JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
44, 3, pp. 485-503, Warsaw 2006

EFFECT OF CHARACTERISTIC LENGTH ON NONLOCAL PREDICTION OF DAMAGE AND FRACTURE IN CONCRETE

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The paper deals with a new nonlocal integral-type model for simulation of an anisotropic, localised damage and for prediction of combined failure modes in a plane-notched concrete specimen. The nonlocal incremental-type model of the elastic-brittle-damage material is an extension of the relevant local model originated by Murakami and Kamiya (1997), modified later to the incremental form by Kuna-Ciskał and Skrzypek (2004). In order to avoid the mesh-dependence and ensure stability and convergence, two localisation limiters are examined: the concept of Nonlocal Averaging (NA) and the additional Cut-off Algorithm (CA), applied to damage conjugate thermodynamic forces. The elastic-brittle damage constitutive equations are formulated in an incremental and nonlocal fashion, by the use of a damage dissipation potential defined in the space of averaged regularised damage variables instead of the corresponding local ones. The Gauss distribution function is taken as the weight function for the definition of a nonlocal continuum. In order to assess how much the new nonlocal model is capable of describing localised strain and damage fields, an example of the plane double-notched specimen of Nooru-Mohammed (1992) is examined. Much emphasis is put to proper choice of the characteristic length of the nonlocal continuum. Convergence of the mesh size is proved for both, the damage incubation period and fracture, when a single localisation limiter (NA) is active.

Key words: nonlocal approach, anisotropic damage, characteristic length, mesh-dependence

1. Introduction

The use of classical local constitutive models is insufficient for problems where a strong strain softening effect occurs. In case of inelastic material behaviour, two dissipative processes are responsible for the strain softening: (visco)plasticity and/or damage (cf. Hansen and Schreyer, 1994; Abu Al-Rub and Voyiadjis, 2003). In the case of Local Models (LM), the stress at a given point is assumed to be uniquely determined by the strain history at this point only. However, as the (visco)plasticity and damage frequently localize over narrow zones of a continuum, statistical homogeneity in a representative volume element is lost. Hence, the characteristic length scale has to be introduced into the nonlocal model (NL) in order to account for the influence of an internal state variable also at neighbouring points. From a computational point of view, in the case of the localised phenomena, ill-posedness of the boundary value problem and mesh sensitivity of finite element computations are met. In particular, incorporation of viscosity retains ellipticity of the problem, such that the well-posedness is preserved because the viscosity implicitly introduces length-scale measures that reduce the strain and damage localisation (cf. Wang *et al.*, 1996; Dornowski and Perzyna, 2000; Glema *et al.*, 2000). On the other hand, if (visco)plasticity is not accounted for, some computational localisation limiters should be used provided by the concept of nonlocal weighted averaging (cf. Bažant, 1984; Pijaudier-Cabot and Bažant, 1987; Bažant and Pijaudier-Cabot, 1988; Jirasek, 1998; Comi, 2001; Comi and Perego, 2004; Voyijadis and Abu Al-Rub, 2002).

If $f(\boldsymbol{x})$ is a local field in a volume V, the corresponding nonlocal field is defined as

$$\overline{f}(\boldsymbol{x}) = \frac{1}{V}\int\limits_V h(\boldsymbol{x},\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{1.1}$$

where $h(\boldsymbol{x},\boldsymbol{\xi})$ is a monotonically decreasing weight function, defined in such a way that a uniform field is not altered by it

$$V(\boldsymbol{\xi}) = \int\limits_V h(r)\, dV \qquad\qquad r = |\boldsymbol{x}-\boldsymbol{\xi}| \tag{1.2}$$

As the weight function $h(r)$, the Gauss distribution function

$$h(r) = \exp\Big(-\frac{r^2}{2l^2}\Big) \tag{1.3}$$

or the bell-shape function

$$h(r) = \begin{cases} \Big(1-\dfrac{r^2}{R^2}\Big)^2 & 0 \leqslant r \leqslant R \\ 0 & R \leqslant r \end{cases} \tag{1.4}$$

are frequently used. In (1.3) l is the internal length of the nonlocal continuum, whereas R is called the interaction length that is related, but not equal, to the internal length l. If the orientation of material fibres is considered, a more complicated averaging operator might be used, where not only the distance between the points $\boldsymbol{x}$ and $\boldsymbol{\xi}$, but also the orientation of principal axes at these points are accounted for (cf. Bažant, 1994).

In the present paper, the nonlocal measures (NA) of the force conjugates $\{\overline{\mathbf{Y}}, \overline{B}\}$ are defined instead of the local ones $\{\mathbf{Y}, B\}$ previously used (LA). The use of another Cut-off Algorithm (CA), originated by Murakami and Liu (1995) $\hat{\mathbf{Y}} = k\mathbf{Y}$, where $k = 1$ if $Y_{eq} \leqslant Y_u$ or $k = Y_u/Y_{eq}$ if $Y_{eq} > Y_u$ is also tested from the point of view of convergence. The nonlocal measure $\overline{\mathbf{Y}}$ of the strain energy release rate $\mathbf{Y}$ was earlier used by Pijaudier-Cabot and Bažant (1987) and by Comi and Perego (2004) for a simple isotropic elastic-damage model.

By contrast, a more extended case is considered in the present paper, where the anisotropic damage measure $\mathbf{D}$ is used and an additional scalar parameter β stands for the damage hardening. The other possibility is to directly average the damage as suggested by Bažant and Pijaudier-Cabot (1988) or strain (cf. Bažant and Liu, 1988). The nonlocal variables $\{\overline{\mathbf{Y}}, \overline{B}\}$, with the cut-off algorithm $\{\hat{\mathbf{Y}}, \hat{B}\}$ if necessary, affect the nonlocal definition of the damage dissipation potential $F(\overline{\mathbf{Y}}, \overline{B})$ instead of the traditional one $F(\mathbf{Y}, B)$, when the local approach is used (cf. Murakami and Kamiya, 1997). The developed model is capable of capturing the damage anisotropy and deactivation (incubation period) as well as the failure mechanism (fracture). The essential point is how to properly choose the internal length parameter of the nonlocal continuum. This length may be assessed by experimental comparison of energy in a specimen where damage is constrained to remain diffuse, and another one where damage localizes to yield a single crack (cf. Mazars and Pijaudier-Cabot, 1989, 1996). It may also be established from the maximum size of the aggregate in concrete d_a, such that $l \approx 3d_a$ holds (cf. also Saouridis and Mazars, 1992). Some particular suggestions can be found in the comparative study on different models for concrete: local (Ottosen) or nonlocal (nonlocal damage and gradient plasticity). The internal length values are thus set to $l \approx 5\,\mathrm{mm}$ for the gradient plasticity model, and to $l \approx 8\,\mathrm{mm}$ for the nonlocal damage model. On the other hand, Comi and Perego (2004) used for the nonlocal concrete model the value $l = 1.1\,\mathrm{mm}$. In what follows, a proper characteristic length is numerically assessed from simulation tests on damage and fracture prediction in the double-notched specimen of Nooru-Mohammed (1992). Different values of characteristic lengths for concrete are numerically tested in the present paper, ranging from 0 mm (local) to 20 mm, to finally assess the value $l = 7.5$ mm as the "optimal" one, to preserve characteristic damage incubation and ultimate localised failure prediction without violating the stability and mesh convergence.

2. Total form of the local elastic brittle damage constitutive model

When a total stress-strain formulation is used, the general thermodynamically based theory of local constitutive and evolution equations of an elastic-brittle damaged material is the key for further extension (cf. Murakami and Kamiya, 1997). The Helmholtz free energy is defined in a local fashion as a function of the elastic strain tensor $\boldsymbol{\varepsilon}^e$, the second rank damage tensor $\mathbf{D}$, and the scalar damage hardening variable β. The Helmholtz free energy, decomposed into the elastic $\psi^e(\boldsymbol{\varepsilon}^e, \mathbf{D})$ and damage $\psi^d(\beta)$ terms, is postulated as a state potential

$$\rho\psi(\boldsymbol{\varepsilon}^e, \mathbf{D}, \beta) = \rho\psi^e(\boldsymbol{\varepsilon}^e, \mathbf{D}) + \rho\psi^d(\beta) \tag{2.1}$$

Following Murakami and Kamiya assumptions, both terms of free energy (2.1) are represented as

$$\begin{aligned} \rho\psi^e(\boldsymbol{\varepsilon}^e, \mathbf{D}) &= \frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{\varepsilon}^e)^2 + \mu \operatorname{tr}(\boldsymbol{\varepsilon}^e \cdot \boldsymbol{\varepsilon}^e) + \eta_1(\operatorname{tr}\boldsymbol{\varepsilon}^e)^2 \operatorname{tr}\mathbf{D} + \\ &+\eta_2 \operatorname{tr}(\boldsymbol{\varepsilon}^e \cdot \boldsymbol{\varepsilon}^e)\operatorname{tr}\mathbf{D} + \eta_3 \operatorname{tr}\boldsymbol{\varepsilon}^e \operatorname{tr}(\boldsymbol{\varepsilon}^e \cdot \mathbf{D}) + \eta_4 \operatorname{tr}(\boldsymbol{\varepsilon}^{*e} \cdot \boldsymbol{\varepsilon}^{*e} \cdot \mathbf{D}) \\ \rho\psi^d(\beta) &= \frac{1}{2}K_d\beta^2 \end{aligned} \tag{2.2}$$

where λ and μ are Lamè constants for the undamaged material, whereas η_1, η_2, η_3, η_4 and K_d are the damage material constants.

In order to properly describe the unilateral damage response under tension or compression, the modified elastic strain tensor $\boldsymbol{\varepsilon}^{*e}$ is defined in the principal strain co-ordinate system

$$\begin{aligned} \varepsilon_I^{*e} &= \langle\varepsilon_I^e\rangle + \zeta\langle-\varepsilon_I^e\rangle = k(\varepsilon_I^e)\varepsilon_I^e \qquad && \zeta \in \langle 0, 1\rangle \\ k(\varepsilon_I^e) &= k_I = H(\varepsilon_I^e) + \zeta H(-\varepsilon_I^e) && I = 1, 2, 3 \end{aligned} \tag{2.3}$$

where $\langle\cdot\rangle$ denotes the Macauley bracket, $H(\cdot)$ is the Heaviside step function, $\varepsilon_I^e (I = 1, 2, 3)$ are the principal values of $\boldsymbol{\varepsilon}^e$, and ζ is an additional material constant responsible for the unilateral damage response effect under tension or compression (cf. Krajcinovic, 1996). For $\zeta = 1$ the modified strain tensor $\boldsymbol{\varepsilon}^{*e}$ is identical to $\boldsymbol{\varepsilon}^{e}$, so that the unilateral damage (crack opening/closure) effect is not accounted for. In contrast, for $\zeta = 0$, the strain tensor $\boldsymbol{\varepsilon}^{*e}$ is modified in such a way that the negative principal strain components are replaced by zeros, whereas the positive ones remain unchanged.

When the general coordinate system is used, the modified strain tensor $\boldsymbol{\varepsilon}^{*e}$ is expressed as follows (cf. Hayakawa and Murakami, 1997)

$$\varepsilon_{ij}^{*e} = \sum_{I=1}^{3} \varepsilon_I^{*e} Q_{Ii} Q_{Ij} = \sum_{I=1}^{3} k(\varepsilon_I^e)\varepsilon_I^e Q_{Ii} Q_{Ij} = B_{ijkl}\varepsilon_{kl}^e \tag{2.4}$$

where $B_{ijkl} = \sum_{I=1}^{3} k(\varepsilon_I) Q_{Ii} Q_{Ij} Q_{Ik} Q_{Il}$ is a fourth rank tensor built of the direction cosines between the principal strain axes and the current spatial system.

The following local, total stress–strain relations of the anisotropic elasticity coupled with damage are furnished from $(2.2)_1$ according to the conventional procedure of the thermodynamic formalism

$$\begin{aligned}\boldsymbol{\sigma} = \frac{\partial(\rho\psi)}{\partial\boldsymbol{\varepsilon}^e} &= [\lambda(\operatorname{tr}\boldsymbol{\varepsilon}^e) + 2\eta_1(\operatorname{tr}\boldsymbol{\varepsilon}^e)(\operatorname{tr}\mathbf{D}) + \eta_3 \operatorname{tr}(\boldsymbol{\varepsilon}^e\cdot\mathbf{D})]\mathbf{I} + \\ &+2[\mu + \eta_2(\operatorname{tr}\mathbf{D})]\boldsymbol{\varepsilon}^e + \eta_3\operatorname{tr}(\boldsymbol{\varepsilon}^e)\mathbf{D} + \eta_4(\boldsymbol{\varepsilon}^{*e}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\varepsilon}^{*e}) : \frac{\partial\boldsymbol{\varepsilon}^{*e}}{\partial\boldsymbol{\varepsilon}^e}\end{aligned} \tag{2.5}$$

When the vector-matrix notation is used, the above reduces to

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{\Lambda}^s(\mathbf{D})]\{\boldsymbol{\varepsilon}^e\} \tag{2.6}$$

where $[\boldsymbol{\Lambda}^s(\mathbf{D})]$ stands for the locally defined secant stiffness matrix.

The thermodynamic force conjugates of the internal state variables $\{\mathbf{D}, \beta\}$ are defined in a local form as follows

$$\begin{aligned}\mathbf{Y} &= -\rho\frac{\partial\psi^e}{\partial\mathbf{D}} = -[\eta_1(\operatorname{tr}\boldsymbol{\varepsilon}^e)^2 + \eta_2\operatorname{tr}(\boldsymbol{\varepsilon}^e\cdot\boldsymbol{\varepsilon}^e)]\mathbf{I} - \eta_3(\operatorname{tr}\boldsymbol{\varepsilon}^e)\boldsymbol{\varepsilon}^e - \eta_4\boldsymbol{\varepsilon}^{*e}\cdot\boldsymbol{\varepsilon}^{*e} \\ B &= \rho\frac{\partial\psi^d}{\partial\beta} = K_d\beta\end{aligned} \tag{2.7}$$

The damage dissipation potential, defined in the space of local force conjugates $\{\mathbf{Y}, -B\}$, is assumed in the form (cf. Murakami and Kamiya, 1997)

$$F(\mathbf{Y}, B) = Y_{eq} - (B_0 + B) = 0 \qquad\qquad Y_{eq} = \sqrt{\frac{1}{2}\mathbf{Y} : \mathbf{L} : \mathbf{Y}} \tag{2.8}$$

where B_0 and B stand for the initial damage threshold and the subsequent damage force conjugate, respectively. The fourth-rank tensor $\mathbf{L}$ is defined here in a simplified way

$$L_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.9}$$

However, in a more general case, it may also be assumed as a tensor function of damage $\mathbf{L}(\mathbf{D})$, linear in $\mathbf{D}$ (cf. Hayakawa and Murakami, 1997; Bielski *et al.*, 2006). Finally, the local damage evolution equations are established from the damage potential by the normality rule

$$\dot{\mathbf{D}} = \dot{\lambda}_d \frac{\partial F}{\partial \mathbf{Y}} \qquad \dot{\beta} = \dot{\lambda}_d \tag{2.10}$$

where the consistency condition is used to eliminate $\dot{\lambda}_d$

$$\dot{\lambda}_d = \frac{\alpha \frac{\partial F}{\partial \mathbf{Y}} : \dot{\mathbf{Y}}}{\frac{\partial B}{\partial \beta}} = \alpha \frac{\mathbf{L} : \mathbf{Y}}{2K_d Y_{eq}} : \dot{\mathbf{Y}} \tag{2.11}$$

3. Nonlocal formulation of the damage dissipation potential and evolution equations

Local state equations (2.1) through (2.6), combined with the local evolution law for $\dot{\mathbf{D}}$ and $\dot{\beta}$, were applied by Kuna-Ciskał and Skrzypek (2004). This approach, however, is not capable of predicting damage evolution in the case of localised damage and strain fields because of a spurious mesh effect. In what follows, the Nonlocal Approach (NA) to damage dissipation and evolution is proposed. In general, to avoid the singularity of $\mathbf{Y}$ at the crack tip when the mesh size tends to zero, the Cut-off Algorithm (CA) may optionally be used in the neighbourhood of the crack tip, according to the scheme (cf. Skrzypek *et al.*, 2005a, 2005b)

$$\hat{\mathbf{Y}} = k\mathbf{Y} \qquad k = \begin{cases} 1 & \text{if} \quad Y_{eq} \leqslant Y_u \\ \dfrac{Y_u}{Y_{eq}} & \text{if} \quad Y_{eq} > Y_u \end{cases} \tag{3.1}$$

where the cut-off factor k is defined as follows

$$k = \frac{Y_u}{Y_{eq}} = \frac{B_0 + B}{Y_{eq}} \tag{3.2}$$

and B_0 stands for the initial damage threshold. The new variable $\hat{\mathbf{Y}}$ is next subjected to the nonlocal treatment (NA) $\overline{\hat{\mathbf{Y}}}$, according to the following formula (cf. e.g. Pijaudier-Cabot and Bažant, 1987; Comi and Perego, 2004)

$$\overline{\hat{\mathbf{Y}}} = \frac{\int\limits_{\Omega_d} \hat{\mathbf{Y}}(\boldsymbol{\xi})\varphi(\boldsymbol{x},\boldsymbol{\xi})\, d\Omega_d}{\int\limits_{\Omega_d} \varphi(\boldsymbol{x},\boldsymbol{\xi})\, d\Omega_d} \qquad \varphi(\boldsymbol{x},\boldsymbol{\xi}) = \exp\Big[-\Big(\frac{d(\boldsymbol{x},\boldsymbol{\xi})}{d^*}\Big)^2\Big] \tag{3.3}$$

where $\varphi(\boldsymbol{x},\boldsymbol{\xi})$ is the weight function and d^* stands for the internal length of the nonlocal continuum.

The damage dissipation potential in the space of nonlocal force conjugates $\{\overline{\hat{\mathbf{Y}}}, -\overline{B}\}$ is assumed in an analogous form as in the local space $\{\mathbf{Y}, -B\}$, where only the isotropic hardening is accounted for (2.8)

$$\overline{F}(\overline{\hat{\mathbf{Y}}}, \overline{B}) = \overline{\hat{Y}}_{eq} - (B_0 + \overline{B}) = 0 \qquad \overline{\hat{Y}}_{eq} = \sqrt{\frac{1}{2}\overline{\hat{\mathbf{Y}}} : \mathbf{L} : \overline{\hat{\mathbf{Y}}}} \tag{3.4}$$

In equation (3.4), $\overline{B}$ stands for the nonlocal damage force conjugate of the nonlocal damage hardening variable $\overline{\beta}$. The nonlocal evolution equations for $\dot{\overline{\beta}}$ and $\dot{\overline{\mathbf{D}}}$ are finally established from the normality rule instead of (2.10)

$$\dot{\overline{\mathbf{D}}} = \dot{\overline{\lambda}}_d \frac{\partial \overline{F}}{\partial \overline{\hat{\mathbf{Y}}}} \qquad \dot{\overline{\beta}} = \dot{\overline{\lambda}}_d \tag{3.5}$$

where the consistency condition is used to calculate $\dot{\overline{\lambda}}_d$

$$\dot{\overline{F}} = 0 = \frac{\partial \overline{F}}{\partial \overline{\hat{\mathbf{Y}}}} : \dot{\overline{\hat{\mathbf{Y}}}} + \frac{\partial \overline{F}}{\partial \overline{B}} \dot{\overline{B}} \qquad \dot{\overline{\lambda}}_d = \alpha \frac{\mathbf{L} : \dot{\overline{\hat{\mathbf{Y}}}}}{2K_d \overline{\hat{Y}}_{eq}} : \dot{\overline{\hat{\mathbf{Y}}}} \tag{3.6}$$

A factor $\alpha = 1$ or $\alpha = 0$ is used for the active or passive damage growth, respectively.

4. Incremental formulation of the nonlocal model and the failure criterion

In the case of a nonlocal continuum, new nonlocal variables $\overline{\mathbf{Y}}$ and $\overline{B}$ or, alternatively, $\overline{\hat{\mathbf{Y}}}$ and $\overline{B}$ are used in the evolution equations only, instead of the local ones $\mathbf{Y}$ and B. Hence, the internal damage variables are also defined in a nonlocal sense, e.g. $\overline{\mathbf{D}}$ and $\overline{\beta}$. Other state variables, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^e$ are not subjected to the nonlocal averaging, however the locally-defined siffness matrix $\boldsymbol{\Lambda}^s(\mathbf{D})$ in (2.6) has to be replaced by the new, nonlocally prescribed secant matrix $\overline{\boldsymbol{\Lambda}}^s(\overline{\mathbf{D}})$ that accounts for damage nonlocality. In the paper by Kuna-Ciskał and Skrzypek (2004), the incremental constitutive equations were derived in a local sense, to enable introduction of the general failure criterion based on Drucker's material stability postulate, and to ensure convergence and numerical stability. When the nonlocal approach is used, the new nonlocal effective tangent stiffness matrix $\overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}})$ has to be defined instead of the local one $\boldsymbol{\Lambda}^t(\boldsymbol{\varepsilon}^e, \mathbf{D})$. To this end, the local secant stiffness matrix $\boldsymbol{\Lambda}^s(\mathbf{D})$ in (2.6) has to be modified in a nonlocal sense, $\overline{\boldsymbol{\Lambda}}^s(\overline{\mathbf{D}})$, accounting for damage nonlocality (3.5). Then, the new nonlocal effective tangent stiffness matrix $\overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}})$ is established to yield the incremental constitutive equation as follows

$$d\boldsymbol{\sigma} = \overline{\boldsymbol{\Lambda}}^s : d\boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^e : \frac{\partial \overline{\boldsymbol{\Lambda}}^s}{\partial \overline{\mathbf{D}}} : d\overline{\mathbf{D}} \tag{4.1}$$

Finally, applying (3.5) to obtain nonlocal damage increments $d\overline{\mathbf{D}}$ on $d\boldsymbol{\varepsilon}^e$, the following incremental state equation is derived (cf. Kuna-Ciskał and Skrzypek 2004)

$$\begin{aligned} d\boldsymbol{\sigma} &= \Big[\overline{\boldsymbol{\Lambda}}^s + \alpha\boldsymbol{\varepsilon}^e : \frac{\partial \overline{\boldsymbol{\Lambda}}^s}{\partial \overline{\mathbf{D}}} : \frac{\partial \overline{\mathbf{D}}}{\partial \boldsymbol{\varepsilon}^e}\Big] : d\boldsymbol{\varepsilon}^e = \overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}}) : d\boldsymbol{\varepsilon}^e \qquad \text{or} \\ \{d\boldsymbol{\sigma}\} &= [\overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}})]\{d\boldsymbol{\varepsilon}^e\} \end{aligned} \tag{4.2}$$

The square bracket in (4.2) represents the new nonlocal effective tangent stiffness $\overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}})$ that follows damage nonlocality $\overline{\mathbf{D}}$ (3.5). In order to introduce the general failure criterion, Drucker's material stability postulate is adopted

$$d\sigma_{ij} d\varepsilon_{ij} > 0 \tag{4.3}$$

Substituting for $d\sigma_{ij}$ formula (4.2) into stability criterion (4.3), we obtain

$$\frac{\partial^2 \overline{\psi}}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} d\varepsilon_{ij} d\varepsilon_{kl} = \overline{H}_{ijkl} d\varepsilon_{ij} d\varepsilon_{kl} > 0 \tag{4.4}$$

The nonlocally defined quadratic form $(\partial^2\overline{\psi}/\partial\varepsilon_{ij}\partial\varepsilon_{kl})d\varepsilon_{ij}d\varepsilon_{kl}$ must be positive definite for arbitrary values of the components $d\varepsilon_{ij}$. Hence, eventually, condition (4.4) requires that the nonlocal Hessian matrix $\overline{\mathbf{H}}$ be positive definite (cf. Chen and Han, 1995).

The element tangent stiffness matrix is used for the quasi-Newton algorithm for the first iteration step of solving non-linear equation (4.2) as long as failure criterion (4.4) holds. The stiffness of the element in the FE mesh that have come to failure is next reduced to zero. As a consequence, the failed element is completely released from stress and an appropriate stress redistribution occurs in the neighbouring elements to ensure the global equilibrium. Note that the above failure criterion, (4.4), assumes a brittle failure mechanism. However, when a broader class of materials is considered, a post-peak softening regime can also be admitted which would result in strain localisation such that a smooth drop in stiffness of elements that come to failure is met (cf. Bielski *et al.*, 2006).

5. Numerical simulation of nonlocal damage and fracture in concrete under plane stress conditions

5.1. Computational algorithm for nonlocal description of damage and fracture in an elastic-damage material

The iteration of the global equilibrium of a system is performed by ABAQUS with the use of Newton-Raphson method. All variables are updated by the end of an increment, after the convergence is achieved. The physical relations are integrated at a point level (Gauss point of an element) by means of the user-supplied subroutine UMAT, starting from the known equilibrium state and for the elastic strain increment given in each iteration. The output information – stresses and all other state variables – is updated by the end of the integration increment and so are both stiffness matrices, $\overline{\boldsymbol{\Lambda}}^s(\overline{\mathbf{D}})$ (secant) and $\overline{\boldsymbol{\Lambda}}^t(\boldsymbol{\varepsilon}^e, \overline{\mathbf{D}})$ (tangent), accounting for damage nonlocality.

The integration is performed with explicit forward Euler's scheme. The derivatives (stiffness) are known at the starting point and kept constant along the increment. Such an approach may successfully be used only for a sufficiently small incremental step.

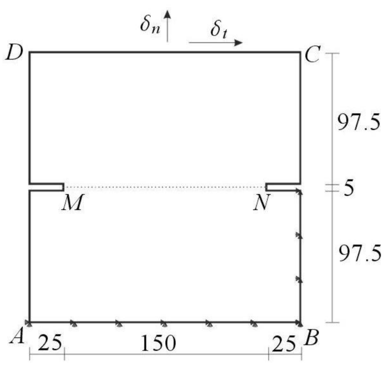
The particular form of the stiffness matrix depends on the state variables as well as on the kind of deformation process taking place through a strain increment. Namely, it depends on whether the process is active or passive. "Active" (loading) denotes a process which implies evolution of the limit surface (damage evolution); "passive" (unloading) stands for changes inside the limit surface (no damage evolution); "neutral" denotes a process tangent to the limit surface (no damage evolution).

The integral-type nonlocal definition of variables is furnished by means of another subroutine URDFIL, which gives an access to the file with results during the analysis. The subroutine is called up at the end of any increment in which new information is written to the results file. The local variables from all integration points are written to an array and then subjected to the nonlocal treatment (NA) according to formulae (3.1) through (3.3). The array is placed in the COMMON block, hence the nonlocal variables from the end of the present increment are accessible in all user routines for the next increment.

When applying the algorithm (CA)+(NA) (see Section 3), one more numerical operation is neccessary. After "cutting-off" CA and "weighted averaging" NA, the value of $\overline{\hat{Y}_{eq}}$ has to be additionally shifted to meet the critical surface at a particular integration point in which it was maximal before cutting; otherwise the damage evolution process would be blocked.

5.2. Material data, geometry and loading

A double-edge notched plane-stress specimen follows an experiment carried out by Nooru-Mohammed experiment (1992) and Nooru-Mohammed *et al.* (1993). The experiment enabled analysis of various combinations of shear and tension under controlled displacement. The model was investigated by di Prisco *et al.* (2000) to simulate fracture by means of three approaches: the local model, the gradient plasticity model and the nonlocal damage model. In what follows, in order to prevail Mode I crack growth under tension, the shear component was excluded ($\delta_t = 0$). The material data for a high strength concrete that describe the basic Murakami-Kamiya model are taken after Murakami and Kamiya (1997) (see Fig. 1).



E [GPa]	21.4
ν	0.2
η_1 [MPa]	-400
η_2 [MPa]	-900
η_3 [MPa]	100
η_4 [MPa]	-23500
ζ	0.1
K_d [MPa]	0.04
B_0 [MPa]	0.0026

Fig. 1. Double-edge notched specimen configuration (cf di Prisco *et al.*, 2000) and material data (cf Murakami and Kamiya, 1997)

5.3. Simulation of fracture in a double-notched specimen

Assuming a uniform normal displacement δ_n applied at the top of the specimen shown in Fig. 1 ($\delta_t = 0$), a complete process of damage growth and fracture is simulated until the ultimate failure of the specimen is predicted. A combined non-symmetric tension/shear failure mode is developed due to the non-symmetric boundary conditions used (Fig. 1). Two zones of failed elements where the ellipticity is lost (checking nonlocal Hessian matrix $\overline{\mathbf{H}}$ (4.4)) are spreading inwards in opposite directions from the notches as long as the ultimate fracture mechanism is not achieved. The releasing of consecutive failed elements from stresses results in stress redistribution in neighbouring (non-failed) elements. The distribution of stresses along the MN line (Fig. 1) becomes more and more non-uniform, finally yielding strong stress localisation in front of two failed zones that come into touch when the ultimate fracture is met.

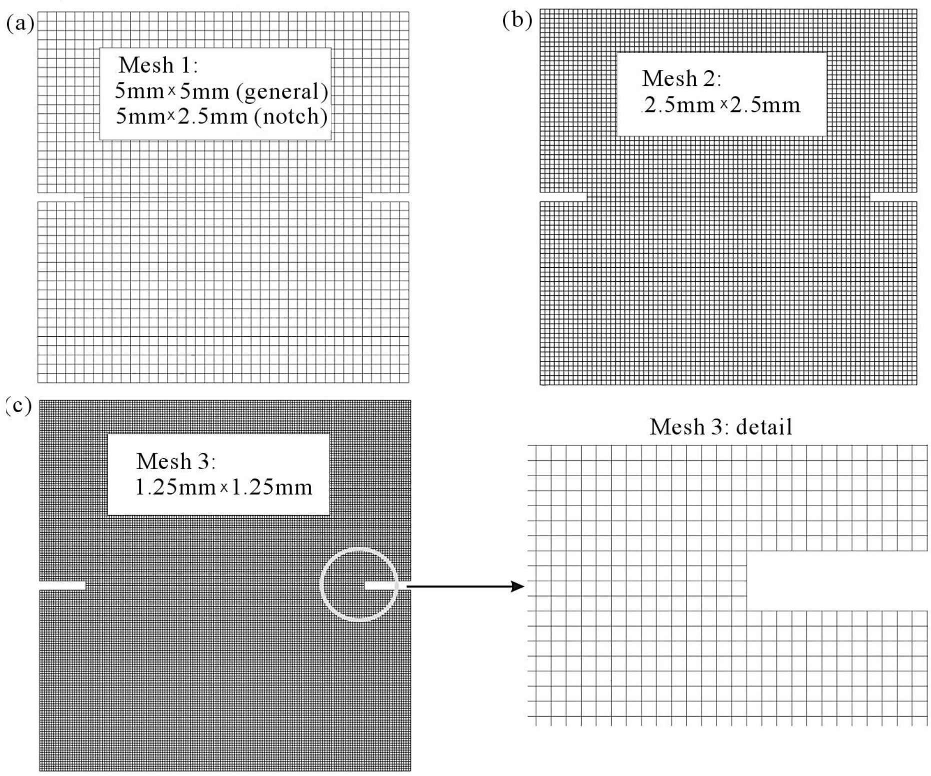


Fig. 2. FEM meshes for convergence tests

To check the influence of characteristic length of nonlocal continuum (4.2) on the damage evolution and fracture processes, the following values of $d^* = 2.5,\ 5.0,\ 7.5,\ 10.0,\ 20.0$ mm are tested. The range of values is taken on the basis of di Prisco *et al.* (2000), where for the nonlocal damage model $d^* \approx 8$ mm is adopted. The finite element size must be lower than the characteristic length to make the nonlocal approach active, so the rectangular mesh 2.5 mm × 2.5 mm is adopted here (Fig. 2a). Other meshes, shown in Fig. 2b and Fig. 2c are defined for the convergence test.

The effect of increasing value of d^* on $\overline{Y}_{eq}$ is shown in Fig. 3 and Fig. 4. The characteristic length defines the area over which Y_{eq} is averaged, according to (3.3). The bigger d^* – the larger area, the more balanced and lower values of the averaged variable around the present integration point (Fig. 3), and the less advanced damage and fracture process at a chosen level of load (Fig. 4).

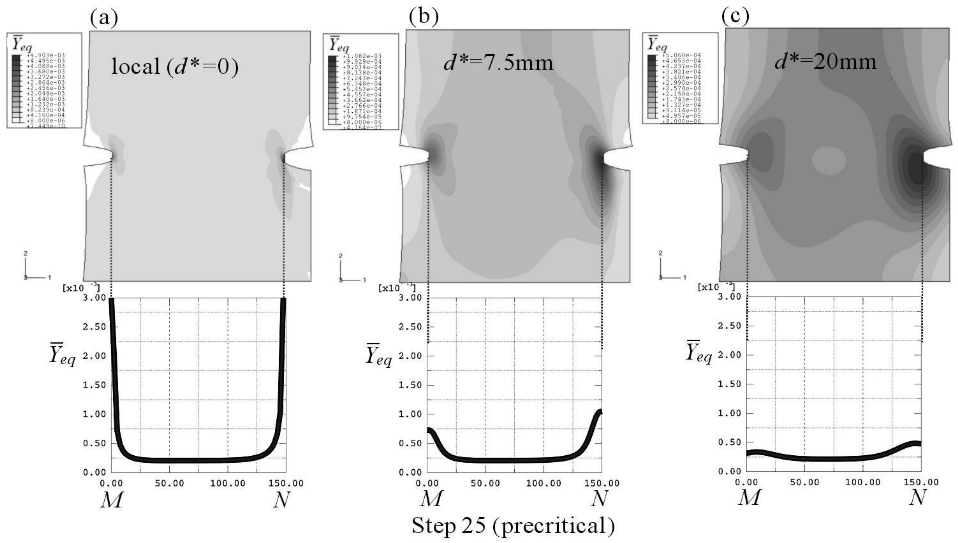


Fig. 3. Effect of the characteristic length on Y_{eq} localization – precritical stage

The increasing value of d^* clearly makes the fracture progress slower, as shown in Fig. 4. Depending on the value of d^*, at the same post-critical loading step (54), a different advance of fracture is met. In other words, an increase of d^* results in an increase of both critical displacements shown in Fig. 5.

The value of the "incubation" displacement (Fig. 5) denotes the displacement of the edge DC (Fig. 1) at the instant of macro-crack appearance, while the "fracture" displacement means the displacement of the same edge at the

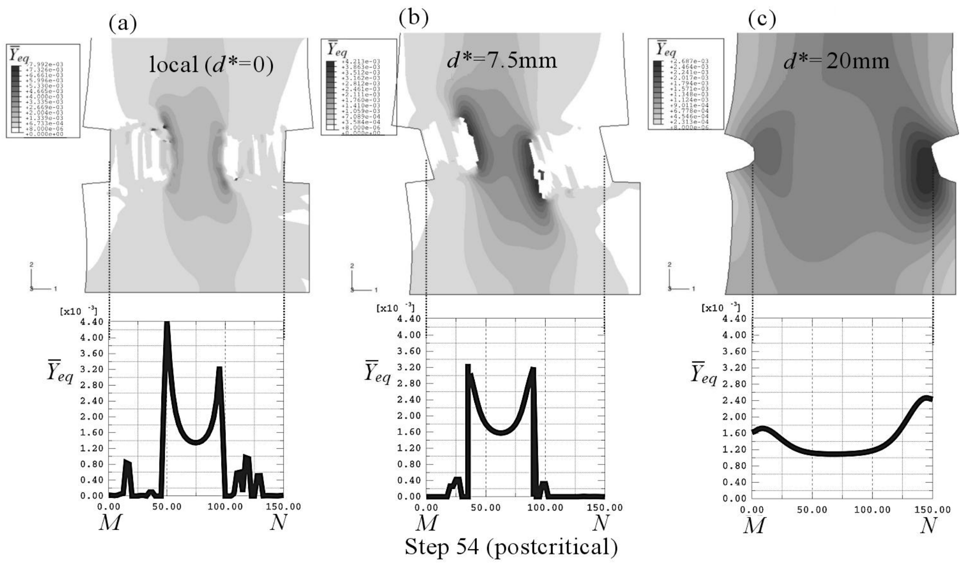


Fig. 4. Effect of the characteristic length on Y_{eq} localization – postcritical stage

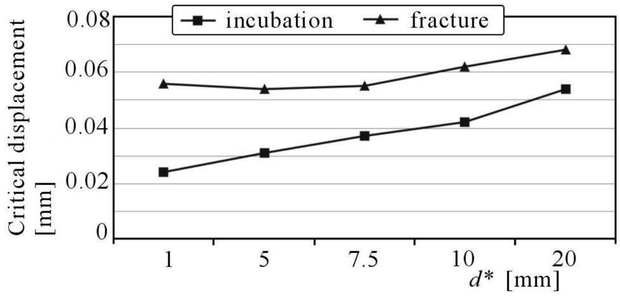


Fig. 5. Effect of the characteristic length on critical displacements: incubation and failure

overall failure of the structural element. It is observed that, for $d^* \leqslant 7.5$ mm, the second critical displacement at fracture becomes (almost) non-sensitive to the characteristic length size.

5.4. Effect of the characteristic length on the equivalent crack shape

Simulations of the crack growth by the nonlocal approach developed in the present paper depend on the characteristic length of a nonlocal continuum (cf. Nooru-Mohammed *et al.*, 1993; Skrzypek *et al.*, 2005a, 2005b). When the (NA) algorithm is used, the increasing width of the crack is observed when the characteristic length grows. At the same time, the direction of macro-cracking changes: the lower d^* the more the crack shape tends to a straight form (see Fig. 6).

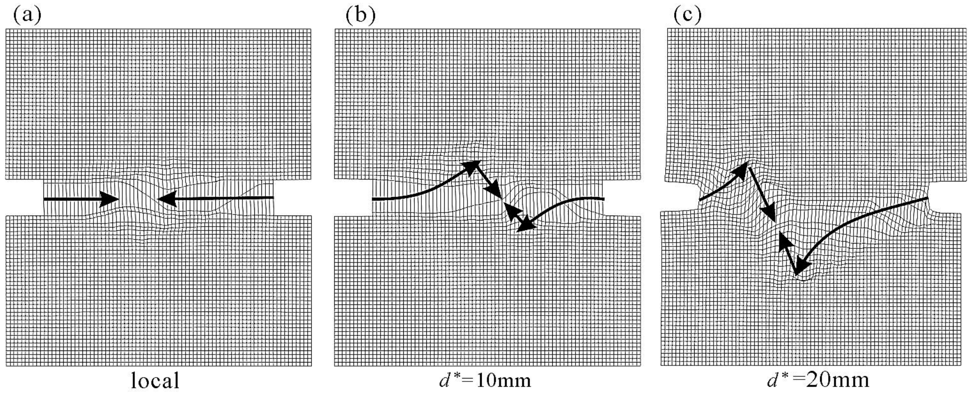


Fig. 6. Effect of d^* on mesh deformation (crack pattern) at failure

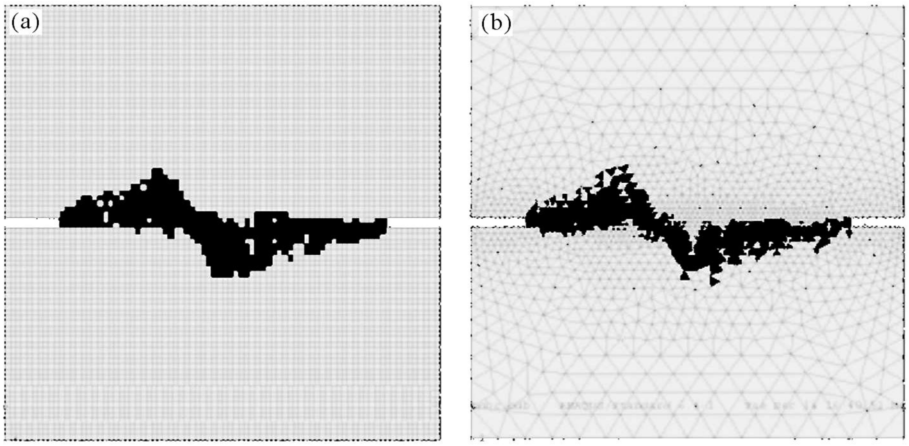


Fig. 7. A test on the mesh shape dependence ($d^* = 7.5\,\text{mm}$): (a) regular rectangular mesh, (b) irregular triangular mesh

Let us consider the mesh-dependence of the numerical simulations of damage and fracture. The local approach is element shape-dependent. For different finite element shapes, different crack patterns are usually obtained. Namely, when the local approach is used, the crack may propagate from the current failed element to the neighbouring one, the edge of which is shared with the corresponding edge of the current failed element. In other words, the element shape restricts possible directions of crack propagation (cf Murakami and Liu, 1995; Kuna-Ciskał, 1999). By contrast, the nonlocal approach presented in this paper allows obtaining the crack pattern independent of the element shape (see Fig. 7).

Eventually, changing the size of an element a convergence test was performed. Results shown in Fig. 8 proved capability of the developed nonlocal model to properly describe both the pre-critical damage evolution during incubation period as well as the post–critical crack growth period. It is shown that the use of an additional localisation limiter, the Cut-off Algorithm (CA), is not necessary to meet a convergence. The use of a single localization limiter, by the Nonlocal Averaging (NA) only, is capable of convergent prediction of both critical displacements – at the crack initiation and the fracture.

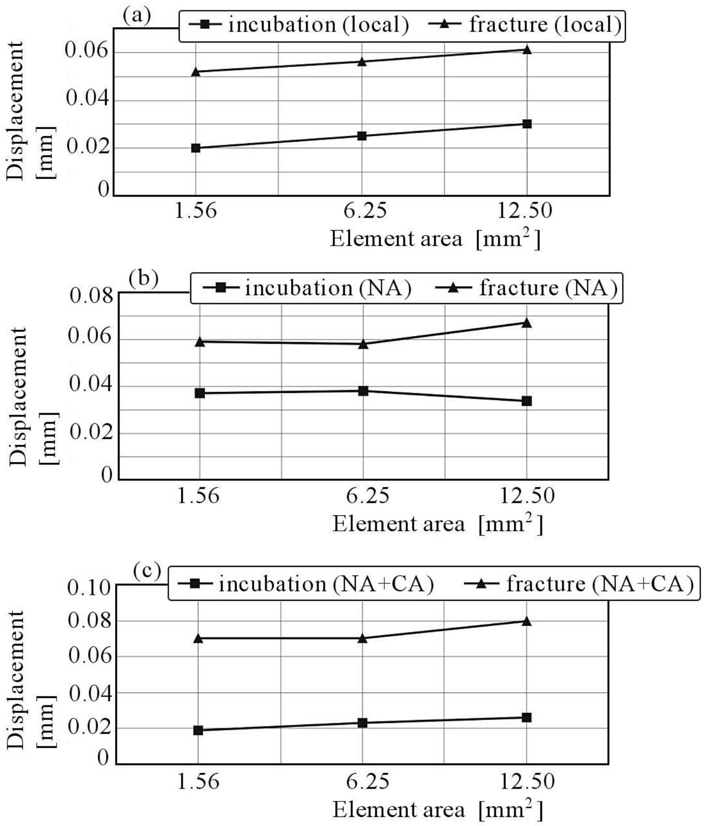


Fig. 8. Convergence tests: (a) no localization limiters (local approach), (b) one localization limiter (NA), (c) two localization limiters (NA+CA)

6. Conclusions: capability of localization limiters

- The modified local Murakami-Kamiya model of an elastic-damage material is capable of qualitative simulating secondary crack growth, but a severe mesh dependence is observed when the local approach is used.
- A sufficient nonlocal extension of the modified Murakami-Kamiya model of the elastic-brittle damage (NMMK) is achieved by the use of the nonlocal damage variable definition (NA). No additional application of the Cut-off Algorithm (CA) is necessary to meet a convergence (mesh independence).
- The appropriate choice of the internal length of a nonlocal continuum depends on the characteristic dimension of the model. If too small. i.e. close to the local approach – the convergence is lost. If too large – the specific crack topology is lost (smeared too much). In the example used for simulation of the crack growth in a double-notched concrete specimen, the notch dimension is $5\,\mathrm{mm}\times25\,\mathrm{mm}$, whereas the "optimal" internal length is established to 7.5 mm.
- The topology of secondary cracks, obtained by simulation when the NMMK model is used, is similar to these obtained in benchmark tests by di Prisco *et al.* The same double-notched sample geometry is incorporated, but the material used in the reference test (concrete) slightly differs from the one used in the present simulation. Besides, simple loading is used, $\delta_t = 0$.
- The mesh size convergence is proved for both critical displacements, at incubation and at fracture.

Acknowledgements

The Grant PB 807/T07/2003/25 and MNiI/SGI2800/PK/079/2005 supported by the State Committee for Scientific Research (KBN) in Poland are gratefully acknowledged.

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Wpływ długości charakterystycznej na prognozowanie rozwoju uszkodzeń i pękania w betonie przy zastosowaniu podejścia nielokalnego

Streszczenie

W pracy opisany został nowy, nielokalny model typu całkowego do symulacji rozwoju anizotropowych uszkodzeń w betonie. Przedstawiony model nielokalny jest rozwinięciem modelu lokalnego zaproponowanego w pracy Murakami i Kamiya (1997), a zmodyfikowanego do formy przyrostowej w pracy Kuna-Ciskał i Skrzypek (2004). W celu uniknięcia zależności rozwiązania numerycznego od siatki MES oraz zapewnienia stabilności i zbieżności zastosowano w obecnej pracy dwa sposoby ograniczania lokalizacji: nielokalne uśrednianie (NA) oraz algorytm obcinania (CA), oba zastosowane do sił termodynamicznych sprzężonych ze zmiennymi stanu uszkodzenia. Równania konstytutywne materiału sprężysto-kruchego zapisane zostały w formie przyrostowej z zastosowaniem zmiennych nielokalnych przy użyciu potencjału dyssypacji zdefiniowanego w przestrzeni uśrednionych zmiennych stanu uszkodzenia. Jako funkcję wagową przyjęto funkcję Gaussa. Przy pomocy opisanego modelu przeprowadzono numaryczną analizę rozwoju uszkodzeń i pękania w płaskim elemencie betonowym z podwójnym karbem, badanym eksperymentalnie przez Nooru-Mohammeda (1992). Przedyskutowano problem odpowiedniego doboru długości charakterystycznej kontinuum nielokalnego oraz jej wpływu na rozwiązanie numeryczne.

Manuscript received December 28, 2005; accepted for print April 20, 2006