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VELOCITY OF ACCELERATION WAVE PROPAGATING IN HYPERELASTIC ZAHORSKI AND MOONEY-RIVLIN MATERIALS

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This paper studies homogeneous static deformation of a incompressible body. It shows a comparative analysis of a wave process in hyperelastic materials which have linear (Mooney-Rivlin material) and nonlinear (Zahorski material) dependences on invariants of the deformation tensor. The numerical analysis clearly demonstrates fundamental quantitative differences in the process of propagation of the acceleration wave. These differences are the consequence of calculation of elastic potentials which has been assumed in the study.

Key words: discontinuous surface, acceleration waves, hyperelastic materials.

1. Introduction

The general principles governing the mechanics of continuum, in particular its motion, are essential to determine body behaviour being influenced by external forces. Consequently, what becomes especially important, is the analysis of propagation of a disturbance being modeled as a "moving discontinuity surface" in the continuum.

An acceleration wave is a surface of discontinuity. Much work has been done on the subject of acceleration waves in hyperelastic materials, for example by Wright (1973), Truesdell and Noll (1965), Varley (1965), Chen (1968) or Jeffrey (1982). Studies on the acceleration wave propagation have been the object of interest in a range of scientific domains which make use of mathematical models of the continuum in their deliberations. For many years, there have been lots of researches to recognize the process of wave propagation as well as accompanying transport of energy and momentum. In the case of acceleration waves, these processes demonstrate great complexity. Their intensities change depending on series of factors. It gives a meaningful sense to the analysis of propagation of a weak discontinuity wave in the continuum.

There has been a progress in this domain since new measurement methods have been applied. In the experimental research, measurements of wave propagation velocity allows one to determine material constants more precisely. The methods being permanently improved, give new possibilities to experimental analysis within both compressible and incompressible materials.

2. Basic dependences

Motion of a three-dimensional continuum is represented by a set of functions

$$x^i = x^i(X^\alpha, t) \qquad i, \alpha = 1, 2, 3 \tag{2.1}$$

The coordinates x^i describe the current position at a time t of a material point in terms of its position X^α in the reference configuration.

The deformation gradient and the particle velocity have the components

$$x^i_\alpha = \frac{\partial x^i}{\partial X^\alpha} \qquad \dot{x}^i = \frac{\partial x^i}{\partial t} \tag{2.2}$$

The left Cauchy-Green tensor is defined by

$$B^{ik} = {x^i}_\alpha {x^k}_\beta g^{\alpha\beta} \tag{2.3}$$

Its principal invariants I_1, I_2 and I_3 are the deformation invariants.

Incompressible, isotropic elastic material is characterized (i) by the internal constraint of incompressibility, $I_3 = 1$ and (ii) by a strain-energy function

$$W = W(I_1, I_2) \tag{2.4}$$

The internal energy W per unit undeformed volume is expressed as function of the two free deformation invariants. In this case, the first Piola-Kirchhoff stress tensor has the components

$${T_{Ri}}^\alpha = \frac{\partial W}{\partial {x^i}_\alpha} + p{X^\alpha}_i = \frac{\partial W}{\partial I_k}\frac{\partial I_k}{\partial {x^i}_\alpha} + p{X^\alpha}_i \tag{2.5}$$

where p is a hydrostatic reaction stress.

Using the propagation condition for the acceleration wave in the reference configuration (Wesołowski, 1974)

$$(A_{i\ k}^{\alpha\beta} N_\alpha N_\beta - \rho_R g_{ik} U^2) A^k + C N_\alpha X^\alpha{}_i = 0 \tag{2.6}$$

and

$$N_\alpha = \frac{1}{J} x^i{}_\alpha n_i \frac{dS}{dS_R} = \frac{U}{u} x^i{}_\alpha n_i \tag{2.7}$$

and identity

$$X^\alpha{}_i x^i{}_\beta \equiv \delta^\alpha{}_\beta \tag{2.8}$$

we obtain the equation

$$(A_{i\ k}^{\alpha\beta} N_\alpha N_\beta - \rho_R \delta_{ik} U^2) A^k + C \frac{U}{u} n_i = 0 \tag{2.9}$$

Multiplying (2.9) by n^i, we have

$$A_{i\ k}^{\alpha\beta} N_\alpha N_\beta A^k n^i - \rho_R \delta_{ik} U^2 A^k n^i + C \frac{U}{u} = 0 \tag{2.10}$$

According to (Wesołowski, 1974), the first order function of the material and coordinates of the vector $\boldsymbol{N}$ normal to the discontinuity surface in the reference configuration is the acoustical tensor

$$Q_{ik} = A_{i\ k}^{\alpha\beta} N_\alpha N_\beta \tag{2.11}$$

Then (2.10) takes the form

$$Q_{ik} n^i A^k - \rho_R A^k n_k + C \frac{U}{u} = 0 \tag{2.12}$$

where

$$C = -Q_{ik} n^i A^k \frac{u}{U} \tag{2.13}$$

because in incompressible bodies there are only transverse waves $(A^k n_k = 0)$.

Substituting (2.13) into (2.10), we obtain

$$(Q_{ik} - Q_{rk} n^r n_i - \rho_R \delta_{ik} U^2) A^k = 0 \tag{2.14}$$

Taking a notation

$$\overset{*}{Q}_{ik} = Q_{ik} - Q_{rk} n^r n_i \tag{2.15}$$

for the reduced acoustical tensor, we obtain the propagation condition second-order discontinuity surface in the reference configuration

$$(\overset{*}{Q}_{ik} - \rho_R \delta_{ik} U^2) A^k = 0 \tag{2.16}$$

3. Homogeneous deformations in isotropic materials

We consider homogenous static deformation of an incompressible body in the coincide Cartesian coordinate systems $\{x_i\}$ and $\{X_\alpha\}$, represented by

$$x_1 = \lambda X_1 \qquad x_2 = \lambda X_2 \qquad x_3 = \frac{1}{\lambda^2} X_3 \tag{3.1}$$

where $\lambda =$ const.

For medium deformation (3.1) being considered, the deformation gradient does not depend on X_α coordinate and time t. For a homogenous material, the material functions are constant in time and space.

The deformation gradient, its converse, and the left and right Cauchy-Green deformation tensor, are

$$[x^i{}_\alpha] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \dfrac{1}{\lambda^2} \end{bmatrix} \qquad \Big[X^\alpha{}_i\Big] = \begin{bmatrix} \dfrac{1}{\lambda} & 0 & 0 \\ 0 & \dfrac{1}{\lambda} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}$$
$$[B^{ij}] = [C_{\alpha\beta}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \dfrac{1}{\lambda^4} \end{bmatrix} \tag{3.2}$$

and the deformation invariants

$$I_1 = 2\lambda^2 + \frac{1}{\lambda^4} \qquad I_2 = \lambda^4 + \frac{2}{\lambda^2} \qquad I_3 = 1 \tag{3.3}$$

We have to define the vector $\boldsymbol{n}$ normal to the discontinuity surface in the current configuration in order to work out the coordinates of the reduced acoustical tensor. We assume that the discontinuity surface in the reference configuration propagates in the X_3 direction (Fig. 1).

The normal vector in the reference configuration takes the form

$$\boldsymbol{N} = [\cos\alpha, \sin\alpha, 0] \tag{3.4}$$

The unit vector in the current configuration (Wesołowski, 1974)

$$n_k = J X^\alpha{}_k N_\alpha \frac{dS_R}{dS} \tag{3.5}$$

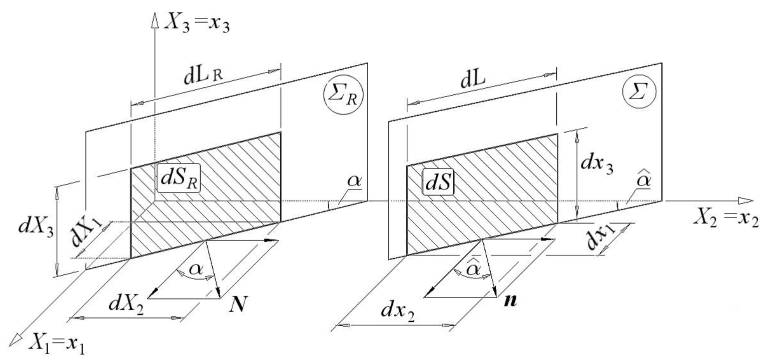

Fig. 1. The surface elements dS_R and dS and the discontinuity surface Σ_R and Σ in the reference and current configuration

According to (Fig. 1), for the deformation described by (3.1), we have

$$\begin{aligned} dS_R &= dL_R \cdot dX_3 = \sqrt{(dX_1)^2 + (dX_2)^2} \cdot dX_3 \\ dS &= dL \cdot dx_3 = \sqrt{\lambda^2(dX_1)^2 + \lambda^2(dX_2)^2} \cdot \frac{dX_3}{\lambda^2} \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we obtain a formula for the coordinates of the vector $\boldsymbol{n}$ in the current configuration

$$n_k = X^{\alpha}{}_k N_\alpha \lambda \tag{3.7}$$

The analysis will be continued for a special isotropic elastic material characterized by the constitutive equation (Zahorski, 1962)

$$W(I_1, I_2) = \sigma\rho_R = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_1^2 - 9) \tag{3.8}$$

where C_1, C_2 and C_3 are constants.

The special case $C_3 = 0$ corresponds to the Mooney-Rivlin material.

For the analyzed material

$$\sigma_i = \frac{\partial\sigma}{\partial I_i} \qquad \sigma_{ik} = \frac{\partial^2\sigma}{\partial I_i \partial I_k} \qquad i,k = 1,2 \tag{3.9}$$

we obtain the properties

$$\begin{aligned} \sigma_1 &= \frac{1}{\rho_R}(C_1 + 2C_3 I_1) \qquad & \sigma_2 &= \frac{C_2}{\rho_R} \\ \sigma_{12} &= \sigma_{21} = \sigma_{22} = 0 \qquad & \sigma_{11} &= \frac{2C_3}{\rho_R} \end{aligned} \tag{3.10}$$

According to (Wesołowski, 1974), the coordinates of the first order functions of the material tensor can be calculated from the equation

$$A_{i\ k}^{\alpha\beta} = \rho_R \frac{\partial^2 \sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta} = \rho_R \Big\{ \sigma_1 \frac{\partial^2 I_1}{\partial x^i{}_\alpha \partial x^k{}_\beta} + \sigma_{11} \frac{\partial I_1}{\partial x^k{}_\beta} \frac{\partial I_1}{\partial x^i{}_\alpha} + \sigma_2 \frac{\partial^2 I_2}{\partial x^i{}_\alpha \partial x^k{}_\beta} \Big\} \tag{3.11}$$

where

$$\begin{aligned} &\frac{\partial I_1}{\partial x^i{}_\alpha} = 2x_i{}^\alpha \qquad \frac{\partial I_2}{\partial x^i{}_\alpha} = 2(I_1 x_i{}^\alpha - B_{ir} x^r{}_\alpha) \\ &\frac{\partial^2 I_1}{\partial x^i{}_\alpha \partial x^k{}_\beta} = 2g_{ik} g^{\alpha\beta} \\ &\frac{\partial^2 I_2}{\partial x^i{}_\alpha \partial x^k{}_\beta} = 2[2x_i{}^\alpha x_k{}^\beta - g_{ik} x_r{}^\alpha x^{r\beta} - x_i{}^\beta x_k{}^\alpha + (I_1 g_{ik} - B_{ik}) g^{\alpha\beta}] \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain the first order function of the material tensor for the material characterized by strain-energy function (3.8)

$$\begin{aligned} A_{i\ k}^{\alpha\beta} &= \rho_R \Big\{ 2\sigma_1 g_{ik} g^{\alpha\beta} + \\ &+ 2\sigma_2 [2x_i{}^\alpha x_k{}^\beta - g_{ik} C^{\alpha\beta} - x_i{}^\beta x_k{}^\alpha + (I_1 g_{ik} - B_{ik}) g^{\alpha\beta}] + 4\sigma_{11} x_i{}^\alpha x_k{}^\beta \Big\} \end{aligned} \tag{3.13}$$

According to (2.11) and including (3.13), we have the acoustical tensor

$$\begin{aligned} Q_{ik} &= \rho_R \Big\{ 2\sigma_1 g_{ik} + 2\sigma_2 (2x_i{}^\alpha x_k{}^\beta - g_{ik} C^{\alpha\beta} - x_i{}^\beta x_k{}^\alpha) N_\alpha N_\beta + \\ &+ 2\sigma_2 (I_1 g_{ik} - B_{ik}) + 4\sigma_{11} x_i{}^\alpha x_k{}^\beta N_\alpha N_\beta \Big\} \end{aligned} \tag{3.14}$$

Substituting the deformation gradient and deformation tensor described by expressions (3.2) into (3.14) and including the vector $\boldsymbol{N}$ from (3.4), we obtain components of the acoustical tensor

$$\begin{aligned} Q_{11} &= 2\Big[C_1 + C_2 \Big(\lambda^2 \cos^2\alpha + \frac{1}{\lambda^4} \Big) + 2C_3 \Big(\frac{1}{\lambda^4} + \lambda^2 (2 + \cos^2\alpha) \Big) \Big] \\ Q_{12} &= Q_{21} = (C_2 + 2C_3) \lambda^2 \sin 2\alpha \\ Q_{22} &= 2\Big[C_1 + C_2 \Big(\lambda^2 \sin^2\alpha + \frac{1}{\lambda^4} \Big) + 2C_3 \Big(\frac{1}{\lambda^4} + \lambda^2 (2 + \sin^2\alpha) \Big) \Big] \\ Q_{33} &= 2\Big[C_1 + C_2 \lambda^2 + 2C_3 \Big(2\lambda^2 + \frac{1}{\lambda^4} \Big) \Big] \\ Q_{13} &= Q_{23} = Q_{31} = Q_{32} = 0 \end{aligned} \tag{3.15}$$

For deformation (3.1) being considered, the vector $\boldsymbol{N} = [\cos\alpha, \sin\alpha]$ according to (3.7) passes into the normal vector $\boldsymbol{n}$ in the current configuration, and this vector is the same like the vector $\boldsymbol{N}$ in the reference configuration

$$\boldsymbol{n} = \boldsymbol{N} = [\cos\alpha, \sin\alpha] \tag{3.16}$$

The coordinates vector $\boldsymbol{n}$ allows one to simplify expression (2.15), thus

$$\overset{*}{Q}_{ik} = Q_{ik} - Q_{1k}n_1n_i - Q_{2k}n_2n_i \qquad i,k = 1,2,3 \tag{3.17}$$

Substituting (3.15) and (3.16) into the above equation, we obtain components of the tensor $\overset{*}{Q}_{ik}$

$$\begin{aligned}
\overset{*}{Q}_{11} &= 2\Big[C_1 + C_2\frac{1}{\lambda^4} + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big]\sin^2\alpha \\
\overset{*}{Q}_{12} &= \overset{*}{Q}_{21} = -\Big[C_1 + C_2\frac{1}{\lambda^4} + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big]\sin 2\alpha \\
\overset{*}{Q}_{22} &= 2\Big[C_1 + C_2\frac{1}{\lambda^4} + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big]\cos^2\alpha \\
\overset{*}{Q}_{33} &= Q_{33} = 2\Big[C_1 + C_2\lambda^2 + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big] \\
\overset{*}{Q}_{13} &= \overset{*}{Q}_{23} = \overset{*}{Q}_{31} = \overset{*}{Q}_{32} = 0
\end{aligned} \tag{3.18}$$

The reduced acoustical tensor matrix $[\overset{*}{Q}_{ik}]$ could be simplified

$$[\overset{*}{Q}_{ik}] = \begin{bmatrix} \overset{*}{Q}_{11} & \overset{*}{Q}_{12} & 0 \\ \overset{*}{Q}_{21} & \overset{*}{Q}_{22} & 0 \\ 0 & 0 & \overset{*}{Q}_{33} \end{bmatrix} \tag{3.19}$$

The expression $\rho_R U^2$ is an eigenvalue of the $\overset{*}{Q}_{ik}$ tensor. From the characteristic equation, we determine the propagation velocity of the discontinuity surface in the reference configuration

$$(U_{1,2})^2 = \frac{1}{2\rho_R}\Big[\overset{*}{Q}_{11} + \overset{*}{Q}_{22} \pm \sqrt{(\overset{*}{Q}_{11} - \overset{*}{Q}_{22})^2 + 4\overset{*}{Q}_{12}\overset{*}{Q}_{21}}\ \Big] \qquad (U_3)^2 = \frac{\overset{*}{Q}_{33}}{\rho_R} \tag{3.20}$$

Having substituted the components $\overset{*}{Q}_{ik}$, see (3.18), into expressions (3.20), we have

$$U_1 = \sqrt{\frac{2}{\rho_R}\Big[C_1 + C_2\frac{1}{\lambda^4} + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big]} \qquad U_2 = 0$$
$$U_3 = \sqrt{\frac{2}{\rho_R}\Big[C_1 + C_2\lambda^2 + 2C_3\Big(2\lambda^2 + \frac{1}{\lambda^4}\Big)\Big]} \tag{3.21}$$

The eingenvector $\boldsymbol{D}^{(1)}$ for the velocity U_1 satisfies the equation

$$\begin{bmatrix} \overset{*}{Q}_{11} - (U_1)^2\rho_R & \overset{*}{Q}_{12} & 0 \\ \overset{*}{Q}_{21} & \overset{*}{Q}_{22} - (U_1)^2\rho_R & 0 \\ 0 & 0 & \overset{*}{Q}_{33} - (U_1)^2\rho_R \end{bmatrix} \begin{bmatrix} D_1^{(1)} \\ D_2^{(1)} \\ D_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.22}$$

from the above equation we directly have

$$\frac{D_1^{(1)}}{D_2^{(1)}} = -\frac{\overset{*}{Q}_{12}}{\overset{*}{Q}_{11} - (U_1)^2\rho_R} = -\frac{\overset{*}{Q}_{22} - (U_1)^2\rho_R}{\overset{*}{Q}_{21}} \tag{3.23}$$

Substituting (3.18) into (3.23), we obtain

$$\frac{D_1^{(1)}}{D_2^{(1)}} = -\frac{\sin\alpha}{\cos\alpha} \tag{3.24}$$

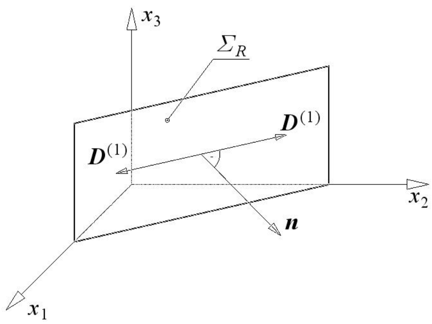

Fig. 2. The unit vector $\boldsymbol{D}^{(1)}$ for the acceleration wave propagating with the velocity U_1 in spatial coordinates

For the vector $\boldsymbol{n}$ normal to the discontinuity surface, the unit vector in the wave amplitude direction $\boldsymbol{D}^{(1)}$ for the propagation velocity U_1 has the coordinates

$$\boldsymbol{D}^{(1)} = [D_1^{(1)}, D_2^{(1)}, D_3^{(1)}] = [-\sin\alpha, \cos\alpha, 0] \tag{3.25}$$

Making analogical deliberation for the velocity U_3, we find that the unit vector $\boldsymbol{D}^{(3)}$ does not depend on the angle α. Then we have

$$\boldsymbol{D}^{(3)} = [D_1^{(3)}, D_2^{(3)}, D_3^{(3)}] = [0, 0, \pm 1] \tag{3.26}$$

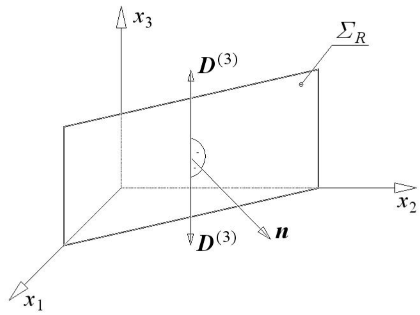

Fig. 3. The unit vector $\boldsymbol{D}^{(3)}$ for the acceleration wave propagating with the velocity U_3 in spatial coordinates

4. Numerical results

The subject of numerical analysis are the expressions for velocity of propagation of the acceleration wave in a hyperelastic material assumed in the calculations. The analysis is based on formulas represented by the velocity as function of the preliminary deformation (extension) λ.

The density of rubber assumed in the analysis is $\rho = 1190\,\mathrm{kg/m^3}$. C_1, C_2 and C_3 constants correspond to one kind of rubber represented by (Zahorski, 1962)

$$C_1 = 6.278 \cdot 10^4\,\mathrm{Pa} \qquad C_2 = 8.829 \cdot 10^3\,\mathrm{Pa} \qquad C_3 = 6.867 \cdot 10^3\,\mathrm{Pa}$$

The velocity distributions U_1 and U_3 of the acceleration wave presented in the Fig. 4 point out considerable both quantitative and qualitative differences between the Zahorski and MooneyRivlin materials. They result from different strain-energy functions of hyperelastic materials assumed in the calculations. This difference is based on the linear (MooneyRivlin material) and nonlinear (Zahorski material) dependence on the Cauchy-Green deformation invariants.

The preliminary strain of rubber in the interval in which both compression and tension are possible ($\lambda \in < 0.5, 2 >$) for the velocity U_1 and U_3 in Zahorski material, increase the velocity of the compression and tension strain.

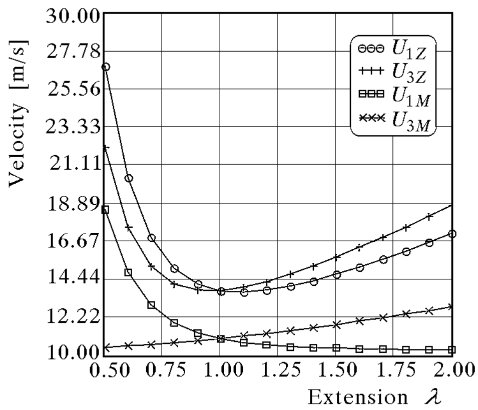

Fig. 4. The propagation velocity of the acceleration wave in the Zahorski material (U_{1Z} and U_{3Z}) and the Mooney-Rivlin material (U_{1M} and U_{3M}) for $\lambda \in< 0.5, 2 >$

For the Mooney-Rivlin material, this increase occurs for compression for the velocity U_1 and for tension for the velocity U_3, however the decrease occurs for the velocity U_1 in the case of tension strain, and for the velocity U_3 in the case of compression strain.

Such dynamical behaviour of the incompressible medium is different than in the case of behaviour of the Murnaghan material (Major, 2001). Consistently, the increase of propagation velocity of the acceleration wave and tension cause a decrease in this velocity.

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Prędkość fali przyspieszenia propagującej w hipersprężystym materiale Zahorskiego i Mooneya-Rivlina

Streszczenie

W pracy rozważana jest jednokrotna deformacja statyczna ciała nieściśliwego. Przedstawiona została analiza porównawcza procesu falowego zachodzącego w materiałach hipersprężystych, które rózni liniowa (materiał Mooneya-Rivlina) i nieliniowa (materiał Zahorskiego) zależność od niezmienników tensora odkształcenia Cauchy-Greena. Przeprowadzona analiza numeryczna wyraźnie wykazała istotne różnice ilościowe w procesie propagacji fali przyspieszenia. Różnice te są następstwem przyjętych do obliczeń potencjałów sprężystych.

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