3rd strength theory and when $k_j = k_{j*}$: $p_3/\sigma_y = (k^2 - 1)/k^2$, and greater than the maximum elastic load-bearing capability of the unautofrettaged cylinder: $p_e/\sigma_y = (k^2 - 1)/(\sqrt{3}k^2)$ or $1 + \sqrt{3}/2 (= 1.866025)$ times the initial yield pressure of the unautofrettaged cylinder. It is seen that $p = (p_3 + p_4)/2$, and this is too big to be coincidental – the authors think. Another interesting thing is that letting Eq. (2.20) equal the entire yield pressure p_y/σ_y or $p/\sigma_y = (2\sqrt{3} + 3)/6(k^2 - 1)/k^2 = p_y/\sigma_y = (2/\sqrt{3}) \ln k$ one obtains Eq. (2.18). p, p_3, p_4 and p_e are plotted in Fig. 4.

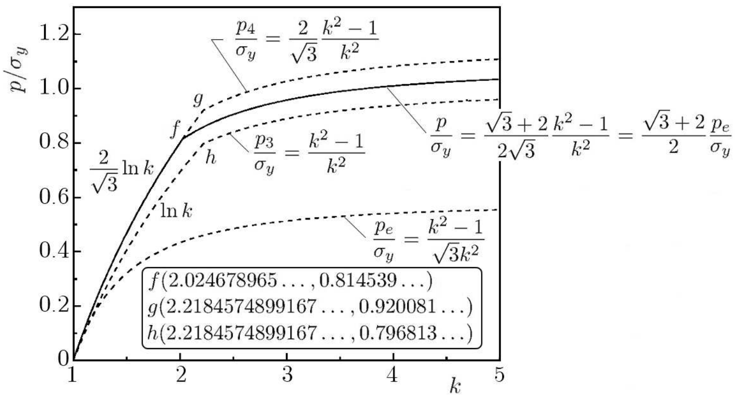

Fig. 4. Comparison between load-bearing capacity

When $k_j = k_{j\theta}$, uniting Eq. (2.17) and Eq. (2.1)-(2.4), one obtains

$$\begin{aligned}
&\frac{\sigma'_z}{\sigma_y} = \frac{1}{\sqrt{3}}\ln\Big(\frac{r}{r_i}\Big)^2 - \frac{1}{2} = \frac{1}{\sqrt{3}}\ln x^2 - \frac{1}{2} &&\frac{\sigma'_r}{\sigma_y} = \frac{1}{\sqrt{3}}\Big(\ln x^2 + \frac{2+\sqrt{3}}{2x^2} - \frac{2+\sqrt{3}}{2}\Big)\\
&\frac{\sigma'_\theta}{\sigma_y} = \frac{1}{\sqrt{3}}\Big(\ln x^2 - \frac{2+\sqrt{3}}{2x^2} + \frac{2-\sqrt{3}}{2}\Big) &&\frac{\sigma'_e}{\sigma_y} = 1 - \frac{\sqrt{3}+2}{2(r/r_i)^2} = 1 - \frac{\sqrt{3}+2}{2x^2}
\end{aligned} \tag{2.21}$$

It is seen that Eqs. (2.21) are unconcerned with k_j and k, meaning that when $\sigma'_{\theta i}$ is controlled, or $k_j = k_{j\theta}$, the residual stresses and their equivalent stress are determined only by the radial relative location (r/r_i) within the plastic wall of the cylinder and independent of k_j and k or at a certain r/r_i, the residual stresses and their equivalent stress within the plastic zone are identical for various k_j and k. From Eq. $(2.21)_4$, there are:
when $x \leqslant \sqrt{(\sqrt{3}+2)/2}$, $\sigma'_e \leqslant 0$, at the inside surface where $x = 1$, $\sigma'_e = -\sqrt{3}/2$ reaching the minimum for $d\sigma'_e/dr > 0$. When $x > \sqrt{(\sqrt{3}+2)/2}$, $\sigma'_e > 0$. Within the whole wall of a cylinder, $|\sigma'_e| < \sigma_y$. At the elastic-plastic juncture where $x = k_j$, $0 < \sigma'_e/\sigma_y = 1 - (\sqrt{3}+2)/(2k_j^2) < 1$, for $k_j^2 > 1 + \sqrt{3}/2$, meaning that σ'_e at the elastic-plastic juncture is tension and can not reach σ_y.

At $r/r_i = \sqrt{(\sqrt{3}+2)/2} = 1.366025\ldots$, $\sigma'_e = 0$, and this is just the location where $\sigma'_\theta = \sigma'_r = \sigma'_z$, that is to say, when $k_j = k_{j\theta}$, no matter how great k is, the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is

$$\Big(\sqrt{\frac{\sqrt{3}+2}{2}}, \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+2}{2} - \frac{1}{2}\Big) = (1.366025\ldots, -0.13984\ldots)$$

where $r/r_i = \sqrt{(\sqrt{3}+2)/2}$ is just Eq. (2.7) when k and k_j meet the relation expressed by Eq. (2.17)

$$\begin{aligned}
&\frac{\sigma'_z}{\sigma_y} = \frac{k_j^2 - (\sqrt{3}+2)/2}{\sqrt{3}k^2} &&\frac{\sigma'_r}{\sigma_y} = \Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y} = \Big(1 - \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y}\\
&\frac{\sigma'_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma'_z}{\sigma_y} = \Big(1 + \frac{k^2}{x^2}\Big)\frac{\sigma'_z}{\sigma_y} &&\frac{\sigma'_e}{\sigma_y} = \frac{k_j^2 - (\sqrt{3}+2)/2}{x^2}
\end{aligned} \tag{2.22}$$

The distribution of equivalent residual stress within the whole wall is demonstrated in Fig. 5, the related parameters are shown in the figure as well.

$k_j(= 1.748442)$ for $k = 3$ in Fig. 2b is determined by Eq. (2.8) or $k_j = k_{j^*} = 1.748442\ldots$. It is seen from the figure that $|\sigma'_{ei}/\sigma_y| = 1$, but $|\sigma'_{\theta i}/\sigma_y| > 1$. For $|\sigma'_{\theta i}/\sigma_y| = 1$, letting $k = 3$ in Eq. (2.17), one obtains $k_j = k_{j\theta} = 1.603502\ldots$. Similarly, letting $k = 4$ in Eq. (2.17) one finds $k_{j\theta} = 1.571211\ldots$, letting $k = 4$ in Eq. (2.8) $k_{j^*} = 1.694172\ldots$. For $k = 3$, $k_{j^*} = 1.748442\ldots$ and $k = 3$, $k_{j\theta} = 1.603502\ldots$, the residual stresses in three directions and their equivalent stress are plotted in Fig. 6a. For $k = 4$, $k_{j^*} = 1.694172\ldots$ and $k = 4$, $k_{j\theta} = 1.571211\ldots$, the residual stresses in the three directions and their equivalent stress are plotted in Fig. 6b.

The stresses caused by the internal pressure p at a general radius location are

$$\frac{\sigma^p_z}{\sigma_y} = \frac{1}{k^2-1}\frac{p}{\sigma_y} \qquad \frac{\sigma^p_r}{\sigma_y} = \Big(1 - \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma^p_z}{\sigma_y} \qquad \frac{\sigma^p_\theta}{\sigma_y} = \Big(1 + \frac{k^2}{(r/r_i)^2}\Big)\frac{\sigma^p_z}{\sigma_y} \tag{2.23}$$

The equivalent stress of the stresses caused by the internal pressure p is

$$\frac{\sigma^p_e}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma^p_\theta}{\sigma_y} - \frac{\sigma^p_r}{\sigma_y}\Big) = \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_y}\Big(\frac{r}{r_i}\Big)^{-2} \tag{2.24}$$

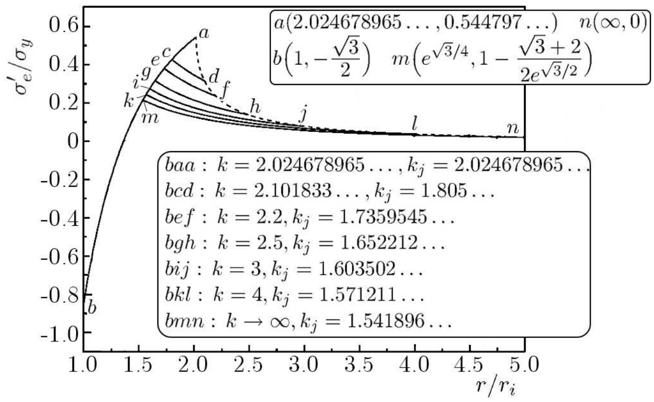

Fig. 5. The distribution of equivalent residual stress within the whole wall

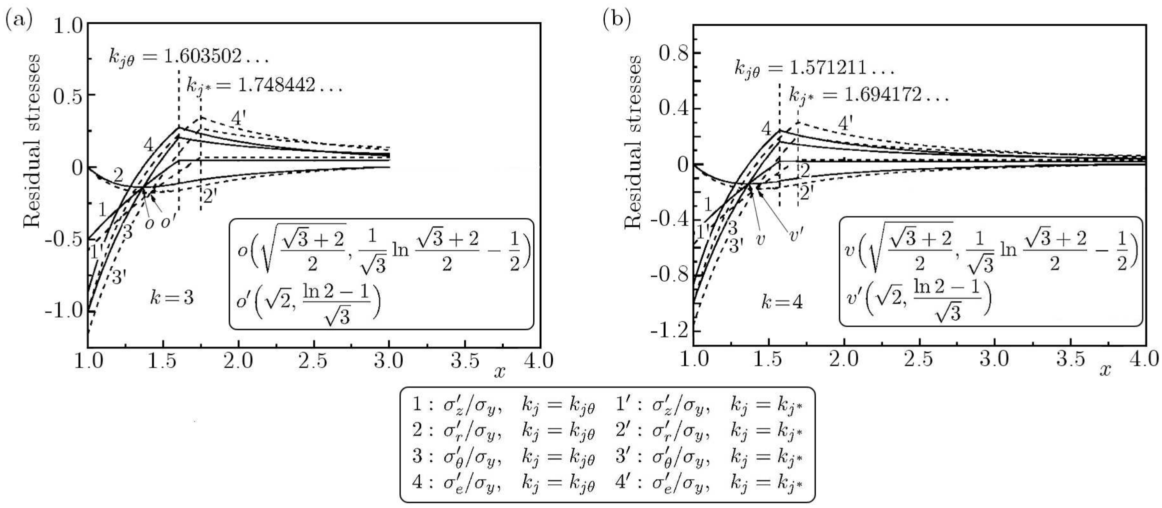

Fig. 6. Residual stresses and their equivalent stress; (a) $k = 3$, $k_{j^*} = 1.748442\ldots$ and $k = 3$, $k_{j\theta} = 1.603502\ldots$, (b) $k = 4$, $k_{j^*} = 1.694172\ldots$ and $k = 3$, $k_{j\theta} = 1.571211\ldots$

The equivalent stress of the total stress σ^T/σ_y (ETS) is

$$\frac{\sigma_e^T}{\sigma_y} = \frac{\sqrt{3}}{2}\frac{\sigma_\theta^T - \sigma_r^T}{\sigma_y} = \frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta + \sigma_\theta^p}{\sigma_y} - \frac{\sigma'_r + \sigma_r^p}{\sigma_y}\Big) = \frac{\sqrt{3}}{2}\Big(\frac{\sigma'_\theta - \sigma'_r}{\sigma_y} + \frac{\sigma_\theta^p - \sigma_r^p}{\sigma_y}\Big) = \frac{\sigma'_e}{\sigma_y} + \frac{\sigma_e^p}{\sigma_y} \tag{2.25}$$

Then, within the plastic zone

$$\frac{\sigma_e^T}{\sigma_y} = 1 - \frac{k^2 - k_j^2 + k^2\ln k_j^2}{(k^2-1)(r/r_i)^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_y}\Big(\frac{r}{r_i}\Big)^{-2} \tag{2.26}$$

within the elastic zone

$$\frac{\sigma_e^T}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)(r/r_i)^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_y}\Big(\frac{r}{r_i}\Big)^{-2} \tag{2.27}$$

At the elastic-plastic juncture $(r/r_i = k_j)$, Eqs. (2.26) and (2.27) both become

$$\frac{\sigma_e^T}{\sigma_y} = \frac{k^2(k_j^2 - 1 - \ln k_j^2)}{(k^2-1)k_j^2} + \frac{\sqrt{3}k^2 p/\sigma_y}{(k^2-1)k_j^2} \tag{2.28}$$

If $k_j = k_{j\theta}$, Eqs. (2.26) and (2.27) become respectively

$$\frac{\sigma_e^T}{\sigma_y} = 1 - \frac{\sqrt{3}+2}{2x^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p/\sigma_y}{x^2} \qquad \frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2 - (\sqrt{3}+2)/2}{x^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p/\sigma_y}{x^2} \tag{2.29}$$

From Eq. $(2.29)_1$, it is known that:

(1) provided $p/\sigma_y > -[4x^2 - (\sqrt{3}+2)]/(2\sqrt{3})(k^2-1)/k^2$ (negative), $\sigma_e^T/\sigma_y > -1$, this is definitely feasible for $p/\sigma_y > 0$ in engineering;

(2) as long as $p/\sigma_y > [(\sqrt{3}+2) - 2x^2]/(2\sqrt{3})(k^2-1)/k^2$, $\sigma_e^T > 0$, while $[(\sqrt{3}+2) - 2x^2]/(2\sqrt{3})(k^2-1)/k^2 < p_e/\sigma_y$, so when $p > p_e$, $\sigma_e^T > 0$;

(3) so long as $p/\sigma_y \leqslant (\sqrt{3}+2)/(2\sqrt{3})(k^2-1)/k^2 = (\sqrt{3}+2)/2p_e/\sigma_y$, $\sigma_e^T \leqslant \sigma_y$. When $x=1$, if $p/\sigma_y = (k^2-1)/2k^2 = (\sqrt{3}/2)p_e/\sigma_y$, then $\sigma_{ei}^T/\sigma_y = 0$.

From Eq. $(2.29)_2$:

(1) provided $p/\sigma_y > -[k_j^2 - (\sqrt{3}+2)/2]/\sqrt{3}(k^2-1)/k^2$ (negative), $\sigma_e^T/\sigma_y > 0$, this is definitely feasible for $p/\sigma_y > 0$ in engineering, so the equivalent residual stress within the elastic zone is always tension;

(2) as long as $p/\sigma_y < [x^2 - k_j^2 + (\sqrt{3}+2)/2]/\sqrt{3}(k^2-1)/k^2$, $\sigma_e^T < \sigma_y$, so when $p/\sigma_y \leqslant (\sqrt{3}+2)/(2\sqrt{3})(k^2-1)/k^2 = (\sqrt{3}+2)/2p_e/\sigma_y$, $\sigma_e^T \leqslant \sigma_y$.

At the inside surface, $x = r/r_i = 1$, then, from Eq. $(2.29)_1$

$$\frac{\sigma_{ei}^T}{\sigma_y} = \frac{\sqrt{3}k^2}{k^2-1}\frac{p}{\sigma_y} - \frac{\sqrt{3}}{2} \tag{2.30}$$

Unless $p/\sigma_y < -(2-\sqrt{3})/(2\sqrt{3})(k^2-1)/k^2$ (negative), σ_{ei}^T can not be lower than $-\sigma_y$. Unless $p/\sigma_y > 2p_e/\sigma_y$, σ_{ei}^T/σ_y can not be higher than σ_y. So, when $0 \leqslant p/\sigma_y \leqslant (\sqrt{3}+2)/2p_e/\sigma_y$, $-1 < \sigma_{ei}^T/\sigma_y \leqslant 1$. Especially, when $p/\sigma_y = (\sqrt{3}+2)/2p_e/\sigma_y$, $\sigma_e^T/\sigma_y \equiv 1$ within the whole plastic zone.

At the elastic-plastic juncture, $x = r/r_i = k_j$, from $(2.29)_1$ or $(2.29)_2$

$$\frac{\sigma_{ej}^T}{\sigma_y} = \frac{k_j^2 - (\sqrt{3}+2)/2}{k_j^2} + \frac{\sqrt{3}k^2}{k^2-1}\frac{p/\sigma_y}{k_j^2} \tag{2.31}$$

Clearly, $\sigma_{ej}^T > 0$ within the whole elastic zone. If $p/\sigma_y \leqslant (\sqrt{3}+2)/2p_e/\sigma_y$, σ_e^T can not be higher than σ_y. So, when $0 \leqslant p/\sigma_y \leqslant (\sqrt{3}+2)/2p_e/\sigma_y$, $0 < \sigma_{ej}^T/\sigma_y \leqslant 1$. Especially, when $p/\sigma_y = (\sqrt{3}+2)/2p_e/\sigma_y$, $\sigma_{ej}^T/\sigma_y = 1$ at the elastic-plastic juncture and $\sigma_e^T/\sigma_y = k_j^2/x^2$ at a general radial location within the elastic zone.

When $p/\sigma_y = (\sqrt{3}+2)/2p_e/\sigma_y$ or the load-bearing capacity is determined by Eq. (2.20), Eqs. (2.29) become respectively

$$\frac{\sigma_e^T}{\sigma_y} \equiv 1 \qquad \frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2}{x^2} \tag{2.32}$$

Figure 7 shows the distribution of the equivalent stress of the total stress.

The supplemental explaination for Fig. 7 is as follows.

(1) Horizontal line *baa*: $k = 2.024678965\ldots = k_{c\theta}$, $k_j = k = k_{j\theta} = 2.024678965\ldots$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y = 1$, x varies from 1 to $k_{j\theta}(= k_{c\theta} = 2.024678965\ldots)$ (from point b to a), and then from $2.024678965\ldots(k_{j\theta})$ to $2.024678965\ldots(k)$ (from point a to a) within the elastic zone (no elastic zone, the "curve" of the equivalent stress of the total stress is actually a point within the elastic zone).

(2) Curve *bcd*: $k = 2.1\ldots$, $k_{j\theta} = 1.806908\ldots$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y = 1$, x varies from 1 to $k_{j\theta}(1.806908\ldots)$ (from point b to c), and then from $1.806908\ldots$ to $2.1\ldots(k)$ (from point c to d) within the elastic zone.

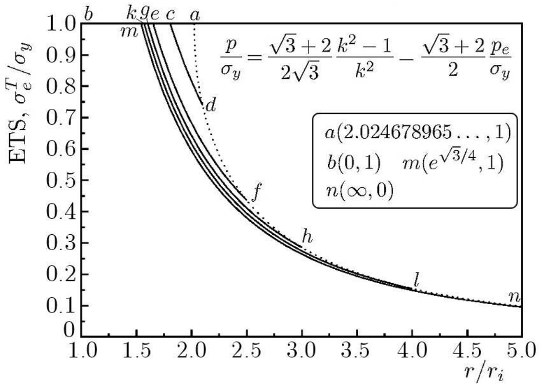

Fig. 7. Distribution of the equivalent stress of the total stress for $p/\sigma_y=(\sqrt{3}+2)/2p_e/\sigma_y$ and $k_j=k_{j\theta}$

(3) Curve bef: $k=2.5\ldots$, $k_{j\theta}=1.6522121\ldots$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y=1$, x varies from 1 to $k_{j\theta}(1.6522121\ldots)$ (from point b to e), and then from $1.6522121\ldots$ to $2.5\ldots(k)$ (from point e to f) within the elastic zone.

(4) Curve bgh: $k=3$, $k_{j\theta}=1.60350225\ldots$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y=1$, x varies from 1 to $k_{j\theta}(1.844363\ldots)$ (from point b to g), and then from $1.844363\ldots$ to $3(k)$ (from point g to h) within the elastic zone.

(5) Curve bkl: $k=4$, $k_{j\theta}=1.57121054\ldots$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y=1$, x varies from 1 to $k_{j\theta}(1.57121054\ldots)$ (from point b to k), and then from $1.57121054\ldots$ to $4(k)$ (from point k to l) within the elastic zone.

(6) Curve bmn: $k\to\infty$, $k_{j\theta}=e^{\sqrt{3}/4}$. Within the plastic zone, σ^T/σ_y is a horizontal line: $\sigma^T/\sigma_y=1$, x varies from 1 to $k_{j\theta}=e^{\sqrt{3}/4}=1.541896\ldots$ (from point b to m), and then from $k_{j\theta}$ to $\infty(k)$ (from point m to n) within the elastic zone.

The prerequisite to the above arguments is that k and k_j meet Eq. (2.17) or $k_j=k_{j\theta}$ and $p/\sigma_y=(\sqrt{3}+2)/(2\sqrt{3})(k^2-1)/(k^2)=(\sqrt{3}+2)/2p_e/\sigma_y$. Grasping these laws is helpful to the design of high and ultrahigh pressure vessels. If $k_j\neq k_{j\theta}$ or $p/\sigma_y\neq(\sqrt{3}+2)/(2\sqrt{3})(k^2-1)/k^2$, the above facts are untenable, and $|\sigma_e^T|$ may exceed σ_y.

When $p/\sigma_y=(\sqrt{3}+2)/2p_e/\sigma_y$, Eqs. (2.23) and (2.24) become

$$\begin{aligned}
&\frac{\sigma_z^p}{\sigma_y}=\frac{\sqrt{3}+2}{2\sqrt{3}k^2} && \frac{\sigma_r^p}{\sigma_y}=\frac{\sqrt{3}+2}{2\sqrt{3}k^2}-\frac{\sqrt{3}+2}{2\sqrt{3}x^2}\\
&\frac{\sigma_\theta^p}{\sigma_y}=\frac{\sqrt{3}+2}{2\sqrt{3}k^2}+\frac{\sqrt{3}+2}{2\sqrt{3}x^2} && \frac{\sigma_e^p}{\sigma_y}=\frac{\sqrt{3}+2}{2x^2}
\end{aligned}\tag{2.33}$$

When $p/\sigma_y=(\sqrt{3}+2)/2p_e/\sigma_y$ and $k_j=k_{j\theta}$, the components of the total stresses are:
— within the plastic zone

$$\begin{aligned}
&\frac{\sigma_z^T}{\sigma_y}=\frac{\ln x^2}{\sqrt{3}}-\frac{1}{2}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2} && \frac{\sigma_r^T}{\sigma_y}=\frac{\ln x^2}{\sqrt{3}}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2}-\frac{\sqrt{3}+2}{2\sqrt{3}}\\
&\frac{\sigma_\theta^T}{\sigma_y}=\frac{\ln x^2}{\sqrt{3}}+\frac{\sqrt{3}+2}{2\sqrt{3}k^2}+\frac{2-\sqrt{3}}{2\sqrt{3}}
\end{aligned}\tag{2.34}$$

From Eqs. $(2.34)_2$ and $(2.34)_3$, $\sigma_e^T/\sigma_y=(\sqrt{3}/2)[\sigma_\theta^T/\sigma_y-\sigma_r^T/\sigma_y]\equiv 1$, the same as from Eq. $(2.32)_1$. This proves the above arguments are correct and reasonable.

Within the elastic zone

$$\begin{aligned}
&\frac{\sigma_z^T}{\sigma_y} = \frac{\sigma_z'}{\sigma_y} + \frac{\sigma_z^p}{\sigma_y} = \frac{k_j^2}{\sqrt{3}k^2} \qquad \frac{\sigma_r^T}{\sigma_y} = \frac{\sigma_r'}{\sigma_y} + \frac{\sigma_r^p}{\sigma_y} = -\frac{k_j^2(x^{-2} - k^{-2})}{\sqrt{3}} \\
&\frac{\sigma_\theta^T}{\sigma_y} = \frac{\sigma_\theta'}{\sigma_y} + \frac{\sigma_\theta^p}{\sigma_y} = \frac{k_j^2(x^{-2} + k^{-2})}{\sqrt{3}}
\end{aligned} \tag{2.35}$$

From Eqs. $(2.35)_2$ and $(2.35)_3$, $\sigma_e^T/\sigma_y = k_j^2/x^2$, the same as from Eq. $(2.32)_2$. This as well proves that the above arguments are correct and reasonable.

Figure 8 shows a comparison between the equivalent stresses of the total stresses under different internal pressures and $k_j = k_{j\theta}$, from which it is known that only when $p/\sigma_y = (\sqrt{3}+2)/(2\sqrt{3})(k^2-1)/k^2 = (\sqrt{3}+2)/2p_e/\sigma_y$ and $k_j = k_{j\theta}$, is the operation state optimum, otherwise, or $p/\sigma_y \neq (\sqrt{3}+2)/2p_e/\sigma_y$ and/or $k_j \neq k_{j\theta}$, either $\sigma_e^T > \sigma_y$ or the load--bearing capacity is lowered or compressive yield occurs. In Fig. 8, curve 1 is just curve bgh in Fig. 7, in this case, $p/\sigma_y = (\sqrt{3}+2)/2p_e/\sigma_y$, $\sigma_e^T \equiv \sigma_y$ within the whole plastic zone. Curve 2: $p/\sigma_y = (\sqrt{3}+2.1)/2p_e/\sigma_y$, $\sigma_{ei}^T > \sigma_y$. Curve 3: $p/\sigma_y = (\sqrt{3}+1.9)/2p_e/\sigma_y$, $0 < \sigma_{ei}^T < \sigma_y$. Curve 4: $p = p_e$, $0 < \sigma_{ei}^T \ll \sigma_y$. Curve 5: $p/\sigma_y = (k^2-1)/2k^2 = (\sqrt{3}/2)p_e/\sigma_y$, $\sigma_{ei}^T/\sigma_y = 0$. Curve 6: $p = 0.8p_e$, $\sigma_{ei}^T/\sigma_y < 0$.

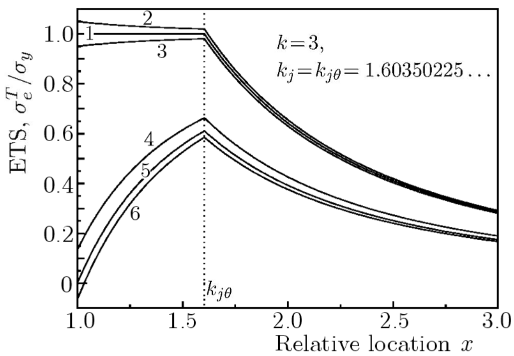

Fig. 8. Comparison between the equivalent stresses of the total stresses under different internal pressures and $k_j = k_{j^*}$

Besides, for certain k, when $k_j < k_{j\theta}$, though the residual stresses are less than when $k_j = k_{j\theta}$, the load-bearing capacity is dropped. For example, for $k = 3$, if $k_j = k_{j\theta}$ $(= 1.60350225\ldots)$, from Eq. (2.19) or (2.20), $p/\sigma_y = 0.9576\ldots$; while if $k_j = 1.5$, from Eq. (2.19), $p/\sigma_y = 0.9012\ldots < 0.9576\ldots$.

3. Conclusions

- For $|\sigma_{ei}'| \leqslant \sigma_y$, the depth of the plastic zone is $k^2 \ln k_{j^*}^2 - k^2 - k_{j^*}^2 + 2 = 0$, where $\sqrt{e} \leqslant k_{j^*} \leqslant k_c = 2.2184574899167\ldots$, when $k \geqslant 2.2184574899167\ldots$, the load-bearing capacity is

$$\frac{p_4}{\sigma_y} = \frac{2}{\sqrt{3}}\frac{k^2-1}{k^2} = 2\frac{p_e}{\sigma_y}$$

- For $|\sigma_{\theta i}'| \leqslant \sigma_y$, the depth of the plastic zone is $2k^2 \ln k_{j\theta}^2 - 2k_{j\theta}^2 - \sqrt{3}k^2 + (2+\sqrt{3}) = 0$, where $e^{\sqrt{3}/4} \leqslant k_{j\theta} \leqslant k_{c\theta} = 2.024678965\ldots$, when $k \geqslant 2.024678965\ldots$, the load-bearing capacity is

$$\frac{p}{\sigma_y} = \frac{\sqrt{3}+2}{2\sqrt{3}}\frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2}\frac{p_e}{\sigma_y}$$

The data 2.024678965... is the solution to $(k^2 \ln k^2)/(k^2-1) = (2+\sqrt{3})/2$.

- For an autofrettaged cylinder, $k_j = k_{j\theta}$ with

$$\frac{p}{\sigma_y} = \frac{\sqrt{3}+2}{2\sqrt{3}} \frac{k^2-1}{k^2} = \frac{\sqrt{3}+2}{2} \frac{p_e}{\sigma_y}$$

is the optimum operation state – not only safe but also highly capable, under the state $\sigma_e^T \equiv \sigma_y$ within the plastic zone and $\sigma_e^T/\sigma_y = k_j^2/x^2 < 1$ within the elastic zone.

- For the same k, $k_{j\theta} < k_{j^*}$, as means that the depth of plastic zone necessary to ensure $|\sigma'_{\theta i}| \leqslant \sigma_y$ is less than that to ensure $|\sigma'_{ei}| \leqslant \sigma_y$.

- Irrespective of k, the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is

$$\Big(\sqrt{\frac{\sqrt{3}+2}{2}}, \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+2}{2} - \frac{1}{2}\Big) = (1.366025\ldots, -0.13984\ldots) \qquad \text{when} \qquad k_j = k_{j\theta}$$

or

$$\Big(\sqrt{2}, \frac{\ln 2 - 1}{\sqrt{3}}\Big) = (1.414214\ldots, -0.17716\ldots) \qquad \text{when} \qquad k_j = k_{j^*}$$

- When $\sigma'_{\theta i}$ is controlled, or $k_j = k_{j\theta}$, the residual stresses and their equivalent stress are determined only by the radial relative location (r/r_i) within the plastic zone and independent of k_j and k, or once the location r/r_i is determined, the residual stresses and their equivalent stress within the plastic zone are identical for any k_j and k.

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References

1. Farrahi G.H., Faghidian S.A., Smith D.J., 2009, Reconstruction of residual stresses in autofrettaged thick-walled tubes from limited measurements, *International Journal of Pressure Vessels and Piping*, **86**, 11, 777-784
2. Gao J.-W., Xie G.-W., Zhou H.-J., et al., 2008, Research on autofrettage technology *Mechanical Research and Application*, **21**, 6, 21-23 [in Chinese]
3. Hameed A., Brown R.D., Hetherington J., 2004, A study of the residual stress distribution in an autofrettaged, thick-walled cylinder with cross-bore, *Journal of Pressure Vessel Technology*, **126**, 4, 497-503
4. Lee E.-Y., Lee Y.-S., Yang Q.-M., et al., 2008, Autofrettage process analysis of compound cylinder based on the elastic-perfectly plastic and strain hardening stress-strain curve, *Journal of Mechanical Science and Technology*, **23**, 3153-3160
5. Lin Y., Dong Q., Jia J., 2009, Autofrettage damage residual stress of thick-walled cylinder, *Science Technology and Engineering*, **9**, 24, 7306-7309 [in Chinese]
6. Maleki M., Farrahi G.H., Haghpanah B., Hosseinian J.E., 2010, Residual stress analysis of autofrettaged thick-walled spherical pressure vessel, *International Journal of Pressure Vessels and Piping*, **87**, 7, 396-401

7. Park J.-H., Lee Y.-S., Kim J.-H., et al., 2008, Machining effect of the autofrettaged compound cylinder under varying overstrain levels, *Journal of Materials Processing Technology*, **201**, 1/3, 491-496

8. Parker A.P., 2001b, Bauschinger effect design procedures for compound tubes containing an autofrettaged layer, *Journal of Pressure Vessel Technology*, **123**, 2, 203-206

9. Parker A.P., 2001a, Autofrettage of open-end tubes – pressures, stresses, strains, and code comparisons, *Journal of Pressure Vessel Technology*, **123**, 3, 271-281

10. The compiling group of "handbook of mathematics", *A handbook of Mathematics*, Beijing: Higher Education Press, 1979 [in Chinese]

11. Yu G., 1990, *Chemical Pressure vessels and Equipment*, Beijing: Chemical Industrial Press [in Chinese]

12. Zheng X., Xuan F., 2010, Investigation on autofrettage and Safety of the thick-walled cylinder under thermo-mechanical loadings, *Journal of Mechanical Engineering*, **46**, 16, 156-161 [in Chinese]

13. Zhu R., 2008a, Results resulting from autofrettage of cylinder? *Chinese Journal of Mechanical Engineering*, **21**, 4, 105-110

14. Zhu R., 2008b, Ultimate load-bearing capacity of cylinder derived from autofrettage under ideal condition? *Chinese Journal of Mechanical Engineering*, **21**, 5, 80-87

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