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ELASTIC ELECTROCONDUCTING SURFACE IN MAGNETOSTATIC FIELD

Marek Rudnicki

Faculty of Civil Engineering, Warsaw University of Technology
e-mail: marr@siwy.il.pw.edu.pl

The dynamical linear theory of a material surface placed in vacuum and subjected to an external strong magnetostatic field is considered. Motion of the surface is described by a position function. The material of the surface is assumed to be an isotropic elastic non-magnetizable electric conductor. The residual stress is taken into account. Displacement-based field equations are obtained in a coordinate-free notation.

Key words: magnetoelasticity, material surface, membrane theory, surface current, real electric conductor, perfect electric conductor, residual stress

1. Introduction

A three-dimensional thin body may be represented by a two-dimensional continuum as a result of reduction of the thickness dimension or by a direct approach. A deformable surface with usual kinematics (one deformation function) serves as a direct model underlying the membrane theory. In this paper, we develop the theory of Gurtin and Murdoch (1975) providing an extension necessary for magnetoelastic interactions. The mechanical part is directly obtained as two-dimensional, however, the electromagnetic part is subsequent to three-dimensional considerations. Displacement of the surface, normal magnetic induction at the surface and scalar potentials of outward magnetic induction are unknowns involved in the final field equations. The MKSA system of units is used.

2. Initial state

2.1. Surface

Let s denote a surface in the three-dimensional Euclidean point space Σ endowed with an appropriate structure (see Gurtin and Murdoch, 1975), especially the tangent space T_p and the unimodular vector field $\boldsymbol{a}_3 : s \to V$, where V is the translation space, such that $\boldsymbol{a}_3(p) \in T_p^{\perp}$ at each point $p \in s$. We use the following notation: $\mathbf{I}(p)$ for the inclusion map from T_p into V, $\mathbf{P}(p)$ for the perpendicular projection from V onto T_p. If $c : s \to R$, where R stands for the reals, $\boldsymbol{u} : s \to V$, $\mathbf{S} : s \to V \otimes V$, where $\mathbf{S}(p) \in V \otimes T_p$, then $\operatorname{grad}_s c(p) \in T_p$, $\operatorname{grad}_s \boldsymbol{u}(p) \in V \otimes T_p$, $\operatorname{grad}_s \mathbf{S}(p) \in V \otimes V \otimes T_p$. Moreover, we have

$$\boldsymbol{u} = \mathbf{P}\boldsymbol{u} + u\boldsymbol{a}_3 \qquad\qquad \mathbf{S} = \mathbf{PS} + \boldsymbol{a}_3 \otimes \boldsymbol{S} \tag{2.1}$$

where $u(p) \in R$ and $\boldsymbol{S}(p) \in T_p$ are defined by

$$u = \boldsymbol{u} \cdot \boldsymbol{a}_3 \qquad\qquad \boldsymbol{S} = \mathbf{S}^{\top}\boldsymbol{a}_3 \tag{2.2}$$

with $\mathbf{S}^{\top}$ being the transpose of the tensor $\mathbf{S}$. Given surface gradients and making use of the following notations

$$\begin{aligned}
&\operatorname{skw}(\boldsymbol{a} \otimes \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}) \qquad && \Lambda(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}) = \boldsymbol{a} \times \boldsymbol{b}\\
&\operatorname{tr}(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} && \operatorname{tr}_{(1,3)}(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b}\\
&\delta_{(2,1,3)}(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}) = \boldsymbol{b} \otimes \boldsymbol{a} \otimes \boldsymbol{c}
\end{aligned} \tag{2.3}$$

where $\times$ and $\cdot$ mean the cross product and the inner product, respectively, we define surface divergence and curl operations as

$$\begin{aligned}
\operatorname{div}_s \boldsymbol{u} &= \operatorname{tr}(\mathbf{P} \operatorname{grad}_s \boldsymbol{u})\\
\operatorname{curl}_s \boldsymbol{u} &= -\Lambda[2\operatorname{skw}(\mathbf{P} \operatorname{grad}_s \boldsymbol{u})]\\
\operatorname{div}_s \mathbf{S} &= \operatorname{tr}_{(1,3)} \mathbf{P} \delta_{(2,1,3)} \operatorname{grad}_s \mathbf{S}
\end{aligned} \tag{2.4}$$

Thus, $\operatorname{div}_s \boldsymbol{u}(p) \in R$, $\operatorname{curl}_s \boldsymbol{u}(p) \in T_p^{\perp}$, and $\operatorname{div}_s \mathbf{S}(p) \in V$.

2.2. Static bias magnetic field

The bias magnetic induction $\boldsymbol{B}$ is governed in a certain neighbourhood of the surface s by equations

$$\operatorname{curl} \boldsymbol{B} = \boldsymbol{0} \qquad\qquad \operatorname{div} \boldsymbol{B} = 0 \tag{2.5}$$

Introduce surface vector fields: $\boldsymbol{L}, \boldsymbol{G} : s \to V$ by

$$\boldsymbol{L} = \boldsymbol{B}\Big|_s \qquad\qquad \boldsymbol{G} = \frac{\partial}{\partial x_3}\boldsymbol{B}\Big|_s \tag{2.6}$$

where x_3 is the metric coordinate in the normal direction. Then, when calculating on the surface s, Eqs (2.5), take the form

$$\begin{aligned} &\mathbf{P}\,\mathrm{grad}_s L - \mathbf{K}(\mathbf{P}\boldsymbol{L}) - \mathbf{P}\boldsymbol{G} = \mathbf{0} \\ &\mathrm{curl}_s \boldsymbol{L} = \mathbf{0} \qquad\qquad \mathrm{div}_s \boldsymbol{L} + G = 0 \end{aligned} \tag{2.7}$$

where $\mathbf{K}$ denotes the Weingarten map.

3. Present state

3.1. Kinematics

Deformation of the surface s during the time interval T is a mapping $\chi : s \times T \to \Sigma$. The displacement corresponding to χ is the field $\boldsymbol{u} : s \times T \to V$ defined by

$$\boldsymbol{u}(p,t) = \chi(p,t) - p \tag{3.1}$$

where t is time. Thus

$$\mathrm{grad}_s \chi = \mathbf{I} + \mathrm{grad}_s \boldsymbol{u} \tag{3.2}$$

where $\mathrm{grad}_s \chi(p,t) \in V \otimes T_p$. The rotation field corresponding to $\boldsymbol{u}$ is a mapping $\boldsymbol{r} : s \times T \to V$ defined by

$$\mathbf{P}\boldsymbol{r} = (\mathrm{grad}_s \boldsymbol{u})^\top \boldsymbol{a}_3 \qquad\qquad r = \frac{1}{2}\boldsymbol{a}_3 \cdot \mathrm{curl}_s \boldsymbol{u} \tag{3.3}$$

The infinitesimal strain reads

$$\mathbf{E} = \mathrm{sym}(\mathbf{P}\,\mathrm{grad}_s \boldsymbol{u}) = \mathrm{sym}[\mathbf{P}\,\mathrm{grad}_s(\mathbf{P}\boldsymbol{u})] + u\mathbf{K} \tag{3.4}$$

where "sym" means the symmetrical part of a tensor.

3.2. Magnetic field outside the surface

Let Ω^+ and Ω^- denote certain outward material-free regions touching the surface s from the upper and lower side, respectively, and $\boldsymbol{b}$ represents induced magnetic induction governed in the regions Ω^+ and Ω^- by the equations

$$\mathrm{curl}\boldsymbol{b} = 0 \qquad\qquad \mathrm{div}\,\boldsymbol{b} = 0 \tag{3.5}$$

accompanied by the continuity condition at the surface s in the form

$$[b] = 0 \tag{3.6}$$

where $[\cdot]$ denotes the jump across the surface. Introducing scalar potentials $\psi^+ : \Omega^+ \times T \to R$ and $\psi^- : \Omega^- \times T \to R$ with the use of the space gradient

$$\boldsymbol{b} = \mathrm{grad}\,\psi \tag{3.7}$$

Eqs (3.5) lead to the Laplace equations in the regions Ω^+ and Ω^-

$$\Delta\psi^+ = 0 \qquad\qquad \Delta\psi^- = 0 \tag{3.8}$$

with the Neumann boundary conditions on the surface s

$$\frac{\partial}{\partial x_3}\psi^+ = b \qquad\qquad \frac{\partial}{\partial x_3}\psi^- = b \tag{3.9}$$

3.3. Electromagnetic field within the surface

The surface current density on the surface s is determined by the relation

$$\boldsymbol{j}^{sur} = \frac{1}{\mu}\boldsymbol{a}_3 \times [\boldsymbol{b}] \tag{3.10}$$

where μ means the magnetic permeability of vacuum. Moreover, the quantities b and $\mathbf{P}\boldsymbol{e}$, where $\boldsymbol{e}$ denotes the electric field, are identical at both sides of the surface s. The corresponding differential equation reads

$$\mathrm{curl}_s(\mathbf{P}\boldsymbol{e}) - \frac{\partial}{\partial t}b = 0 \tag{3.11}$$

Making use of the inverted Ohm law

$$\mathbf{P}\boldsymbol{e} = \frac{1}{\lambda}\mathbf{P}\boldsymbol{j}^{sur} + \mathbf{P}(\boldsymbol{L} \times \boldsymbol{v}) = \frac{1}{\mu\lambda}\mathbf{P}(\boldsymbol{a}_3 \times \mathbf{I}\,\mathrm{grad}_s[\psi]) + \mathbf{P}(\boldsymbol{L} \times \boldsymbol{v}) \tag{3.12}$$

where $\boldsymbol{v}$ denotes the velocity vector and λ is the electric surface conductivity, Eq (3.11) becomes

$$\varDelta_s(\psi^+ - \psi^-) - \mu\lambda\frac{\partial}{\partial t}b + \mu\lambda\frac{\partial}{\partial t}[L\,\mathrm{div}_s\boldsymbol{u} + \boldsymbol{G}\cdot\boldsymbol{u} - (\mathbf{P}\boldsymbol{L})\cdot(\mathbf{P}\boldsymbol{r})] = 0 \qquad (3.13)$$

where $\varDelta_s$ stands for the surface Lagrangian. In the case of perfect conduction, Eq (3.13) simplifies to the relation

$$b = L\,\mathrm{div}_s\boldsymbol{u} + \boldsymbol{G}\cdot\boldsymbol{u} - (\mathbf{P}\boldsymbol{L})\cdot(\mathbf{P}\boldsymbol{r}) \qquad (3.14)$$

3.4. Electromagnetic momentum and energy

The following linearized identity is derivable from three-dimensional Maxwell equations when simplified by neglecting the displacement current

$$\boldsymbol{f}^L = \mathrm{div}\,\mathbf{T}^M \qquad (3.15)$$

where $\boldsymbol{f}^L$ and $\mathbf{T}^M$ are the electromagnetic force and magnetic stress, respectively, defined by (see Costen and Adamson, 1965)

$$\boldsymbol{f}^L = \boldsymbol{j}\times\boldsymbol{B} \qquad \mathbf{T}^M = \frac{1}{\mu}(\boldsymbol{b}\otimes\boldsymbol{B} + \boldsymbol{B}\otimes\boldsymbol{b}) - w^M\mathbf{1} \qquad (3.16)$$

where, in turn, $\boldsymbol{j}$ is the conduction current density, $\mathbf{1}$ denotes the identity on V, and w^M means the electromagnetic energy density in the form

$$w^M = \frac{1}{\mu}\boldsymbol{B}\cdot\boldsymbol{b} \qquad (3.17)$$

Similarly, the power per unit volume lost by the fields equals

$$P^M = -\,\mathrm{div}\,\boldsymbol{S}^M - \frac{\partial}{\partial t}w^M = 0 \qquad (3.18)$$

where

$$\boldsymbol{S}^M = \frac{1}{\mu}\boldsymbol{e}\times\boldsymbol{B} \qquad (3.19)$$

denotes the Poynting vector. In an integral form, the electromagnetic momentum and energy laws are

$$\int\limits_V \boldsymbol{f}^L = \int\limits_{\partial V}\mathbf{T}^M\boldsymbol{n} \qquad \int\limits_{\partial V}\boldsymbol{S}^M\boldsymbol{n} + \int\limits_V \frac{\partial}{\partial t}w^M = 0 \qquad (3.20)$$

where $\boldsymbol{n}$ represents the outward unit vector normal to the surface ∂V. In the limit for the surface s, setting $\boldsymbol{n} = \boldsymbol{a}_3$, the electromagnetic momentum law reduces to

$$\boldsymbol{f}^{sur} = [\mathsf{T}^M]\boldsymbol{a}_3 \tag{3.21}$$

Using Eqs (3.16), (3.10) and (3.17), we find

$$[\mathsf{T}^M] = (\boldsymbol{j}^{sur} \times \boldsymbol{a}_3) \otimes \boldsymbol{B} + \boldsymbol{B} \otimes (\boldsymbol{j}^{sur} \times \boldsymbol{a}_3) - [(\boldsymbol{j}^{sur} \times \boldsymbol{a}_3) \cdot \boldsymbol{B}]\mathbf{1} \tag{3.22}$$

Hence

$$[\mathsf{T}^M]\boldsymbol{a}_3 = (\boldsymbol{j}^{sur} \times \boldsymbol{a}_3)(\boldsymbol{B} \cdot \boldsymbol{a}_3) - [(\boldsymbol{j}^{sur} \times \boldsymbol{a}_3) \cdot \boldsymbol{B}]\boldsymbol{a}_3 = \boldsymbol{j}^{sur} \times \boldsymbol{B} \tag{3.23}$$

Similarly,

$$[\boldsymbol{S}^M] = \frac{1}{\mu}[\boldsymbol{e}] \times \boldsymbol{B} = \frac{1}{\mu}[e]\boldsymbol{a}_3 \times \boldsymbol{B} \tag{3.24}$$

Thus, the electromagnetic energy law for the surface s takes the form

$$[\boldsymbol{S}^M] \cdot \boldsymbol{a}_3 = 0 \tag{3.25}$$

3.5. Stress-based equations of motion

The stress equation of motion of a material surface has the local form

$$\operatorname{div}_s\mathsf{S} + \boldsymbol{f}^{mech} + \boldsymbol{f}^{sur} = \rho\frac{\partial^2}{\partial t^2}\boldsymbol{u} \tag{3.26}$$

where S denotes the surface stress tensor, ρ is the mass density per unit area, and $\boldsymbol{f}^{mech}$ stands for the mechanical force. Using Eqs (2.1), (2.6), (3.10) and (3.7), Eq (3.26) may be put in a more detailed form

$$\begin{aligned}
&\mathsf{P}\operatorname{div}_s(\mathsf{PS}) + \mathsf{K}\boldsymbol{S} + \mathsf{P}\boldsymbol{f}^{mech} + \frac{1}{\mu}L\operatorname{grad}_s[\psi] = \rho\frac{\partial^2}{\partial t^2}(\mathsf{P}\boldsymbol{u})\\
&\operatorname{div}_s\mathsf{S} - \mathsf{K}\cdot(\mathsf{PS}) + f^{mech} - \frac{1}{\mu}(\mathsf{P}\boldsymbol{L})\operatorname{grad}_s[\psi] = \rho\frac{\partial^2}{\partial t^2}u
\end{aligned} \tag{3.27}$$

where ”·” denotes the inner product of two tensors.

3.6. Stress-strain relation

The constitutive relation for the stress S reads

$$\mathsf{S} = (\operatorname{grad}_s\chi)\{\mathsf{S}^{res} + \mathsf{C}[\mathsf{E}]\} \tag{3.28}$$

where S^{res} is the residual stress and C denotes the elasticity tensor. If the material is isotropic relative to the reference configuration, then

$$\mathsf{S}^{res} = \sigma \mathbf{1}_s \qquad \mathsf{C}[\mathsf{E}] = \lambda_L(\operatorname{tr}\mathsf{E})\mathbf{1}_s + 2\mu_L\mathsf{E} \tag{3.29}$$

where λ_L and μ_L are Lame constants, and $\mathbf{1}_s(p)$ is the identity on T_p. Making use of Eqs (2.1) and (3.2), we arrive at

$$\mathsf{PS} = \sigma\mathbf{1}_s + \sigma\mathsf{P}\operatorname{grad}_s\boldsymbol{u} + \lambda_L(\operatorname{tr}\mathsf{E})\mathbf{1}_s + 2\mu_L\mathsf{E} \qquad \boldsymbol{S} = \sigma\mathsf{P}\boldsymbol{r} \tag{3.30}$$

3.7. Displacement-based equations of motion

Now assume that σ, λ_L and μ_L are constant on the surface s. Then, making use of Eqs (3.30) and (3.4), Eqs (3.27) are transformed to the displacement-based form

$$\begin{aligned}
&(\sigma+2\mu_L)\mathsf{P}\operatorname{div}_s[\mathsf{P}\operatorname{grad}_s(\mathsf{P}\boldsymbol{u})] + \lambda_L\operatorname{grad}_s\operatorname{div}_s(\mathsf{P}\boldsymbol{u}) -\\
&-2\mu_L\boldsymbol{a}_3\times[\mathsf{I}\operatorname{grad}_s(\boldsymbol{a}_3\operatorname{curl}_s\boldsymbol{u})] - \sigma\mathsf{K}[\mathsf{K}(\mathsf{P}\boldsymbol{u})] +\\
&+2(\sigma+\mu_L)\mathsf{K}\operatorname{grad}_s u + 2H\lambda_L\operatorname{grad}_s u + 2(\sigma+2\mu_L+\lambda_L)(\operatorname{grad}_s H)u +\\
&+\mathsf{P}\boldsymbol{f}^{mech} + \frac{1}{\mu}L\operatorname{grad}_s(\psi^+ - \psi^-) = \rho\frac{\partial^2}{\partial t^2}(\mathsf{P}\boldsymbol{u})\\
&\sigma\Delta_s u - (\sigma+2\mu_L)(\mathsf{K}\cdot\mathsf{K})u - \lambda_L(2H)^2u - 2H\sigma - 2\sigma(\operatorname{grad}_s H)\cdot(\mathsf{P}\boldsymbol{u}) -\\
&-2\lambda_L H\operatorname{div}_s(\mathsf{P}\boldsymbol{u}) - 2(\sigma+\mu_L)\mathsf{K}\cdot[\mathsf{P}\operatorname{grad}_s(\mathsf{P}\boldsymbol{u})] +\\
&+\boldsymbol{f}^{mech} - \frac{1}{\mu}(\mathsf{P}\boldsymbol{L})\cdot\operatorname{grad}_s(\psi^+ - \psi^-) = \rho\frac{\partial^2}{\partial t^2}u
\end{aligned} \tag{3.31}$$

where H is the mean curvature.

4. Conclusions

- In order to incorporate magnetoelastic effects in the theory of a material surface, the concept of surface electric current is required, even in the case of real conduction.
- The obtained model is not entirely two-dimensional because Eqs (3.8) are needed for completeness.

- The lack of a term including normal bias magnetic induction in the second equation of motion seems to be the most significant difference occurring within the electromagnetic part between the presented model and shell-like models based on the electromagnetic thickness hypotheses (cf. Rudnicki, 1995).

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Elektroprzewodząca powierzchnia sprężysta w polu magnetostatycznym

Streszczenie

Przedmiotem rozważań jest teoria liniowa powierzchni materialnej umieszczonej w próżni i poddanej działaniu silnego zewnętrznego pola magnetostatycznego. Ruch powierzchni opisuje funkcja położenia. Założono, że materiał powierzchni jest izotropowy, sprężysty, niemagnetyzowalny i przewodzący prąd elektryczny. Uwzględniono naprężenia rezydualne. Otrzymano równania rozwiązujące z użyciem przemieszczeń.

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