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PIECEWISE LINEAR luz(...) AND tar(...) PROJECTIONS. PART 1 – THEORETICAL BACKGROUND

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The paper presents definitions and theorems for luz(...) and tar(...) piecewise linear projections. These projections and their original mathematical apparatus are very useful for modelling of nonlinear systems, eg systems with freeplay or friction.

Key words: non-linearties, piecewise linear systems, algebraic and differential equations

1. Introduction

Nonlinear systems which can be modelled using piecewise linear equations are called "piecewise linear systems". Oftentimes, the piecewise linearity is a result of non-linear approximation of a function. But piecewise linear characteristics with well-marked points of "fractures" can be consequences of variational principles referring to physical processes with constrains – see Grzesikiewicz (1990) for examples. So, from the mathematical point of view, such a piecewise linearity might be a result of some optimization task with limits (see example below).

Theorem 1.1. The optimisation task $y_{opt}(x,a): \ \min Q_y(x-y) \wedge y \in [-a,a]$, where $Q(x-y)$ is a convex function has the solution

$$y_{opt}(x,a) = \begin{cases} a & \text{if} \quad x \geqslant a \\ x & \text{if} \quad -a < x < a \\ -a & \text{if} \quad x \leqslant -a \end{cases}$$

Proof

This task is solved by analysis of the function family $q(y) = Q(x - y)$ (where x is the parameter) on the Oyq plane with the limits $y \in [-a, a]$. It is shown in Fig. 1 for $Q(x - y) = |x - y|$.

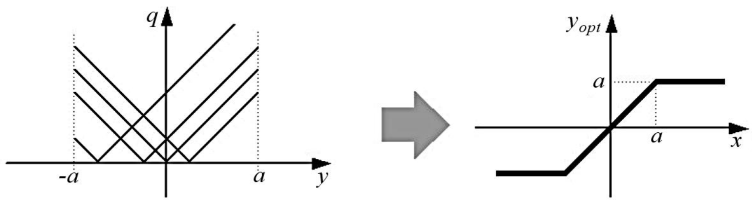

Fig. 1. Topological solution to the optimization task

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The theory of piecewise linear systems contains methods of modelling, static and dynamic analysis, numerical procedures for algebraic and differential piecewise linear equations, etc. This has been developed for about 30 years with connection to the non-linear theory of electrical circuits, non-linear control theory and, recently, also by the way of works on the non-linear contact theory of discrete mechanical systems (details and bibliographic information is given by Żardecki (2001, 2005)).

There are two methods of modelling piecewise linear systems (Kevenaar and Leenaerts, 1992):

- In the first method the model is described by linear equations varied for all ranges of its piecewise linearity. The ranges can be conditioned by constraints or (the simplest case) given by fracture points of characteristics;
- In the second method the model is given in a compact analytic form for full range of variability by piecewise linear equations based on linear and nonlinear forms of the "module" and "sign" type. The nonlinear characteristics are superceded by function series without logic operators and definition step-by-step.

The first approach is more all-matching, unfortunately it leads to long-drawn-out descriptions. Such models are very difficult for analytical transformation and reduction, especially when they have implicit forms. Such inconveniences can be avoided when the second manner of modelling is applied. Of course, synthesis of the compact form of a model can be difficult and, theoretically, even impossible. But creation of an analytical piecewise linear model give a chance for its analytical simplification and reduction in terms of its apparent variables. Such analytical transformations are easier to be accomplished by application of special mathematic apparatus prepared before. Piecewise linear projections are very helpful for simplification of numerical simulations procedures. In the case of multidimensional systems with deep closed loops and acting in variable structures with disentanglement constraints, when the full compact type of a model is impossible to formulate, a mixed manner of the modelling is preferred.

Analytical description of piecewise linear projections makes use of functions and pseudo-functions which are called basic projections. They can be created by elementary projections or their simple compounds and combinations, eg. $y = x$, $y = \operatorname{sgn}(x)$, $y = |x| = x\operatorname{sgn}(x)$, and so on. The new projections can be treated also as basic ones for specific applications. In the case of modelling of mechanical systems with freeplay and friction, they ought to refer to stiffness characteristics with "dead zone" and to Coulomb's friction characteristics. Such characteristics have a lot of topological likenesses. This was a fundamental remark for the author's idea of creation the $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ piecewise linear projections with their special mathematical apparatus. Fundamentals for $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ apparatus were introduced in the author's dissertation (Żardecki, 1992) and extended by Żardecki (2001).

This paper contains the main points of the theory including recent unpublished theorems (with proofs) concerning algebraic and differential equations and inclusions. Application of the $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ theory to the modelling of systems with freeplay and friction are presented in the second part (Żardecki, 2006).

2. Definitions and introduction $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ projections

Definition 2.1. For $-\infty < x < +\infty$ and $a \geqslant 0$

$$\operatorname{luz}(x,a) = x + \frac{|x-a| - |x+a|}{2}$$
$$\operatorname{tar}(x,a) = x + a\operatorname{sgh}(x)$$

where

$$\operatorname{sgh}(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ s^* \in [-1,1] & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

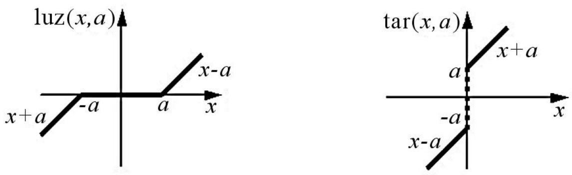

Fig. 2. Geometric interpretation of luz (...) and tar (...) projections

The cross-invertibility of the luz (...) and tar (...) projection is their main attribute.

Theorem 2.1. (On invertibility, formal proof by Żardecki (2001))

$$\begin{aligned} \operatorname{luz}(x,a) &= \operatorname{tar}^{-1}(x,a) \\ \operatorname{tar}(x,a) &= \operatorname{luz}^{-1}(x,a) \end{aligned}$$

For all x, the $\operatorname{luz}(x,a)$ and $\operatorname{tar}(x,a)$ are like anti-functions, when a has a non-negative value. Note that such attribute is not true (see Fig. 3) for projections $\operatorname{luz}(x,-a)$ and $\operatorname{tar}(x,-a)$ defined according to the presented formulas (details by Żardecki (2001)).

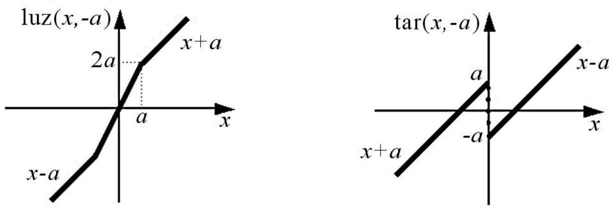

Fig. 3. Geometric interpretation of projections with a negative parameter

The luz (...) and tar (...) can be treated as cases of a more general $\operatorname{talu}(x,a_1,a_2)$ projection.

Definition 2.2. For $-\infty < x < +\infty$ and $a_1, a_2 \geqslant 0$

$$y = \operatorname{talu}(x,a_1,a_2) = \begin{cases} x-(a_1-a_2) & \text{if } \; x \geqslant a_1 \\ a_2 s^* & \text{if } \; -a_1 \leqslant x \leqslant a_1 \\ x+(a_1-a_2) & \text{if } \; x \leqslant -a_1 \end{cases}$$

where $s^* \in [-1,1]$.

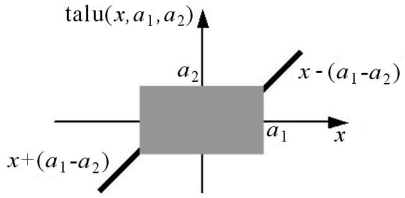

Fig. 4. Geometric interpretation of talu(...) projection

So

$$\operatorname{talu}(x, a, 0) = \operatorname{luz}(x, a) \qquad \operatorname{talu}(x, 0, a) = \operatorname{tar}(x, a)$$
$$\operatorname{talu}(x, 0, 0) = \operatorname{luz}(x, 0) = \operatorname{tar}(x, 0) = x$$

The tar(...) and talu(...) projections have inequivalent areas. It means that dynamic models using tar(...) and talu(...) must be mathematically treated as inclusion models. Going off the inequivalence (by additional dependencies), description of the inclusion in such a model is replaced by varied structural equations.

3. Basic mathematical apparatus of luz(...) and tar(...) projections

The luz(...) and tar(...) projections have interesting properties. Their formulas compose some mathematical apparatus. It is given by theorems and remarks presented below. Their proofs were published by Żardecki (2001).

Attention: Constants $a, b, c, k, \ldots$ appearing in the theorems are non-negative.

Theorem 3.1. (On oddness)

$$\operatorname{luz}(-x, a) = -\operatorname{luz}(x, a)$$
$$\operatorname{tar}(-x, a) = -\operatorname{tar}(x, a)$$

Theorem 3.2. (On multiplication by a positive constant)

$$k \operatorname{luz}(x, a) = \operatorname{luz}(kx, ka)$$
$$k \operatorname{tar}(x, a) = \operatorname{tar}(kx, ka)$$

Note: In the case of multiplication by some negative constant, Theorem 3.2 should be combined with Theorem 3.1, eg.:

$$-k\,\mathrm{luz}\,(x,a) = \mathrm{luz}\,(-kx, ka)$$

Theorem 3.3. (On compounds)

$$\begin{aligned}
&\mathrm{luz}\,(\,\mathrm{luz}\,(x,a),b) = \mathrm{luz}\,(x,a+b)\\
&\mathrm{luz}\,(\,\mathrm{tar}\,(x,a),b) = \begin{cases} \mathrm{luz}\,(x,b-a) & \text{if} \quad b>a\\ x & \text{if} \quad b=a\\ \mathrm{tar}\,(x,a-b) & \text{if} \quad b<a \end{cases}\\
&\mathrm{luz}\,(\,\mathrm{talu}\,(x,a,b),c) = \begin{cases} \mathrm{luz}\,(x,a+c-b) & \text{if} \quad c\geqslant b\\ \mathrm{talu}\,(x,a,b-c) & \text{if} \quad c<b \end{cases}\\
&\mathrm{tar}\,(\,\mathrm{luz}\,(x,a),b) = \mathrm{talu}\,(x,a,b)\\
&\mathrm{tar}\,(\,\mathrm{tar}\,(x,a),b) = \mathrm{tar}\,(x,a+b)\\
&\mathrm{tar}\,(\,\mathrm{talu}\,(x,a,b),c) = \mathrm{talu}\,(x,a,b+c)
\end{aligned}$$

Note: On the base of Theorem 3.3 (as well as Theorem 2.1), we can describe

$$\begin{aligned}
&\mathrm{luz}\,(\,\mathrm{tar}\,(x,a),a) = x\\
&\mathrm{tar}\,(\,\mathrm{luz}\,(x,a),a) = x
\end{aligned}$$

Theorem 3.4. (On linear combination of $\mathrm{luz}\,(\ldots)$ projections)

$$k_1\,\mathrm{luz}\,(x,a_1) \pm k_2\,\mathrm{luz}\,(x,a_2) =$$
$$= \begin{cases} k_1[\,\mathrm{luz}\,(x,a_1) - \mathrm{luz}\,(x,a_2)] + (k_1 \pm k_2)\,\mathrm{luz}\,(x,a_2) & \text{if} \quad a_2>a_1\\ (k_1 \pm k_2)\,\mathrm{luz}\,(x,a_1) & \text{if} \quad a_2=a_1\\ \pm k_2[\,\mathrm{luz}\,(x,a_2) - \mathrm{luz}\,(x,a_1)] + (k_1 \pm k_2)\,\mathrm{luz}\,(x,a_1) & \text{if} \quad a_2<a_1 \end{cases}$$

Note: Replacement of the ordinary combination of $\mathrm{luz}\,(\ldots)$-type function by a special concatenate series is the essence of this formula. Such a form makes calculation of substitutive characteristics for piecewise linear systems easy (Żardecki, 1995).

Theorem 3.5. (On linear combination of tar(...) projections)

$$k_1 \operatorname{tar}(x, a_1) \pm k_2 \operatorname{tar}(x, a_2) =$$
$$= \begin{cases} (k_1 \pm k_2) \operatorname{tar}\Big(x, \dfrac{k_1 a_1 \pm k_2 a_2}{k_1 \pm k_2}\Big) & \text{if } \dfrac{k_1 a_1 \pm k_2 a_2}{k_1 \pm k_2} > 0 \\ (k_1 \pm k_2) x & \text{if } \dfrac{k_1 a_1 \pm k_2 a_2}{k_1 \pm k_2} = 0 \\ (k_1 \pm k_2)\Big[2x - \operatorname{tar}\Big(x, \Big|\dfrac{k_1 a_1 \pm k_2 a_2}{k_1 \pm k_2}\Big|\Big)\Big] & \text{if } \dfrac{k_1 a_1 \pm k_2 a_2}{k_1 \pm k_2} < 0 \end{cases}$$

<u>Note:</u> In the case of summation, this formula simplifies to a compact formula

$$k_1 \operatorname{tar}(x, a_1) + k_2 \operatorname{tar}(x, a_2) = (k_1 + k_2) \operatorname{tar}\Big(x, \frac{k_1 a_1 + k_2 a_2}{k_1 + k_2}\Big)$$

Theorem 3.6. (On disentanglement of the feedback system with luz(...))

If $\operatorname{luz}(y, b) = k \operatorname{luz}(x - y, a)$ then

$$\operatorname{luz}(y, b) = \frac{k}{k+1} \operatorname{luz}(x, a + b)$$
$$\operatorname{luz}(x - y, a) = \operatorname{luz}(x, a + b) - \operatorname{luz}(y, b)$$
$$y = \frac{k}{k+1} \operatorname{talu}\Big(x, a + b, \frac{k+1}{k} b\Big)$$
$$x = \frac{k+1}{k} \operatorname{talu}\Big(y, b, \frac{k}{k+1}(a + b)\Big)$$
$$\operatorname{luz}(y, b) \xrightarrow{k \to \infty} \operatorname{luz}(x, a + b)$$

<u>Note:</u> For linear system (when $a = b = 0$) it means self-evident dependence:

If $y = k(x - y)$ then

$$y = \frac{k}{k+1} x$$

From Theorem 3.6 we can create another formulas, for example:

If $y = k \operatorname{luz}(x - y, a) + c$ then

$$y = \frac{k}{k+1} \operatorname{luz}(x - c, a) + c$$

and so on.

Theorem 3.7. (On disentanglement of the feedback system with $\operatorname{tar}(\ldots)$)

If $\operatorname{tar}(y,b) = k\operatorname{tar}(x-y,a)$ then

$$y = \begin{cases} x - \dfrac{1}{k+1}\operatorname{luz}(x, ka-b) & \text{if } ka > b \\ \dfrac{k}{k+1}x & \text{if } ka = b \\ \dfrac{k}{k+1}\operatorname{luz}\Big(x, \dfrac{b-ka}{k}\Big) & \text{if } ka < b \end{cases}$$

$$x = \begin{cases} y + \dfrac{1}{k}\operatorname{luz}(y, ka-b) & \text{if } ka > b \\ \Big(1+\dfrac{1}{k}\Big)y & \text{if } ka = b \\ \Big(1+\dfrac{1}{k}\Big)\operatorname{tar}\Big(y, \dfrac{b-ka}{k+1}\Big) & \text{if } ka < b \end{cases}$$

$$y \stackrel{k\to\infty}{\longrightarrow} x$$

Theorem 3.8. (On disentanglement of the feedback system with $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$)

If $\operatorname{luz}(y,b) = k\operatorname{tar}(x-y,a)$ then

$$y = x - \frac{1}{k+1}\operatorname{luz}(x, ka+b) \qquad\qquad x = y + \frac{1}{k}\operatorname{luz}(y, ka+b)$$

$$y \stackrel{k\to\infty}{\longrightarrow} x$$

Theorem 3.9. (On disentanglement of the feedback system with $\operatorname{tar}(\ldots)$ and $\operatorname{luz}(\ldots)$)

If $\operatorname{tar}(y,b) = k\operatorname{luz}(x-y,a)$ then

$$y = \frac{k}{k+1}\operatorname{luz}\Big(x, \frac{ka+b}{k+1}\Big) \qquad\qquad x = \frac{k+1}{k}\operatorname{tar}\Big(y, \frac{ka+b}{k+1}\Big)$$

$$y \stackrel{k\to\infty}{\longrightarrow} \operatorname{luz}(x,a)$$

The main advantage of the elaborated mathematical apparatus for piecewise linear systems is the possibility of finding rather simple mathematical dependences. Formulas concerning algebraic operations are analogous to well known formulas of standard linear systems. The mathematical apparatus of $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$ coheres also with topological procedures basing on graphs or block diagrams. For example, Theorem 3.6 expressed by block-diagram symbols is illustrated in Fig. 5.

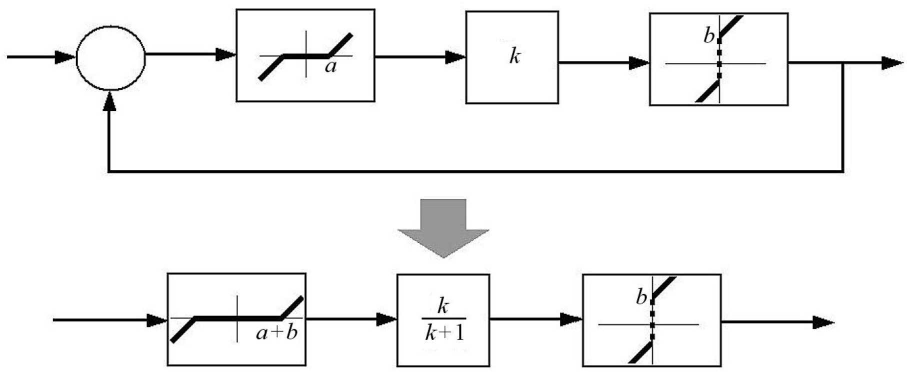

Fig. 5. Block-diagram interpretation of Theorem 3.6

The analytical formulas enable reduction of cascade piecewise systems. For example

$$\mathrm{tar}\,(\,\mathrm{luz}\,(\,\mathrm{luz}\,(x,a),b),c) = \,\mathrm{tar}\,(\,\mathrm{luz}\,(x,a+b),c) = \,\mathrm{talu}\,(x,a+b,c)$$

The formulas concerning disentanglement of feedback system with $\mathrm{luz}\,(\ldots)$ or/and $\mathrm{tar}\,(\ldots))$ projections enables transformation and simplification of complex models governed by piecewise linear algebraic equations. This important matter will be discussed in the next section.

4. Algebraic equations with $\mathrm{luz}\,(\ldots)$ and $\mathrm{tar}\,(\ldots)$

Oftentimes, in multi-dimensional piecewise linear systems, output variables are not explicity dependent on input variables, and they are liable to constraints given by involved piecewise linear algebraic equations (treated as static subsystems). The problem of their clearing turns out to be very important for effective numerical simulation. If such piecewise linear constraints are composed of $\mathrm{luz}\,(\ldots)$ and $\mathrm{tar}\,(\ldots)$ projections, an analytical disentanglement may be unexpectedly easy to carry out. The basic mathematical apparatus can be applied directly (for example Theorem 3.6) to one-dimensional constraint equations. For two-dimensional equations, theorems presented below are a new chance.

Attention: constants a, b, c, k_1, k_2 in the following theorems are non-negative.

Theorem 4.1. If

$$y + k_1 \operatorname{luz}(y - x, a) = f$$
$$x - k_2 \operatorname{luz}(y - x, a) = g$$

then

$$y = f - \frac{k_1}{k_1 + k_2 + 1} \operatorname{luz}(f - g, a)$$
$$x = g + \frac{k_2}{k_1 + k_2 + 1} \operatorname{luz}(f - g, a)$$

Proof

From the first equation

$$\frac{1}{k_1}(f - y) = \operatorname{luz}(y - x, a) \qquad \text{hence} \qquad x = y - \operatorname{tar}\Big(\frac{f - y}{k_1}, a\Big)$$

From the first and second equations

$$\frac{f - y}{k_1} = \frac{x - g}{k_2} \qquad \text{or} \qquad x = \frac{k_2}{k_1}(f - y) + g$$

hence

$$y - \operatorname{tar}\Big(\frac{f - y}{k_1}, a\Big) = \frac{k_2}{k_1}(f - y) + g$$
$$y - g = \frac{1}{k_1} \operatorname{tar}(f - y, k_1 a) + \frac{k_2}{k_1}(f - y)$$

On the basis of Theorem 3.5

$$y - g = \Big(\frac{1}{k_1} + \frac{k_2}{k_1}\Big) \operatorname{tar}\Big(f - y, \frac{k_1 a}{\frac{1}{k_1} + \frac{k_2}{k_1}}\Big)$$

hence

$$\operatorname{luz}(y - g, k_1 a) = \frac{1 + k_2}{k_1}(f - y)$$
$$f - y = \frac{k_1}{k_2 + 1} \operatorname{luz}(f - g - (f - y), k_1 a)$$

From Theorem 3.6

$$f - y = \frac{\frac{k_1}{k_2+1}}{\frac{k_1}{k_2+1} + 1} \operatorname{luz}(f - g, k_1 a)$$

hence

$$y = f - \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}(f - g, a)$$
$$x = \frac{k_2}{k_1}\Big(f - f + \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}(f - g, a)\Big) + g$$

and finally

$$x = g + \frac{k_2}{k_2 + k_1 + 1} \operatorname{luz}(f - g, a)$$

■

<u>Note:</u> If $k_1 = k$, $k_2 = pk$ (linear dependence k_1 and k_2), then

$$y \overset{k\to\infty}{\longrightarrow} f - \frac{1}{p+1} \operatorname{luz}(f - g, a) \qquad\qquad x \overset{k\to\infty}{\longrightarrow} g + \frac{p}{p+1} \operatorname{luz}(f - g, a)$$

Theorem 4.2. If

$$\begin{aligned} y + k_1 \operatorname{tar}(y - x, a) &= f \\ x - k_2 \operatorname{tar}(y - x, a) &= g \end{aligned}$$

then

$$\begin{aligned} y &= \frac{k_2 f + k_1 g}{k_2 + k_1} + \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}\Big(\frac{f - g}{k_2 + k_1}, a\Big) \\ x &= \frac{k_2 f + k_1 g}{k_2 + k_1} - \frac{k_2}{k_2 + k_1 + 1} \operatorname{luz}\Big(\frac{f - g}{k_2 + k_1}, a\Big) \end{aligned}$$

<u>Proof</u>

From the first equation

$$\frac{f - y}{k_1} = \operatorname{tar}(y - x, a)$$

hence

$$x = y - \operatorname{luz}\Big(\frac{f - y}{k_1}, a\Big)$$

From the first and second equation

$$\frac{f - y}{k_1} = \frac{x - g}{k_2} \qquad \text{or} \qquad x = \frac{k_2}{k_1}(f - y) + g$$

hence

$$\begin{aligned} & y - \operatorname{luz}\Big(\frac{f - y}{k_1}, a\Big) = \frac{k_2}{k_1}(f - y) + g \\ & k_1 y - k_2(f - y) - k_1 g = \operatorname{luz}(f - y, k_1 a) \\ & (k_1 + k_2)y - (k_2 f + k_1 g) = \operatorname{luz}(f - y, k_1 a) \\ & y - \frac{k_2 f + k_1 g}{k_1 + k_2} = \frac{1}{k_1 + k_2} \operatorname{luz}(f - y, k_1 a) \end{aligned}$$

also

$$y - \frac{k_2 f + k_1 g}{k_1 + k_2} = \frac{1}{k_1 + k_2} \operatorname{luz}\Big(f - \frac{k_2 f + k_1 g}{k_1 + k_2} - \Big(y - \frac{k_2 f + k_1 g}{k_1 + k_2}\Big), k_1 a\Big)$$

On the basis of Theorem 3.6

$$y - \frac{k_2 f + k_1 g}{k_1 + k_2} = \frac{\frac{1}{k_1+k_2}}{\frac{1}{k_1+k_2} + 1} \operatorname{luz}\Big(f - \frac{k_2 f + k_1 g}{k_1 + k_2}, k_1 a\Big)$$
$$y - \frac{k_2 f + k_1 g}{k_1 + k_2} = \frac{1}{k_1 + k_2 + 1} \operatorname{luz}\Big(\frac{k_1(f-g)}{k_1 + k_2}, k_1 a\Big)$$

hence

$$y = \frac{k_2 f + k_1 g}{k_2 + k_1} + \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}\Big(\frac{f-g}{k_2 + k_1}, a\Big)$$

So

$$x = \frac{k_2}{k_1}\Big(f - \frac{k_2 f + k_1 g}{k_2 + k_1} - \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}\Big(\frac{f-g}{k_1 + k_2}, a\Big)\Big) + g$$

hence

$$x = \frac{k_2 f + k_1 g}{k_2 + k_1} - \frac{k_2}{k_2 + k_1 + 1} \operatorname{luz}\Big(\frac{f-g}{k_2 + k_1}, a\Big)$$

■

Note: If $k_1 = k$, $k_2 = pk$ (linear dependence k_1 and k_2), then

$$y \overset{k\to\infty}{\longrightarrow} \frac{pf+g}{p+1} \qquad x \overset{k\to\infty}{\longrightarrow} \frac{pf+g}{p+1}$$

The variables f and g can be treated as input variables and y and x – as output ones for two-dimensional static systems that were initialy described by equations with entangled outputs. Those entanglements disappear thanks to the presented theorems.

Sometimes, as a result of mathematical modelling, one obtains some multidimensional model with redundant variables. The theorem presented below might be very useful for analytical reduction of the model.

Theorem 4.3. If

$$\operatorname{luz}(y - w, a) = k_1 \operatorname{luz}(w - u, c)$$
$$\operatorname{luz}(u - x, b) = k_2 \operatorname{luz}(w - u, c)$$

then

$$\operatorname{luz}(y - w, a) = \frac{k_1}{k_2 + k_1 + 1} \operatorname{luz}(y - x, a + b + c)$$
$$\operatorname{luz}(u - x, b) = \frac{k_2}{k_2 + k_1 + 1} \operatorname{luz}(y - x, a + b + c)$$

Proof

From the first equation, we have

$$u = w - \operatorname{tar}\Big(\frac{1}{k_1}\operatorname{luz}(y-w,a),c\Big)$$

On the basis of both equations

$$\operatorname{luz}(u-x,b) = \frac{k_2}{k_1}\operatorname{luz}(w-u,c)$$

hence

$$\operatorname{luz}\Big(w-x-\operatorname{tar}\Big(\frac{1}{k_1}\operatorname{luz}(y-w,a),c\Big),b\Big) = \frac{k_2}{k_1}\operatorname{luz}(y-w,a)$$

After inversion

$$w-x-\operatorname{tar}\big(\frac{1}{k_1}\operatorname{luz}(y-w,a),c\Big) = \operatorname{tar}\Big(\frac{k_2}{k_1}\operatorname{luz}(y-w,a),b\Big)$$

hence

$$w-x = \frac{1}{k_1}\operatorname{tar}(\operatorname{luz}(y-w,a),k_1c) + \frac{k_2}{k_1}\operatorname{tar}\Big(\operatorname{luz}(y-w,a),\frac{k_1}{k_2}b\Big)$$

On the basis of Theorem 3.5

$$w-x = \Big(\frac{1}{k_1}+\frac{k_2}{k_1}\Big)\operatorname{tar}\Big(\operatorname{luz}(y-w,a),\frac{c+b}{\frac{1}{k_1}+\frac{k_2}{k_1}}\Big)$$

hence

$$w-x = \operatorname{tar}\Big(\frac{1+k_2}{k_1}\operatorname{luz}(y-w,a),b+c\Big)$$

$$\operatorname{luz}(w-x,b+c) = \frac{1+k_2}{k_1}\operatorname{luz}(y-w,a)$$

$$\operatorname{luz}(y-w,a) = \frac{k_1}{k_2+1}\operatorname{luz}(w-x,b+c)$$

or

$$\operatorname{luz}(y-w,a) = \frac{k_1}{k_2+1}\operatorname{luz}(y-x-(y-w),b+c)$$

On the basis of Theorem 3.7

$$\operatorname{luz}(y-w,a) = \frac{\frac{k_1}{k_2+1}}{\frac{k_1}{k_2+1}+1}\operatorname{luz}(y-x,a+b+c)$$

hence

$$\operatorname{luz}(y-w,a) = \frac{k_1}{k_2+k_1+1}\operatorname{luz}(y-x,a+b+c)$$

The proof of the first part of theorem is ended.

The proof of the second part runs similarly. From the second equation

$$w = u + \operatorname{tar}\left(\frac{1}{k_2}\operatorname{luz}(u - x, b), c\right)$$

Taking into account that

$$\begin{aligned}
\operatorname{luz}(y - w, a) &= \frac{k_1}{k_2}\operatorname{luz}(u - x, b) \\
\operatorname{luz}\left(y - u - \operatorname{tar}\left(\frac{1}{k_2}\operatorname{luz}(u - x, b), c\right), a\right) &= \frac{k_1}{k_2}\operatorname{luz}(u - x, b) \\
y - u - \operatorname{tar}\left(\frac{1}{k_2}\operatorname{luz}(u - x, b), c\right) &= \operatorname{tar}\left(\frac{k_1}{k_2}\operatorname{luz}(u - x, b), a\right)
\end{aligned}$$

hence

$$\begin{aligned}
y - u &= \operatorname{tar}\left(\frac{k_1 + 1}{k_2}\operatorname{luz}(u - x, b), a + c\right) \\
\operatorname{luz}(u - x, b) &= \frac{k_2}{k_1 + k_2}\operatorname{luz}(y - u, a + c) \\
\operatorname{luz}(u - x, b) &= \frac{k_2}{k_1 + k_2}\operatorname{luz}(y - x - (u - x), a + c)
\end{aligned}$$

On the basis of Theorem 3.7 also

$$\operatorname{luz}(u - x, b) = \frac{k_2}{k_1 + k_2 + 1}\operatorname{luz}(y - x, a + b + c)$$

■

Note: If $k_1 = k$, $k_2 = pk$ (linear dependence), then

$$\begin{aligned}
\operatorname{luz}(y - w, a) &\overset{k\to\infty}{\longrightarrow} \frac{1}{p + 1}\operatorname{luz}(y - x, a + b + c) \\
\operatorname{luz}(u - x, b) &\overset{k\to\infty}{\longrightarrow} \frac{p}{p + 1}\operatorname{luz}(y - x, a + b + c)
\end{aligned}$$

There is possible a formulation of analogous theorems for systems of equations with constraints containing the $\operatorname{tar}(\ldots)$ or mixed pair $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$ projections. Such analytical formulas have rather complicated forms.

In comments to Theorems 4.1-4.3 we have considered also peculiar cases when the coefficients k_1, k_2 were extremely large, but linearly dependent. Such an outwardly impossible situation takes place in the case when a mathematical model of so the called stiff dynamic system is set. For stiff systems, degeneration of equations of motion can be done by parametric operations. Such a simplification of the model is easy to execute using the proved formulas for disentanglement.

5. Basic properties of ordinary differential equations and inclusions with luz(...) and tar(...) projections

In this Section, we investigate basic mathematical properties of dynamical systems described by equations and inclusions with the luz(...) and tar(...) projections.

The necessity of taking into account not only equations but also inclusions result from the indetermination (even though at the beginning of our study) of $\operatorname{tar}(0,a)$. So, since $\operatorname{tar}(0,a) = as^* \in [-a,a]$):

- instead of formula $\dot{x}(t) = f(\ldots, \operatorname{tar}(x(t),a),\ldots)$ (differential state equation) we have $\dot{x}(t) \in f(\ldots, \operatorname{tar}(x(t),a),\ldots)$ (differential state inclusion),
- instead of formula $0 = f(\ldots, \operatorname{tar}(x(t),a),\ldots)$ (function equation) we have $0 \in f(\ldots, \operatorname{tar}(x(t),a),\ldots)$ (function inclusion).

Transformation of inclusion description requires an individual approach. In cases when the tar(...) projections are elements of a single inclusion, the theorems presented below can be very useful.

Attention: parameters $a, b, \ldots$ appearing in the following theorems are nonnegative.

Theorem 5.1. Inclusion $\dot{x}(t) \in y(t) - b\operatorname{tar}(x(t),a)$ for which $\operatorname{tar}(0,a)$ performs the optimization task

$$\operatorname{tar}(0,a)_{opt}: \quad \min_{\operatorname{tar}(0,a)} Q(\dot{x}) \quad \wedge \quad \operatorname{tar}(0,a) \in [-a,a]$$

where $Q(\ldots)$ is a convex function is equivalent to:

– differential equation with singularity $s^* \in [-1,1]$

$$\dot{x}(t) = y(t) - b\operatorname{tar}(x(t),a)$$

where

$$bas^* = b(\operatorname{tar}(0,a))_{opt} = y(t) - \operatorname{luz}(y(t),ba)$$

– differential variable-structure equation

$$\dot{x}(t) = \begin{cases} y(t) - b\operatorname{tar}(x(t),a) & \text{if} \quad x(t) \neq 0 \\ \operatorname{luz}(y(t),ba) & \text{if} \quad x(t) = 0 \end{cases}$$

Proof

The differential inclusion $\dot{x}(t) \in y(t) - \operatorname{tar}(x(t), a)$ is equivalent to the differential equation

$$\dot{x}(t) = \begin{cases} y(t) - b\operatorname{tar}(x(t), a) & \text{if} \quad x(t) \neq 0 \\ y(t) - b\operatorname{tar}(0, a) & \text{if} \quad x(t) = 0 \end{cases}$$

Because of $\operatorname{tar}(0, a) = as^*$, where $s^* \in [-1, 1]$, we have $b\operatorname{tar}(0, a) \in [-ba, ba]$.

On the basis of Theorem 1.1, applying $\operatorname{luz}(\ldots)$ notation, the task

$$b(\operatorname{tar}(0, a))_{opt} :$$

$$\min_{b\operatorname{tar}(0,a)} Q(\dot{x}(t)) = \min_{b\operatorname{tar}(0,a)} Q(y(t) - b\operatorname{tar}(0, a)) \quad \wedge \quad b\operatorname{tar}(0, a) \in [-ba, ba]$$

has the solution $b(\operatorname{tar}(0, a))_{opt} = y(t) - \operatorname{luz}(y(t), ba)$. Therefore, also

$$\dot{x}(t) = \begin{cases} y(t) - b\operatorname{tar}(x(t), a) & \text{if} \quad x(t) \neq 0 \\ y(t) - b(\operatorname{tar}(0, a))_{opt} = \operatorname{luz}(y(t), ba) & \text{if} \quad x(t) = 0 \end{cases}$$

■

Both forms of the description are equivalent. Determination of $\operatorname{tar}(0, a)$ in the optimization task $b(\operatorname{tar}(0, a))_{opt} = y(t) - \operatorname{luz}(y(t), ba)$ caused a new situation in which for $x(t) = 0$ the macro-projection $b\operatorname{tar}(x, a)$ is replaced by a new piecewise linear macro-projection based on the variable $y(t)$ and the $\operatorname{luz}(\ldots)$ projection.

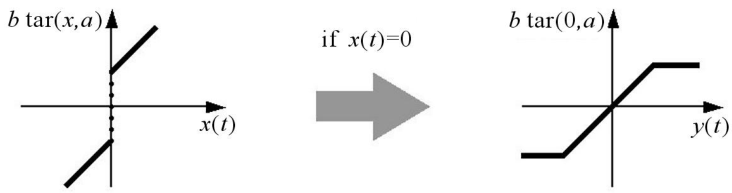

Fig. 6. Determination of $\operatorname{tar}(x, a)$ projection for $x = 0$

Finally, for $x(t) = 0$, the state equation has been described as $\dot{x}(t) = \operatorname{luz}(y(t), ba)$. Analysing this form, we ascertain that it express a practical rule: "for $x(t) = 0$ if $y(t) \in [-ba, ba]$ the blocked state $(\dot{x}(t) = 0)$ is held as far as $y(t) \notin [-ba, ba]$". The calculation of $b(\operatorname{tar}(0, a))_{opt} = y(t) - \operatorname{luz}(y(t), ba)$ on the basis of formal minimization $Q(\dot{x}(t))$ is equivalent with application of the heuristic rule describing "motion blockade". Such replacement of the formal approach (optimization) by the well known heuristic rule is an important practical method for resolving inclusion problems.

Theorem 5.2. The inclusion $0 \in y(t) - b\,\mathrm{tar}\,(x(t), a)$ is equivalent to the equation

$$x(t) = \mathrm{luz}\,\Big(\frac{1}{b}y(t), a\Big)$$

Proof

The inclusion $0 \in y(t) - b\,\mathrm{tar}\,(x(t), a)$ is equivalent to

$$0 = \begin{cases} y(t) - b\,\mathrm{tar}\,(x(t), a) & \text{if} \quad x(t) \neq 0 \\ y(t) - b\,\mathrm{tar}\,(0, a) & \text{if} \quad x(t) = 0 \end{cases}$$

For $x(t) \neq 0$ from $y(t) - b\,\mathrm{tar}\,(x(t), a) = 0$, we obtain

$$\mathrm{luz}\,\Big(\frac{y(t)}{b}, a\Big) = x(t)$$

For $x(t) = 0$ from $y(t) - b\,\mathrm{tar}\,(0, a) = 0$, we obtain

$$\frac{y(t)}{b} = 0 = x(t) \qquad \text{or} \qquad \mathrm{luz}\,\Big(\frac{y(t)}{b}, a\Big) = 0 = x(t)$$

So, in fact

$$x(t) = \mathrm{luz}\,\Big(\frac{y(t)}{b}, a\Big)$$

for all $x(t)$. ■

Theorem 5.3. Degeneration of the inclusion $\varepsilon\dot{x}(t) \in y(t) - b\,\mathrm{tar}\,(x(t), a)$ by $\varepsilon \to 0$ gives the equation

$$x(t) = \mathrm{luz}\,\Big(\frac{y(t)}{b}, a\Big)$$

Proof

$$\varepsilon\dot{x}(t) \in y(t) - b\,\mathrm{tar}\,(x(t), a) \xrightarrow{\varepsilon \to 0} 0 \in y(t) - b\,\mathrm{tar}\,(x(t), a)$$

On the basis of Theorem 6.2, we obtain the final result. ■

On the basis of Theorem 5.3, we conclude that parametric reduction of the inclusive model is deprived of its ambiguousness.

When the argument of the $\mathrm{tar}\,(\ldots)$ projection is given by a linear combination of variables, Theorems 5.1-5.3 can be used directly, but for a modified form of the inclusion model. For a typical two-variable model, this is presented in the proof of Theorem 5.4.

Theorem 5.4. The inclusion

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \in \begin{bmatrix} y_1(t) - b_1 \operatorname{tar}(x_1(t) - x_2(t), a) \\ y_2(t) + b_2 \operatorname{tar}(x_1(t) - x_2(t), a) \end{bmatrix}$$

for which $\operatorname{tar}(0, a)$ performs the optimization task

$$\operatorname{tar}(0, a)_{opt} : \quad \min_{\operatorname{tar}(0,a)} Q(\dot{x}_1(t) - \dot{x}_2(t)) \quad \wedge \quad \operatorname{tar}(0, a) \in [-a, a]$$

$Q(\dots)$ is a convex function is equivalent to:

– differential equation with singularity $s^*_{12} \in [-1, 1]$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) - b_1 \operatorname{tar}(x_1(t) - x_2(t), a) \\ y_2(t) + b_2 \operatorname{tar}(x_1(t) - x_2(t), a) \end{bmatrix}$$

where

$$\begin{aligned} as^*_{12} &= \operatorname{tar}(0, a)_{opt} = \\ &= \frac{1}{b_1 - b_2} [y_1(t) - y_2(t) - \operatorname{luz}(y_1(t) - y_2(t), (b_1 - b_2)a)] \end{aligned}$$

– differential variable-structure equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) - b_1 \operatorname{tar}(x_1(t) - x_2(t), a) \\ y_2(t) + b_2 \operatorname{tar}(x_1(t) - x_2(t), a) \end{bmatrix} \qquad \text{if } x_1(t) \neq x_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) - \dfrac{b_1}{b_1 - b_2} [y_1(t) - y_2(t) - \operatorname{luz}(y_1(t) - y_2(t), (b_1 - b_2)a)] \\ y_2(t) + \dfrac{b_2}{b_1 - b_2} [y_1(t) - y_2(t) - \operatorname{luz}(y_1(t) - y_2(t), (b_1 - b_2)a)] \end{bmatrix}$$
$$\text{if } x_1(t) = x_2(t)$$

Proof

We create a new equation for the variable $x_{12}(t) = x_1(t) - x_2(t)$. Subtracting the state equations

$$\dot{x}_{12}(t) = y_1(t) - y_2(t) - (b_1 - b_2) \operatorname{tar}(x(t), a)$$

For $x_{12}(t) = x_1(t) - x_2(t) = 0$ using Theorem 6.1, we obtain

$$(b_1 - b_2) \operatorname{tar}(0, a)_{opt} = (b_1 - b_2) as^*_{12} = y_1(t) - y_2(t) - \operatorname{luz}(y_1(t) - y_2(t), (b_1 - b_2)a)$$

hence

$$\operatorname{tar}(0, a)_{opt} = \frac{1}{b_1 - b_2} [y_1(t) - y_2(t) - \operatorname{luz}(y_1(t) - y_2(t), a)]$$

After substitution we obtain final results. ■

In more complicated cases, when model equations contain multiple components $\operatorname{tar}(x_i, a_i)$ and $\operatorname{tar}(x_i - x_j, a_{ij})$, creation of new inclusions for the variables $x_{ij} = x_i - x_j$ leads to a unified multidimensional form of the model

$$[\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]^\top \in$$
$$\in \begin{bmatrix} y_1(t) - b_{11} \operatorname{tar}(x_1(t), a_1) - b_{12} \operatorname{tar}(x_2(t), a_2) + \ldots - b_{1n} \operatorname{tar}(x_n(t), a_n) \\ y_2(t) - b_{21} \operatorname{tar}(x_1(t), a_1) - b_{22} \operatorname{tar}(x_2(t), a_2) + \ldots - b_{2n} \operatorname{tar}(x_n(t), a_n) \\ \vdots \\ y_n(t) - b_{n1} \operatorname{tar}(x_1(t), a_1) - b_{n2} \operatorname{tar}(x_2(t), a_2) + \ldots - b_{nn} \operatorname{tar}(x_n(t), a_n) \end{bmatrix}$$

Such a model should be completed by a rule of calculation of the unknown $\operatorname{tar}(0, a_i)$, eg. the optimization task based on general physical principles. But in many cases, calculation of $\operatorname{tar}(0, a_i)$ can be resolved practically using heuristic procedures. Such a procedure (here the S-S procedure) for description of the so-called "stick-slip" process in the multidimensional model is presented below. It makes use of the fact that for $x_i(t) = 0$ a motion is blocked $(\dot{x}_i(t) = 0)$ only for $s_i^*(t) \in [-1, 1]$. The saturation formula $s_i^*(t) = s_i^{**}(t) - \operatorname{luz}(s_i^{**}(t), 1)$ expressing the limitations on $s_i^*(t)$, where $s_i^{**}(t)$ refers to blocked state, enables description of such "stick-slip" conditions in the state $x_1(t) = 0$, $x_2(t) = 0$,..., $x_n(t) = 0$.

Definition 5.1. The S-S procedure for disentanglement of the inclusion system and calculation of the "stick-slip" process:

1. Determination of the "stick-slip" variables x_i for which $x_i(t_k) = 0$ at $t = t_k$
2. Determination of the "stick-slip" subsystem of the equations $\dot{x}_i(t_k) = \ldots$
3. Setting $\operatorname{tar}(x_i(t), a_i)|_{x_i(t)=0} = a_i s_i^* = a(s_i^{**} - \operatorname{luz}(s_i^{**}, 1))$ in the "stick-slip" subsystem
4. Calculation of s_i^{**} from the "stick-slip" subsystem for $\dot{x}_i(t_k) = 0$
5. Calculation of $\operatorname{tar}(x_i(t), a_i)|_{x_i(t)=0} = a_i s_i^* = a(s_i^{**} - \operatorname{luz}(s_i^{**}, 1))$
6. Calculation of $\dot{x}_i(t_k)$ from the system equations.

Note that applying the S-S procedure, for the state $x_1(t) = 0$, $x_2(t) = 0$,..., $x_n(t) = 0$, we obtain

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_1(t) - b_{11}a_1s_1^{**} - b_{12}a_2s_2^{**} + \ldots - b_{1n}a_ns_n^{**} \\ y_2(t) - b_{21}a_1s_1^{**} - b_{22}a_2s_2^{**} + \ldots - b_{2n}a_ns_n^{**} \\ \vdots \\ y_n(t) - b_{n1}a_1s_1^{**} - b_{n2}a_2s_2^{**} + \ldots - b_{nn}a_ns_n^{**} \end{bmatrix}$$

$$[\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]^\top =$$

$$= \begin{bmatrix} y_1(t) - b_{11}a_1[s_1^{**}(t) - \operatorname{luz}(s_1^{**}(t),1)] - b_{12}a_2[s_2^{**}(t) - \operatorname{luz}(s_2^{**}(t),1)] + \ldots \\ y_1(t) - b_{21}a_1[s_s^{**}(t) - \operatorname{luz}(s_1^{**}(t),1)] - b_{22}a_2[s_2^{**}(t) - \operatorname{luz}(s_2^{**}(t),1)] + \ldots \\ \vdots \\ y_n(t) - b_{n1}a_1[s_1^{**}(t) - \operatorname{luz}(s_1^{**}(t),1)] - b_{n2}a_2[s_2^{**}(t) - \operatorname{luz}(s_2^{**}(t),1)] + \ldots \end{bmatrix}$$

and finally

$$[\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n]^\top =$$

$$= \begin{bmatrix} -b_{11}a_1 \operatorname{luz}(s_1^{**}(t),1) - b_{12}a_2 \operatorname{luz}(s_2^{**}(t),1) + \ldots - b_{1n}a_n \operatorname{luz}(s_n^{**}(t),1) \\ -b_{21}a_1 \operatorname{luz}(s_1^{**}(t),1) - b_{22}a_2 \operatorname{luz}(s_2^{**}(t),1) + \ldots - b_{2n}a_n \operatorname{luz}(s_n^{**}(t),1) \\ \vdots \\ -b_{n1}a_1 \operatorname{luz}(s_1^{**}(t),1) - b_{n2}a_2 \operatorname{luz}(s_2^{**}(t),1) + \ldots - b_{nn}a_n \operatorname{luz}(s_n^{**}(t),1) \end{bmatrix}$$

If for all $i = 1, 2, \ldots, n$ the calculated $s_i^{**} \in [-1, 1]$, the all $\operatorname{luz}(s_i^{**}, 1) = 0$ and all $\dot{x}_i(t) = 0$. It means total blockade of the system. If the calculated $s_i^{**} \notin [-1, 1]$, then $\operatorname{luz}(s_i^{**}, 1) \neq 0$ as well. Thus the right sides of the equations are non-zero wherethrough the blockade state of that variables can be terminated. Obviously, this way one can concern only some variables. As a result of the standard ODE (Ordinary Differential Equation) solver procedure, a new dynamic state is calculated. This new state may contain new singularities caused by some $x_i(t) = 0$. The S-S procedure is used once again.

For full analysis, the S-S procedure should be applied alternately and independently for every combination of the singularity states, i.e.:

– for $x_1(t) = 0$, $x_2(t) \neq 0$, $x_3(t) \neq 0$,..., $x_n(t) \neq 0$
– for $x_1(t) \neq 0$, $x_2(t) = 0$, $x_3(t) \neq 0$,..., $x_n(t) \neq 0$
$\vdots$
– for $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) \neq 0$,..., $x_n(t) \neq 0$
$\vdots$
– for $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) = 0$,..., $x_n(t) \neq 0$
$\vdots$
– for $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) = 0$,..., $x_n(t) = 0$

Finally, we obtain multi-structural equations which take into consideration all singular situations. Such a model can be complicated but ready to use in simulations.

The S-S procedure seems to be an attractive proposition for solving the inclusion problems which appear in multi-body mechanical systems with blocked motion, for example – in mechanisms with multiple dry friction (stick-slip problems). Obviously, in such cases, the S-S procedure leads to the same results as the formal solution to the optimization task based on the Gauss "acceleration energy function". This is described in the second part of the paper.

6. Final remarks

The paper presents the concept, definitions and theorems concerning the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections. These piecewise linear projections have very interesting mathematical properties. Basic formulas, eg. on compounds, linear combination, disentanglement of feedback systems constitute surprisingly simple "algebra" apparatus. The theorems as well as the S-S procedure concerning differential inclusions enable efficient analysis of piecewise linear dynamic systems described with the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$.

The $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections seem to be an interesting idea for investigators working on piecewise linear models. Applications concerning nonlinear mechanical systems with freeplay (backlash, clearance) and friction (Coulomb's fiction with stiction) are discussed in the second part of the paper (Żardecki, 2006).

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References

1. Grzesikiewicz W., 1990, Dynamics of mechanical systems with constraints, *Prace Naukowe Politechniki Warszawskiej. Mechanika*, **117** [in Polish]

2. Kevenaar T.A.M., Leenaerts D.M.W., 1992, A comparison of piecewise-linear model descriptions, *IEEE Transactions on Circuit and Systems*, **39**, 12

3. Żardecki D., 1992, Structural sensitivity analysis of mathematical models of dynamical systems with application to car-steerability model, Dissertation: WAT, Warszawa [in Polish]

4. Żardecki D., 1995, The method of notation and computation of rigidity characteristics of the nonlinear spring systems, *Proceedings of Conference AUTO-PROGRES'95*, Jachranka 1995, PIMOT, [in Polish]

5. Żardecki D., 2001, The $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$ projections – a theoretical background and an idea of application in a modeling of discrete mechanical systems with backlashes or frictions, *Biuletyn WAT*, **L**, 5 [in Polish]

6. Żardecki D., 2005, Piecewise-linear modeling of dynamic systems with freeplay and friction, *Proceedings of 8th DSTA Conference*, Łódź, TU of Łódź

7. Żardecki D., 2006, Piecewise linear $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$ projection. Part 2 – Application in modelling of dynamic systems with freeplay and friction, *Journal of Theoretical and Applied Mechanics*, **44**, 1

Przedziałami liniowe odwzorowania $\operatorname{luz}(\ldots)$ i $\operatorname{tar}(\ldots)$. Część 1 – Podstawy teoretyczne

Streszczenie

Artykuł przedstawia definicje i twierdzenia dotyczące przedziałami liniowych odwzorowan $\operatorname{luz}(\ldots)$ i $\operatorname{tar}(\ldots)$. Odwzorowania i ich oryginalny aparat matematyczny są bardzo użyteczne dla modelowania układów nieliniowych, np. układów z luzem i tarciem.

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