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MODELLING AND CONTROL OF MECHATRONIC SYSTEMS BY THE DESCRIPTOR APPROACH

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In recent years, the analysis and synthesis of control systems in a descriptor form has been established. The general description of dynamical systems by differential-algebraic equations (DAE) is important for many applications in various disciplines, but particularly in mechatronics. In this contribution, the pros and cons of the modelling of mechatronic systems by differential-algebraic equations are discussed with application of subsystem modelling. Additionally, the actual state of the art simulation, analysis and design of descriptor systems are presented.

Key words: descriptor systems, mechatronic systems, optimal control

1. Modelling

The investigation of dynamical systems in mechanical or electrical engineering usually requires mathematical modelling of the system behaviour. The increasing complexity of these processes leads on the one hand to the development of computer programs automatically generating the governing system equations (Schiehlen, 1990) for multibody systems, or on the other hand to an increase of modular subsystem modelling of which the complete model is composed. Usually, this interconnection-oriented modelling describes dynamic behaviour of single components by differential equations and the coupling of the subsystems by algebraic equations. All over, a mathematical model is represented by a combined set of differential and algebraic, i.e. by differential-algebraic equations (DAE). In control engineering, we speak about singular control systems or descriptor systems (Luenberger, 1977).

Mechatronic systems usually consist of a large number of mechanical, hydraulic, electrical and electronic components where the subsystem modelling represents the most clear and manageable way of modelling that maintains the physical character of the components. For example, electrical networks can be considered to be composed of subsystems of network elements (like resistors, capacitors, inductors described by different types of equations) and by coupling due to Kirchhoff's laws (described by algebraic equations) (Kampowski *et al.*, 1992; Mathis, 1992). In mechanical systems, differential equations usually describe the dynamics of subsystems and algebraic equations characterise couplings by constraints such as joints. A general approach to handle mechanical systems as an interconnected set of dynamic modules has been given in Rüekgauer and Schiehlen (1997). In the following, three examples of the descriptor modelling are dealt with for illustration.

1.1. Lagrange's equations of the first kind

Lagrange's equations of the first and second kind are well established in analytical mechanics (Rosenberg, 1977). They describe dynamic behaviour of discrete systems, particularly of multibody systems. The difference of the two kinds consists of the manipulation of kinematic constraints. If a kinematic description of the system has been performed by generalised coordinates consistent with the constraints, Lagrange's equation of the second kind can be applied which leads to a set of differential equations only. But if a redundant set of coordinates is used to describe kinematically the system regarding still some constraints explicitly, then the Lagrange's equations of the first kind hold. In the case of holonomic constraints

$$\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{0} \tag{1.1}$$

we have

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\Big) - \frac{\partial L}{\partial \boldsymbol{q}} = \mathbf{Q} + \mathbf{F}^\top \boldsymbol{\lambda} \tag{1.2}$$

where the Lagrangian function $L = T - U$ consists of the kinetic and potential energies T and U, $\mathbf{Q}$ represents nonconservative forces acting on the system. $\mathbf{F} = \mathbf{F}(\boldsymbol{q}) = \partial \boldsymbol{f}/\partial \boldsymbol{q}^\top$ is the Jacobian matrix of the constraints and $\boldsymbol{\lambda}$ is the vector of Lagrange's multiplyers. They represent the constraint forces if the column vectors of $\mathbf{F}^\top$ are normalized. While the variables $\boldsymbol{q}$ describe motion of the system, the Lagrange's multiplyers $\boldsymbol{\lambda}$ give some information on the load of the mechanical structure. Therefore, critical loads due to motion may be considered simultaneously. Equations (1.1) and (1.2) represent a system of DAE. If $\mathbf{Q}$ includes some actuators to control the multibody system, then a descriptor system is under consideration.

1.2. Subsystem modelling

If the interconnection-oriented modelling approach is applied (Müller, 1995), usually the dynamics of N subsystems is described by sets of differential equations

$$\dot{\boldsymbol{x}}_i = \boldsymbol{a}_i(\boldsymbol{x}_i, \boldsymbol{u}_i) \qquad i = 1, \dots, N \tag{1.3}$$

where $\boldsymbol{x}_i$ are the internal state vectors and $\boldsymbol{u}_i$ the control vectors of the corresponding subsystems. The couplings among the subsystems may be obtained kinematically by "constraints" or kinetically by "forces" leading to

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= \boldsymbol{a}_i(\boldsymbol{x}_i, \boldsymbol{u}_i) + \sum_{j=1}^{N} \boldsymbol{a}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j) + \sum_{j=1}^{N} \mathbf{L}_{ij}(\boldsymbol{x}_j)\boldsymbol{\lambda}_j \\ \mathbf{0} &= \sum_{j=1}^{N} \boldsymbol{f}_{ij}(\boldsymbol{x}_j) \qquad i = 1, \dots, N \end{aligned} \tag{1.4}$$

The additional terms compared to (1.3) are the kinetic couplings $\boldsymbol{a}_{ij}$ between subsystems no. i and j, and the kinematic couplings $(1.4)_2$ which have to be considered in dynamic balance equations $(1.4)_1$ by some Lagrange's multipliers $\boldsymbol{\lambda}_j$ with some input matrices $\mathbf{L}_{ij}$ due to coupling requirements. How $\mathbf{L}_{ij}$ is defined more precisely depends on physical principles behind the system discipline; equations (1.1) and (1.2) show an example of mechanical systems. All over, equations (1.4) represent again the descriptor system.

1.3. Tracking control

In mechatronics, often the problem of tracking control arises, e.g. the prescribed path control of a robot. In this case, the process dynamics may be described in the state space by

$$\dot{\boldsymbol{x}} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) \tag{1.5}$$

and it is asked for the control $\boldsymbol{u}$ which guarantees that some output variables $\boldsymbol{y} = \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t})$ follow a prescribed reference path $\boldsymbol{y}_{ref}(t)$

$$\mathbf{0} = \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) - \boldsymbol{y}_{ref} \tag{1.6}$$

This descriptor system (1.5) and (1.6) can be described by

$$\begin{bmatrix} \mathbf{I}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\overline{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}(\boldsymbol{x}, \overline{\boldsymbol{x}}, \boldsymbol{t}) \\ \boldsymbol{c}(\boldsymbol{x}, \overline{\boldsymbol{x}}, \boldsymbol{t}) - \boldsymbol{y}_{ref}(t) \end{bmatrix} \tag{1.7}$$

which defines explicitly the desired tracking control

$$\boldsymbol{u}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I}_u \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\overline{\boldsymbol{x}}} \end{bmatrix} \tag{1.8}$$

where $\mathbf{I}_x$, $\mathbf{I}_u$ are identity matrices of $\dim(\boldsymbol{x})$ and $\dim(\boldsymbol{u})$, respectively.

The exact tracking control (1.7) and (1.8) can be extended to an asymptotic tracking control introducing for the tracking error

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{y}_{ref}(t) \tag{1.9}$$

additional error dynamics

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}(\boldsymbol{e}) \qquad\qquad \boldsymbol{f}(\mathbf{0}) = \mathbf{0} \tag{1.10}$$

which are asymptotically stable in the large. Then (1.7) and (1.8) are replaced by

$$\begin{aligned} &\begin{bmatrix} \mathbf{I}_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \\ \dot{\overline{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}(\boldsymbol{x}, \overline{\boldsymbol{x}}, \boldsymbol{t}) \\ \boldsymbol{f}(\boldsymbol{e}) \\ \boldsymbol{c}(\boldsymbol{x}, \overline{\boldsymbol{x}}, \boldsymbol{t}) - \boldsymbol{y}_{ref}(t) - \boldsymbol{e} \end{bmatrix} \\ &\boldsymbol{u}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I}_u \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \\ \dot{\overline{\boldsymbol{x}}} \end{bmatrix} \end{aligned} \tag{1.11}$$

1.4. Advantages

With respect to tasks of the modelling of mechatronic systems, the descriptor approach has many advantages. It is a very natural way to model process dynamics. It refers much more to physical behaviour of a system and gives more physical insight. The interpretation of results is also more simple than in a case of a more abstract description by state space models. In the opposite, the state space system approach was mainly required by mathematical tools available until 1980 to simulate, to analyse and to design such systems. But today, also many tools have been prepared for the simulation, analysis and control design of descriptor systems.

2. Simulation

As long as it was not possible to simulate descriptor systems, the very efficient and very accurate state space approach was still superior according to well established tools of numerical integration of ordinary differential equations. But in the 1970s, the simultaneous numerical solution to differential and algebraic equations was firstly considered (Gear, 1971). Step by step, numerical system solvers were developed. For index-1-problems (see below), the code DASSL was presented (Petzold, 1983), stimulating more research also for higher index problems. The first code for mechanical index-3-systems was presented by Führer (1988). In the meantime, a lot of efficient solvers for DAE have been developed (cf. Brenan *et al.*, 1989; Hairer and Wanner, 1991; Simeon, 1994). In the Ph.D. thesis by Rükgauer (1997) on the modular simulation of mechatronic systems, several solvers were compared resulting in the recommendation of the codes SDOP853 and SDOPPRI5 which are modified versions of Runge-Kutta solvers for ordinary differential equations including projection steps with respect to constraints of the algebraic equation. A survey on solvers of higher index DAEs was given by Arnold (1988). With respect to these results, today a number of stable and efficient DAE solvers exist and can be applied as naturally as ODE solvers for state space models. Such solvers are included in many program packages to generate and simulate equations of motion of dynamical systems, e.g. in ADAMS and SIMPACK (cf. Schiehlen, 1990), for multibody systems.

3. Analysis and synthesis

The tools for the analysis and control design of descriptor systems have been developed enormously in the last two decades. As usual, linear theory was in the foreground of the discussion, but results on nonlinear problems were reported as well. In the following, the well established results of the linear theory are touched only shortly, but the optimal nonlinear control design of descriptor systems will be discussed in more detail.

3.1. Linear systems

Linear time-invariant descriptor systems are presented by

$$\begin{aligned} \mathbf{E}\dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \mathbf{C}\boldsymbol{x}(t) + \mathbf{D}\boldsymbol{u}(t) \end{aligned} \tag{3.1}$$

where $\boldsymbol{x}$ is an n-dimensional descriptor vector, $\boldsymbol{u}$ denotes the r-dimensional control input vector, and $\boldsymbol{y}$ characterises the m-dimensional measurement output vector. The matrices E, A are $n \times n$-matrices, and B, C, D have dimensions $n \times r$, $m \times n$, $m \times r$, respectively. The essential property of descriptor systems is that E is a singular matrix.

$$\operatorname{rank} \mathsf{E} < n \tag{3.2}$$

such that Eq. $(3.1)_1$ consists of differential and algebraic equations.

The basic tool in discussing Eq. $(3.1)_1$ is the theory of matrix pencils $(s\mathsf{E} - \mathsf{A})$ by Weierstrass and Kronecker in the 19th century (cf. Dai, 1989), separating the system into a few subsystems with different properties. Assuming unique behaviour of $(3.1)_1$ for all control inputs, i.e. assuming that the matrix pencil is regular

$$p(s) \equiv \det(s\mathsf{E} - \mathsf{A}) \not\equiv \mathbf{0} \tag{3.3}$$

then system $(3.1)_1$ is strictly equivalent to the Weierstrass-Kronecker canonical form

$$\begin{aligned} \dot{\boldsymbol{x}}_1(t) &= \mathsf{A}_1 \boldsymbol{x}_1(t) + \mathsf{B}_1 \boldsymbol{u}(t) \\ \mathsf{N}_k \dot{\boldsymbol{x}}_2(t) &= \boldsymbol{x}_2(t) + \mathsf{B}_2 \boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \mathsf{C}_1 \boldsymbol{x}_1(t) + \mathsf{C}_2 \boldsymbol{x}_2(t) \end{aligned} \tag{3.4}$$

Equation $(3.4)_1$ represents the 'slow' subsystem of the dimension n_1, and the n_2-dimensional 'fast'subsystem is described by $(3.4)_2$. The $n_2 \times n_2$-matrix N_k is nilpotent of the degree k ($\mathsf{N}_k^{k-1} \neq \mathbf{0}$, $\mathsf{N}_k^k = \mathbf{0}$) defining the index k of the linear descriptor system.

According to the separation into two subsystems, the controllability and observability investigations split off into at least two different concepts of the so-called R/I-controllability and -observability guaranteeing different properties of the feedback control (cf. Dai, 1989; Lewis, 1986). The results of many investigations in the 1980's were summarized in these two references.

The stability can be discussed in terms of eigenvalues of the matrix pencil $(s\mathsf{E} - \mathsf{A})$, i.e. by the roots of characteristic polynomial (3.3), or equivalently by eigenvalues of the system matrix A_1 of the slow subsystem $(3.4)_1$. Another approach is based on the generalized matrix equation

$$\mathsf{A}^\top \mathsf{P} \mathsf{E} + \mathsf{E}^\top \mathsf{P} \mathsf{A} = -\mathsf{Q} \tag{3.5}$$

where definiteness properties of P and Q with respect to certain subspaces assure stability (Müller, 1993).

First results on the design of the linear feedback control by pole placement were presented by Dai (1989). But the main problem of the synthesis of the feedback control consists in the possibility of non-proper system behaviour. This can be seen immediately by the consistent solution to the fast subsystem $(3.4)_2$ (cf. Dai, 1989)

$$\boldsymbol{x}_2(t) = -\mathbf{B}_2\boldsymbol{u}(t) - \mathbf{N}_k\mathbf{B}_2\dot{\boldsymbol{u}} - \ldots - \mathbf{N}_k^{j-1}\mathbf{B}_2\boldsymbol{u}^{j-1}(t) \tag{3.6}$$

which includes generally higher-order time-derivatives of the control input until to the order $j-1$ with $j \leqslant k$ and $\mathbf{N}_k^j\mathbf{B}_2 = \mathbf{0}$. The two cases have to be distinguished, where solution (3.6) depends either only on $\boldsymbol{u}(t)$ but not on its derivatives $\dot{\boldsymbol{u}}(t), \ldots, \boldsymbol{u}^{j-1}(t)$ (i.e. $j=1$) or on $\boldsymbol{u}(t)$ and its derivatives $\dot{\boldsymbol{u}}(t), \ldots, \boldsymbol{u}^{j-1}(t)$ (i.e. $j>1$) according to the general case (3.6). In the first case, the system is called *proper*, in the second case *non-proper* according to related proper and non-proper transfer matrix functions. System $(3.1)_1$ is proper if and only if in representations $(3.4)_{1,2}$, the equation

$$\mathbf{N}_k\mathbf{B}_2 = \mathbf{0} \tag{3.7}$$

holds. The distinction between proper and non-proper descriptor systems and its consequences were discussed by Müller (1998a, 1999a, 2000). Regarding proper and non-proper systems, the linear quadratic optimal regulator problem (Müller, 1998a, 1999a, 2000) and the descriptor state estimation problem (Müller,1999b) were discussed in detail and properly solved. Some initial results on the robust control design exist as well. The H_∞-control problem was considered in Masubuchi *et al.* (1997), Rehm (2003), Takaba *et al.* (1994). In case of stabilizable, detectable, impulse-controllable and impulse-observable descriptor systems necessary and sufficient conditions for the solvability of the control problem are given.

For the analysis and systhesis of linear descriptor systems, usual theoretical tools are (more or less) available and related program packages have been developed (Bunse-Gerstner *et al.*, 2000; Varga, 2000).

3.2. Nonlinear systems

First results on nonlinear descriptor systems were reported in Bajic (1992), Campbell *et al.* (1999), Müller (1998b). Also, first attempts of the optimal control design were given in Cobb (1983), Jonckheere (1988). Particularly, the linear-quadratic optimal regulator for descriptor systems was dealt with in Bender and Laub (1987).

More recently, theoretical papers on optimal control problems with state constraints have been published, see Dmitruk (1993), Hartl *et al.* (1995), Stefani and Zezza (1996). However, usually the papers put severe restrictions on the type of the control problem (such as properness or index one) in order to secure some necessary optimality conditions. Such restrictions are often not satisfied by real practical problems, therefore the above-mentioned results cannot be applied in practice. The main weakness here is the lack of distinction between proper and non-proper descriptor systems. If such distinction is regarded, the optimal control problems can be correctly dealt with. For the linear-quadratic optimal control design of linear descriptor systems this has been fully implemented in Müller (1999a, 2000). In a recent paper (Müller, 2003), the nonlinear optimal control problem has been discussed in detail.

In the present contribution, we will present a survey on these results as mentioned above.

Controlled time-invariant finite-dimensional descriptor systems can be described in a semi-explicit form by

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \end{aligned} \tag{3.8}$$

where $\boldsymbol{x}_i$, $i = 1, 2$ are n_i-dimensional vectors and $n_1 + n_2 = n$. Usually $\boldsymbol{u}$ is an r-dimensional control input vector.

The problem of the optimal control of nonlinear descriptor systems consists in the control design for (3.8) with respect to the performance criterion

$$J = \int_0^T f_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u})\, dt \ \Rightarrow \ minimum \tag{3.9}$$

where $\boldsymbol{u}$ belongs to a set U of bounded or unbounded control functions: $\boldsymbol{u} \in U$. Additionally, boundary conditions for the optimal control design have to be considered. The boundary conditions are given either geometrically or there are dynamical boundary conditions which have to be determined by an optimization procedure. In both cases they have to be consistent with algebraic equations $(3.8)_2$, which have to be satisfied. The corresponding boundary conditions will be not regarded explicitly in the following, because the handling follows from usual rules of the optimal control if the descriptor control problem has been properly formulated. Furthermore, we confine ourselves to principles of correct formulation of the optimal control design of nonlinear descriptor systems. The notation is simplified assuming that all algebraic equations $(3.8)_2$ have a uniform index, i.e. that they have the same index k. Then the underlying set of ordinary differential equations runs as

$$\dot{\boldsymbol{x}}_2 = -\Big(\frac{\partial}{\partial \boldsymbol{x}_2^\top} L_{\boldsymbol{f}_1}^{k-1}(\boldsymbol{f}_2)\Big)^{-1} L^k(\boldsymbol{f}_2) = \overline{\boldsymbol{f}}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}, \dots, \boldsymbol{u}^{(s)}) \tag{3.10}$$

on a manifold defined by algebraic equations (invariants)

$$\frac{d^j \boldsymbol{f}_2}{dt^j} \equiv L^j(\boldsymbol{f}_2) = \boldsymbol{0} \qquad j = 0, \dots, k-1 \tag{3.11}$$

where L, $L_{\boldsymbol{f}_1}$ are suitably defined operators

$$\begin{aligned} &L(\cdot) = L_{\boldsymbol{f}_1}(\cdot) + L_{\dot{\boldsymbol{u}}}(\cdot) + \frac{\Delta}{\Delta t}(\cdot) \\ &L_{\boldsymbol{f}_1}(\cdot) = \frac{\partial(\cdot)}{\partial \boldsymbol{x}_1^\top}\boldsymbol{f}_1 \qquad L_{\boldsymbol{u}^{(j)}}(\cdot) = \frac{\partial(\cdot)}{\partial \boldsymbol{u}^\top}\boldsymbol{u}^{(j)} \\ &\frac{\Delta}{\Delta t} L_{\boldsymbol{u}^{(j)}}(\cdot) = L_{\boldsymbol{u}^{(j+1)}}(\cdot) \end{aligned} \tag{3.12}$$

The first integrals $L^j(\boldsymbol{f}_2) = \boldsymbol{0}$, $j = 0, \dots, k-2$, of (3.10) depend on $\boldsymbol{x}_1$ but not on $\boldsymbol{x}_2$. The function $L^{k-1}(\boldsymbol{f}_2)$ depends on $\boldsymbol{x}_2$ for the first time such that the related Jacobian is regular and differential equation (3.10) can be derived. Also $L^{k-1}(\boldsymbol{f}_2) = \boldsymbol{0}$ is a first integral of (3.10). Additionally, the function $L^j(\boldsymbol{f}_2)$ depends generally on the time-derivatives of the control input $\boldsymbol{u}$: $\boldsymbol{u}, \dot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(j)}$, $j = 0, \dots, s$, $0 \leqslant s \leqslant k$. If $\boldsymbol{u}$ appears explicitly in $L^p(\boldsymbol{f}_2) = \boldsymbol{0}$ for the first time, then $s = k-p$ holds. System representations $(3.8)_1$ and (3.10) have to be supplemented by consistent initial or boundary conditions satisfying invariants (3.11)

$$L^j(\boldsymbol{f}_2)\Big|_{t=0} = \boldsymbol{0} \qquad L^j(\boldsymbol{f}_2)\Big|_{t=T} = \boldsymbol{0} \qquad j = 0, \dots, k-1 \tag{3.13}$$

The different representations of the control system in the preceding Section show that the system behaviour may depend not only on the control input $\boldsymbol{u}$ but also on its time-derivatives $\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(s-1)}, \boldsymbol{u}^{(s)}$. Although DAE description (3.8) shows explicitly only the input $\boldsymbol{u}$, there may be hidden effects related to time-derivatives $\dot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(s)}$ as it is shown by representations $(3.8)_1$, (3.10), (3.11). This situation is very different from the common state space discussions. Any control design method has to take care of this unconventional problem. Therefore, it is necessary to clarify this unusual situation. For these systems, the notion of "properness" is introduced according to the definition of frequency domain for linear systems as it was done in Section 3.1. Descriptor system $(3.8)_1$ is called proper if the solution $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ does not depend on

$\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(s-1)}$ but only on $\boldsymbol{u}$ (and/or on weighted integrals of $\boldsymbol{u}$). Otherwise the system is called non-proper. The notion of proper behaviour can be sharpened for strictly proper systems, if system $(3.8)_1$ is proper and additionally $\boldsymbol{x}_2$ depends on $\boldsymbol{x}_1$ but not on $\boldsymbol{u}$, according to $L^{k-1}(\boldsymbol{f}_2) = \boldsymbol{0}$. Obviously, the description of the dynamical system by equations $(3.8)_1$, (3.10) with the additional boundary conditions is the most convenient one to perform the optimization with respect to performance criterion (3.9) applying well-known methods of calculus of variations or the maximum principle of Pontryagin (Funk, 1970; Leitman, 1981; Pontryagin *et al.*, 1962). The only one problem, which still appears, is correct handling of the time-derivatives $\dot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(s)}$ in the non-proper case. But this problem is easily solved by introducing some extended variables

$$\boldsymbol{\xi}_1 = \boldsymbol{u}, \quad \boldsymbol{\xi}_2 = \dot{\boldsymbol{u}}, \quad \dots, \quad \boldsymbol{\xi}_s = \boldsymbol{u}^{(s-1)}, \quad \boldsymbol{v} = \boldsymbol{u}^{(s)} \tag{3.14}$$

defining a multi-dimensional integrator chain

$$\dot{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_2, \quad \dot{\boldsymbol{\xi}}_2 = \boldsymbol{\xi}_3, \quad \dots, \quad \dot{\boldsymbol{\xi}}_{s-1} = \boldsymbol{\xi}_s, \quad \dot{\boldsymbol{\xi}}_s = \boldsymbol{v} \tag{3.15}$$

By these variables, we are able to state the optimal control problem properly: Minimize the performance criterion, i.e.

$$J = \int\limits_0^T f_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\xi}_1)\, dt \;\Rightarrow\; \mathit{minimum} \tag{3.16}$$

regarding the differential constraints of an extended dynamical system resulting from

$$\dot{\boldsymbol{x}}_e = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\xi}_1) \\ \overline{\boldsymbol{f}}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\xi}_1, \dots, \boldsymbol{v}) \\ \boldsymbol{\xi}_2 \\ \vdots \\ \boldsymbol{\xi}_s \\ \boldsymbol{v} \end{bmatrix} \tag{3.17}$$

where the extended state vector is defined as

$$\boldsymbol{x}_e^\top = \begin{bmatrix} \boldsymbol{x}_1^\top & \boldsymbol{x}_2^\top & \boldsymbol{\xi}_1^\top & \cdots & \boldsymbol{\xi}_s^\top \end{bmatrix} \tag{3.18}$$

The new (fictitious) control input is $\boldsymbol{v}$. The original control constraints $\boldsymbol{u} \in U$ appear as state constraints $\boldsymbol{\xi}_1 \in U$. The given geometrical consistent boundary conditions have to be supplemented.

The necessary conditions for the optimal control design follow from the application of the conditions of Pontryagin's maximum principle, which was shown in Müller (2003). The necessary condition for the solution of optimal control problem (3.16) and (3.17) is the existence of a non-vanishing adjoint vector

$$\begin{bmatrix} \boldsymbol{\lambda}_1^\top & \boldsymbol{\lambda}_2^\top & \boldsymbol{\Psi}_1^\top & \cdots & \boldsymbol{\Psi}_s^\top \end{bmatrix} \neq \mathbf{0} \tag{3.19}$$

satisfying the relations

$$\begin{array}{lll} \dot{\boldsymbol{\lambda}}_1 = -\dfrac{\partial H_{np}}{\partial \boldsymbol{x}_1} & \mathbf{0} = -\dfrac{\partial H_{np}}{\partial \boldsymbol{x}_2} & \dot{\Psi}_1 = -\dfrac{\partial H_{np}}{\partial \boldsymbol{\xi}_1} \\ \dot{\boldsymbol{\Psi}}_i = -\dfrac{\partial H_{np}}{\partial \boldsymbol{\xi}_i} = -\boldsymbol{\Psi}_{i-1} & i = 2,\dots,s & \end{array} \tag{3.20}$$

and

$$H_{np\max} = \underset{v}{Max}\; H_{np} : \quad \boldsymbol{\Psi}_s^\top \boldsymbol{v} \;\Rightarrow\; \underset{v}{Maximum} \tag{3.21}$$

where the Hamiltonian reads

$$H_{np} = \boldsymbol{\lambda}_1^\top \boldsymbol{f}_1 + \boldsymbol{\lambda}_2^\top \boldsymbol{f}_2 + \boldsymbol{\Psi}_1^\top \boldsymbol{\xi}_2 + \cdots + \boldsymbol{\Psi}_s^\top \boldsymbol{v} - f_0 \tag{3.22}$$

Therefore, for non-proper descriptor systems it is necessary to include higher-order time-derivatives into the optimization procedure according to integrator chain (3.14) and (3.15). Additionally, the control constraint $\boldsymbol{u} \in U$ has to be considered as a constraint $\boldsymbol{\xi}_1 \in U$ of extended state (3.18). A practical problem of maximum condition (3.21) is that the constraint on $\boldsymbol{v}$ is not given in advance. Therefore, it is strongly recommended to think about a properly defined optimization problem. Condition (3.21) is an indirect hint that the problem is not well-posed from the very beginning. In many applications, one may be easily convinced that the constraint $\boldsymbol{u} \in U$ is not reasonable but the constraint $\boldsymbol{v} \in V$ does make sense. If the condition $\boldsymbol{u} \in U$ is replaced by $\boldsymbol{v} \in V$, then a usual optimization problem arrives, which is solved by (3.20) and (3.21). After all, a two-point-boundary problem has to be considered to get the optimal (fictitious) control $\boldsymbol{v}$. According to integrator chain (3.14) and (3.15) this is not static (proportional) control but dynamic control.

A special case of the above result, see (3.20) and (3.21), is a solution to the unconstrained optimization problem under sufficient smoothness conditions such that calculus of variations may be applied. In Müller (2003) it was shown that conditions (3.20) and (3.21) can be essentially simplified to necessary conditions of the existence of non-vanishing adjoint variables $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ satisfying the differential-algebraic equations

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H_r}{\partial \boldsymbol{x}_1} \qquad \mathbf{0} = -\frac{\partial H_r}{\partial \boldsymbol{x}_2} \qquad \mathbf{0} = -\frac{\partial H_r}{\partial \boldsymbol{u}} \tag{3.23}$$

Here, the reduced Hamiltonian

$$H_r = H_r(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \boldsymbol{\lambda}_1^\top \boldsymbol{f}_1 + \boldsymbol{\lambda}_2^\top \boldsymbol{f}_2 - f_0 \tag{3.24}$$

is applied. The boundary conditions of the adjoint variables follow usual rules and are not considered explicitly.

The solution to the unconstrained optimization problem consists in the solution to a two-point-boundary problem with respect to two sets (3.8) and $(3.23)_{1,2}$ of differential-algebraic equations having regard to the control from $(3.23)_3$. Unconstrained optimization problems can be solved by the usual approach of the calculus of variations independently of the proper or non-proper behaviour of dynamical system (3.8).

4. Conclusions

In this contribution, an effort to characterise the state of the art of modelling, analysing and designing dynamical processes, particularly mechatronic systems by the descriptor system approach has been made. Without any doubts, the modelling of dynamical systems by differential-algebraic equations has many advantages and is superior to the state space modelling. The simulation tools for DAEs are well established and are comparable with ODE solvers. For linear descriptor systems, the tools for analysis and the control design are available, too. For nonlinear descriptor systems the optimal control design has been shown, but still more research work has to be performed.

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Modelowanie i sterowanie układów mechatronicznych metodą deskrypcyjną

Streszczenie

W ostatnich latach sformułowano i spopularyzowano problem analizy i syntezy układów sterujących w postaci deskrypcyjnej. Ogólny opis układów dynamicznych za pomocą równań różniczkowo-algebraicznych (DAE) ma ogromne znaczenie aplikacyjne w różnych dziedzinach nauki, w szczególności w zakresie mechatroniki. W prezentowanej pracy przedyskutowano wszystkie „za" i „przeciw" modelowania układów mechatronicznych równaniami różniczkowo-algebraicznymi z zastosowaniem podziału opisywanego układu na podsystemy. Ponadto przedstawiono najnowocześniejsze metody symulacji, analizy i projektowania układów deskrypcyjnych.

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