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ON MODELLING OF PERIODIC PLATES WITH INHOMOGENEITY PERIOD OF AN ORDER OF THE PLATE THICKNESS

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In this contribution, a new averaged non-asymptotic model of Reissner-type plates with a periodic non-homogeneous structure is proposed. This model is obtained by the tolerance averaging technique (TAT) and makes it possible to investigate the effect of the period length parameter on the overall plate behaviuor (the length-scale effect). A new element is applying TAT directly to the equation of 3D-theory of elasticity of solids with periodic structures. Then, taking into account the Hencky-Bolle kinematic assumption, a non-asymptotic 2D-model of plates with periods of an order of the plate thickness is derived. The proposed model is applied to the analysis of some vibration problems.

Key words: modelling, composite plates, dynamics

1. Introduction

The subject of analysis are medium thickness (Reissner-type) rectangular elastic plates with a periodic non-homogeneous structure in directions parallel to the plate mid-plane. The geometry of the above plates, apart from the global mid-plane length dimensions L_1, L_2 and constant thickness d, is characterised by the length l which determines the period of structure inhomogeneity, $l = \min(l_1, l_2)$. A fragment of the aforementioned plate is shown in Fig. 1.

Direct applications of Reissner-type plate equations to the analysis of special problems of periodic plates are rather difficult due to the highly oscillating and possibly non-continuous form of the coefficients, cf. Bensousson *et al.* (1978). Thus, a problem arises here how to formulate an approximate 2D-model of a periodic plate described by an equation with certain averaged

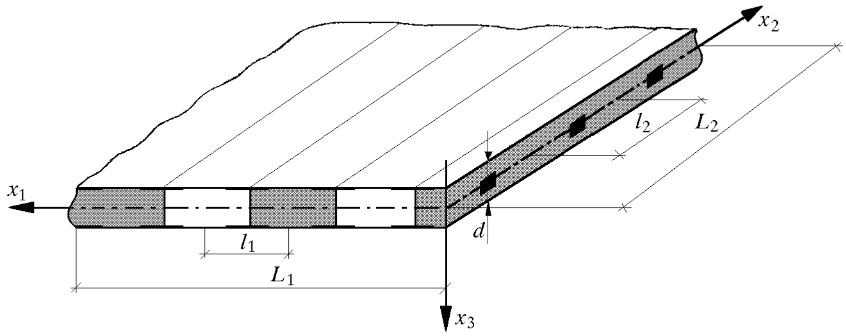

Fig. 1. An example of a medium thickness periodic plate

constant coefficients. This problem can be solved by using the homogenisation theory of PDEs with periodic coefficients, see Caillerie (1984), Kohn and Vogelius (1984). Homogenized models of the Reissner-type plates were studied by Lewiński and Telega (2000) and Lewiński (1991, 1992). However, homogenized equations cannot describe the effect of the period l on the overall plate behaviour, the so called length-scale effect.

The main aim of this paper is to formulate a new non-asymptotic model of medium thickness periodic plates which is free of the aforementioned drawback. To this end, we shall take into account the modelling approach proposed by Woźniak *et al.* (2004), which is a certain generalization of the tolerance averaging technique presented by Woźniak and Wierzbicki (2000). So far, the tolerance averaging technique has been applied to the modelling of medium thickness plates, cf. Woźniak and Baron (1995), Baron (2002, 2005). In contrast to the results derived in the aforementioned papers, where the period of plate inhomogeneity was assumed to be large when compared to the plate thickness, the obtained model describe the behaviour of Reissner-type prestressed plates with periods of an order of the plate thickness. This model is obtained by the tolerance averaging technique, applied directly to 3D-equations of linear elastodynamics. Using the Hencky-Bolle kinematic assumption, we shall derive a non-asymptotic 2D-model of medium thickness periodic plates. In contrast to the homogenized 2D-model, it takes into account the effect of plate rotational inertia on the dynamic response and enables one to determine higher-order vibration frequencies caused by the plate periodic inhomogeneity. The presented general results are illustrated by the analysis of some vibration problems.

Throughout the paper, subscripts $\alpha, \beta, \ldots$ $(i, j, \ldots)$, run over $1, 2$ $(1, 2, 3)$, where superscripts $A, B, \ldots$ take the values $1, 2, \ldots, N$. The summation convention holds related to all aforementioned indices.

2. Preliminaries

Let $0x_1x_2x_3$ be an orthogonal Cartesian coordinate system in a physical space E^3, and Ω is a region occupied by the solid under consideration in its reference state. Let $\Delta(\boldsymbol{x}) = \Delta + \boldsymbol{x}$ be a periodic cell of the central of point $\boldsymbol{x} \in E^3$. By l_i we denote the period of the solid inhomogeneity in the direction of the x_i-axis. It will be assumed that l_i are sufficiently small when compared to the minimum characteristic length dimension of Ω. It is possible to consider three special cases of the non-homogeneity, cf. Woźniak *et al.* (2004). In this paper, considerations will be restricted to the bending of plates with a bi-directional periodic structure. Therefore, for a solid periodic in the x_1 and x_2-axis directions, we shall introduce the averaging operator

$$\langle f\rangle(\boldsymbol{x}) = \frac{1}{l_1 l_2} \int\limits_{\Delta(\boldsymbol{x})} f(y_1, y_2, x_3)\; dy_1 dy_2$$
$$\boldsymbol{x} = \{x_i\} \in \Omega_0 \qquad\qquad \Omega_0 = \{\boldsymbol{x} \in E^3,\;\; \Delta(\boldsymbol{x}) \subset \Omega\} \tag{2.1}$$

for an arbitrary integrable function f defined on Ω.

The basic concept is that of a slowly varying function of the argument $\boldsymbol{x}$. It is a function satisfying the following tolerance averaging approximation (TAA)

$$\langle Ef\rangle(\boldsymbol{x}) \simeq \langle f\rangle(\boldsymbol{x})F(\boldsymbol{x}) \tag{2.2}$$

which has to hold for every integrable function f; where $\simeq$ is a certain tolerance relation, see Woźniak and Wierzbicki (2000). If condition (2.2) holds for all continuous derivatives of F (which exist) then we shall write $F(\cdot) \in SV_\Delta(T)$. By T we denote the set of all tolerance relations in the problem under consideration.

3. Modelling technique

Let $u_i(x_j, t)$, $x_j \in \Omega$, be a displacement field at the time t from the reference configuration of the periodic solid. The solid material is assumed to be elastic and the components A_{ijkl} of the elastic moduli tensor as well the mass density ρ depend on x_j and are periodic functions with respect to the x_1 and x_2 coordinates.

Let $\sigma^0 = \sigma^0_{ji}$ be a tensor of the initial stress, b_i stands for body forces. From the principle of stationary action for the functional depending on the displacement field components, we obtain the following linearized equations of motion for a prestressed solid

$$(A_{ijkl}u_{k,l})_{,j} + \sigma^0_{kl}u_{i,kl} - \rho\ddot{u}_i + \rho b_i = 0 \tag{3.1}$$

Equations (3.1) have highly oscillating (frequently non-continuous) coefficients A_{ijkl} and ρ. In most cases, the prestressing field tensor σ^0_{ji} is also periodic and non-continuous.

Treating Eqs. (3.1) as a starting point, we formulate an approximate model of the solid under consideration, which will be represented by equations with constant coefficients. The proposed modelling technique is based on two assumptions. To formulate these assumptions, we introduce the following decomposition of displacements

$$u_i(\boldsymbol{x},t) = u^0_i(\boldsymbol{x},t) + r_i(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \Omega_0 \tag{3.2}$$

where u^0 is an averaged part of the displacement defined by

$$u^0_i(\boldsymbol{x},t) = \langle u_i\rangle(\boldsymbol{x},t) = [\langle\rho\rangle(x_3)]^{-1}\langle\rho u_i\rangle(\boldsymbol{x},t) \tag{3.3}$$

and $r_i(\cdot,t)$ is a part of the residual displacement field.

The first assumption states that in the macroscopic description of these class of considered problems, the averaged displacement field is slowly-varying for every t

$$u^0_i(\cdot,x_3,t) \in SV_\Delta(T)$$

On the ground of (2.2), we obtain $\langle\rho r_i\rangle \simeq 0$. It follows that ri can be interpreted as a fluctuation displacement field caused by the periodic non-homogeneous structure of the solid.

The second assumption states that the fluctuation of the displacement field, represented by r_i and caused by the non-homogeneous periodic structure, conforms to this structure. It means that in every cell $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_0$, these fluctuations can be approximated by periodic functions in the form of finite sums

$$r_i(\boldsymbol{x},t) = h^A_i(x_\alpha)V^A(\boldsymbol{x},t)$$
$$A = 1,2,\ldots,N \quad \text{summation convention holds}$$

where $V^A(\cdot,x_3,t)$ for every t are slowly-varying functions $V^A(\cdot,x_3,t) \in$ $\in SV_\Delta(T)$ and $h^A_i(x_\alpha)$ are periodic linear independent functions such that $\langle h^A_i\rangle = 0$.

The functions $V^A(\cdot,x_3,t)$ constitute new kinematical variables called fluctuation amplitudes, and $h^A_i(\cdot)$ are assumed to be known *a priori* and are referred to as mode-shape functions. In general, $h_i^A(\cdot)$ represent free periodic vibrations of the 3D-periodic cell and can be treated as eigenvectors related to a certain eigenvalue problem. An alternative specification of the mode-shape functions based on mass discretization of the periodic cell is also possible.

In order to derive the governing equations for fields u_i^0, V^A, we shall introduce a displacement field u_i in the form given below into the action functional

$$u_i(\boldsymbol{x},t) = u_i^0(\boldsymbol{x},t) + h_i^A(x_\alpha)V^A(\boldsymbol{x},t) \tag{3.4}$$

Taking into account that $u_i^0(\cdot,x_3,t) \in SV_\Delta(T)$, $V^A(\cdot,x_3,t) \in SV_\Delta(T)$ we shall use in calculations the tolerance averaging approximation given by (2.2). We will also restrict considerations to the problem in which $A_{ijkl}(\cdot)$ and $\rho(\cdot)$ are even, and $h_i^A(\cdot)$ are odd functions. In such a case, applying the principle of stationary action, for $b_i =$ const, we obtain the following system of equations for u_i^0 and V^A

$$\begin{aligned}
&\langle\rho\rangle\ddot{u}_i^0 - (\langle A_{ijkl}\rangle u_{k,l}^0 + \langle A_{ijk\alpha}h_{k,\alpha}^A\rangle V^A)_{,j} - \langle\sigma_{kl}^0\rangle u_{i,kl}^0 - \langle\rho\rangle b_i = 0\\
&\langle\rho h_i^A h_i^B\rangle\ddot{V}^B - (\langle A_{i3k3}h_i^A h_k^B\rangle V_{,3}^B)_{,3} + \langle A_{i\alpha k\beta}h_{i,\alpha}^A h_{k,\beta}^B\rangle V^A +\\
&\qquad + \langle A_{ijk\alpha}h_{k,\alpha}^A\rangle u_{i,j}^0 + \langle\sigma_{\alpha\beta}^0 h_{k,\alpha}^A h_{k,\beta}^B\rangle V^B = 0
\end{aligned} \tag{3.5}$$

Equations (3.5) have constant coefficients and hence represent a certain macroscopic model of a prestressed periodic solid. The solutions $u_i^0(\cdot,t)$, $V^A(\cdot,t)$ have physical sense only if $u_i^0(\cdot,t) \in SV_\Delta(T)$and $V^A(\cdot,t) \in SV_\Delta(T)$ for every t. These equations cannot be used in analysis of boundary-value problems. The boundary conditions for V^A may not be derived as approximations of boundary conditions for the displacement $u_i = u_i^0 + h_i^A V^A$.

4. Applications to medium thickness plates

Let $\Pi = (0,L_1) \times (0,L_2)$ be a rectangle with the dimensions L_1 and L_2 on the plane $0x_1x_2$. Assume that equations (3.5) hold in a region $\Omega = \Pi \times (-d/2,d/2)$ occupied by a Reissner-type un-deformed plate with a constant thickness d. Let us also assume that the plate has a plane periodic structure, and hence $\Delta = (-l_1/2, l-1/2) \times (-l_2/2, l_2/2)$ is a 2D-periodicity cell on the $0x_1x_2$-plane. Moreover, let the plate be homogeneous in the direction of the x_3-axis and be made of periodically distributed materials along the mid-plane. The dimensions l_α are of an order of the plate thickness d and sufficiently small with respect to L_α, $d \ll L_\alpha$.

Setting $\boldsymbol{x} = (x_1, x_2)$, $z = x_3$, we shall use denotation

$$\begin{aligned} &\Delta(\boldsymbol{x}) = \Delta + \boldsymbol{x} \\ &\Pi_0 = \{\boldsymbol{x} \in \Pi : \quad \Delta(\boldsymbol{x}) \subset \Pi\} \\ &\Omega_0 = \{(\boldsymbol{x}, z) \in \Omega : \quad \Delta(\boldsymbol{x}, z) \subset \Omega\} \end{aligned}$$

Instead of operator (2.1), we introduce the following two kinds of averaging of an arbitrary integrable function $f(\cdot)$

$$\begin{aligned} &\langle f\rangle(\boldsymbol{x}, z) = \frac{1}{l_1 l_2} \int\limits_{\Delta(\boldsymbol{x})} f(y_1, y_2, z)\; dy_1 dy_2 \qquad x \in \Pi_0 \qquad -\frac{d}{2} \leqslant z \leqslant \frac{d}{2} \\ &\langle f\rangle(\boldsymbol{x}) = \frac{1}{d} \int\limits_{-d/2}^{d/2} \langle f\rangle(\boldsymbol{x}, z)\; dz \end{aligned} \tag{4.1}$$

For Δ-periodic function f, $\langle f\rangle$ is constant.

Assuming that the planes $z =$ const are elastic symmetry planes, we define

$$\begin{aligned} &C_{\alpha\beta\gamma3} = A_{\alpha\beta\gamma3} - A_{\alpha\beta33} A_{33\gamma\delta} (A_{3333})^{-1} \\ &B_{\alpha\beta} = K A_{\alpha3\beta3} \end{aligned}$$

where K is the shear coefficient of the medium-thickness plate theory.

We introduce the Hencky-Bolle kinematics assumption in the known form

$$u_\alpha(\boldsymbol{x}, z, t) = z\vartheta_\alpha(\boldsymbol{x}, t) \qquad u_3(\boldsymbol{x}, z, t) = w(\boldsymbol{x}, t) \tag{4.2}$$

where $w(\cdot, t)$ are displacements of points of the mid-plane Π, whereas $\vartheta_\alpha(\cdot, t)$ are independent rotations.

Taking into account the modelling assumptions, outlined in the previous Section, there exist decompositions of ϑ_α and w into slowly varying averaged parts ϑ^0_α, w^0 and residual displacements approximated by finite sums $h^A_i(\boldsymbol{x}) V^A(x_j, t)$. Assuming that $h^A_3(\cdot) = 0$ and $V^A = z\psi^A(\boldsymbol{x}, t)$, we obtain

$$\begin{aligned} &u_\alpha(\boldsymbol{x}, z, t) = z\vartheta^0_\alpha(\boldsymbol{x}, t) + z h^A_\alpha(\boldsymbol{x}) \psi^A(\boldsymbol{x}, t) \\ &u_3(\boldsymbol{x}, z, t) = w^0_\alpha(\boldsymbol{x}, t) \end{aligned} \tag{4.3}$$

where ϑ^0_α, w^0, ψ^A are basic unknowns which are slowly varying for every time t.

Substituting the right-hand sides of (4.3) into the action functional, we obtain from (3.5) the following system of equations for the mid-plane deflection $w^0(\boldsymbol{x},t)$, rotations $vt^0_\alpha(\boldsymbol{x},t)$ and 2D-fluctuation amplitudes $\psi^A(\boldsymbol{x},t)$

$$
\begin{aligned}
&j\langle\rho\rangle\ddot{\vartheta}^0_\alpha - j(\langle C_{\alpha\beta\gamma\delta}\rangle\vartheta^0_{\gamma,\delta})_{,\beta} + \langle B_{\alpha\beta}\rangle(\vartheta^0_\alpha + w^0_{,\alpha}) - j(\langle C_{\alpha\beta\gamma\delta}h^A_{\gamma,\delta}\rangle\psi^A)_{,\beta} + \\
&\qquad -j(\langle\sigma^0_{\gamma\beta}\rangle\vartheta^0_{\alpha,\gamma})_{,\beta} + \langle z\sigma^0_{\gamma3}\rangle\vartheta^0_{\alpha,\gamma} = 0 \\
&\langle\rho\rangle\ddot{w}^0 - [\langle B_{\alpha\beta}\rangle(\vartheta^0_\alpha + w^0_{,\alpha})]_{,\beta} - (\langle\sigma^0_{\gamma\beta}\rangle w^0_{,\gamma})_{,\beta} = 0 \qquad (4.4)\\
&\underline{j\langle\rho h^A_\alpha h^B_\alpha\rangle}\ddot{\psi}^B + (j\langle C_{\alpha\beta\gamma\delta}h^B_{\gamma,\delta}\rangle + \underline{\langle B_{\alpha\beta}h^A_\alpha h^B_\beta\rangle})\psi^B + j\langle C_{\alpha\beta\gamma\delta}h^A_{\gamma,\delta}\rangle\vartheta^0_{\alpha,\beta} + \\
&\qquad +j\langle\sigma^0_{\alpha\beta}h^A_{\gamma,\alpha}h^B_{\gamma,\beta}\rangle\psi^B = 0
\end{aligned}
$$

where $j = d^2/12$ and, for the sake of simplicity, we have neglected the body forces.

The characteristic feature of the derived system of equations (4.4) is that the fluctuation amplitudes ψ^A are governed by the system of ordinary differential equations involving only time derivatives of ψ^A. Hence, these variables do not enter into the boundary conditions and play the role of certain internal variables. Let us observe that the underlined coefficients $\langle\rho h^A_\alpha h^B_\alpha\rangle$, $\langle B_{\alpha\beta}h^A_\alpha h^B_\beta\rangle$ are values of an order of the period length. Thus, equations (4.4) describe the effect of the period length on the overall behaviour of the plate. This inhomogeneity period is of an order of the plate thickness. Neglecting the terms involving the period length, we can eliminate fluctuation variables ψ^A from (4.4) and hence obtain a system of equations for ϑ^0_α and w^0 as the basic unknowns. It can be shown that this system represents a certain approximation of the homogenized 2D-model of the periodic plate under consideration. For a homogeneous plate ρ, $C_{\alpha\beta\gamma\delta}$, $B_{\alpha\beta}$ are constant and hence $\langle C_{\alpha\beta\gamma\delta}h^A_{\gamma,\delta}\rangle = 0$. In this case, equation $(4.4)_3$ yields $\psi^A = 0$ provided that $\sigma^0_{ij} = 0$, the initial conditions for ψ^A are homogeneous and (4.4) takes the form of known Hencky-Bolle plate equations.

Equations (4.4) represent the non-asymptotic averaged 2D-model of the Reissner-type prestressed plates with a plane periodic structure. In contrast to the 2D-models of plates obtained from the equations of the plate theory by Baron and Woźniak (1995), Baron (2000, 2002, 2005), the above model was derived from the macroscopic 3D-model of a periodic solid, and hence can be applied to problems in which period lengths are of the same order as the plate thickness.

5. Dynamic behaviour of medium-thickness plates

The aforementioned results will be now applied to the analysis of free bending vibrations of a plate with a periodic non-homogeneous structure only in the direction of the x_2-axis. The plate under consideration is simply supported on the edges $x_2 = 0$, $x_2 = L$ and subjected to the initial stress $N = \int_{-d/2}^{d/2} \sigma_{22}^0 \, dz$, see Fig. 2. The plate is made of two linear elastic, isotropic and homogeneous materials. It will be assumed that all unknown functions depend on time and variable x_2, exclusively. It is a plate with a one-directional periodic structure which can be treated as a special case of plates with bi-directional periodic structures.

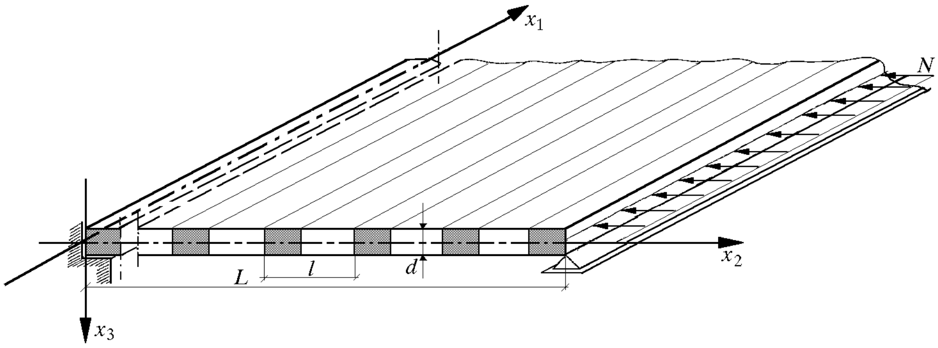

Fig. 2. A plate with a one-directional periodic structure

Considering the isotropy (in this special case also orthotropy) of the plate, it is denoted $C = C_{2222}$, $D = B_{22} = K_2 A_{2323}$.

In the first approximation, we can introduce only one vector of shape functions $h^1 = (0, h(x_2))$, where $h(\cdot)$ is a saw-like l-periodic function shown in Fig. 3.

Thus, in this example, we shall deal with only one fluctuation amplitude $\psi^1(x_2, t) = \psi(x_2, t)$, and formulae (4.2) in the form

$$\begin{aligned} u_1(x_2, z, t) &= 0 \\ u_2(x_2, z, t) &= z\vartheta_\alpha^0(x_2, t) + zh(x_2)\psi(x_2, t) \\ u_3(x_2, z, t) &= w^0(x_2, t) \end{aligned} \tag{5.1}$$

Under these conditions, equations (4.4) reduce to the system of three equations for the averaged plate deflection $w^0(x_2, t)$, rotation $\vartheta = \vartheta_2^0(x_2, t)$ and

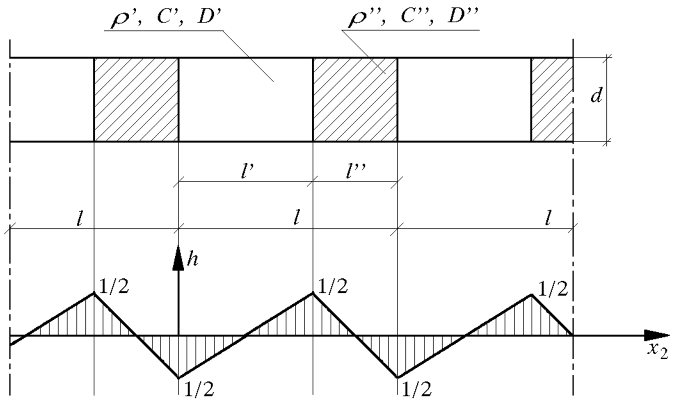

Fig. 3. Diagrams of the function $h(\cdot)$

fluctuation amplitude $\psi(x_2,t)$

$$
\begin{aligned}
&j\langle\rho\rangle\ddot{\vartheta}^0-\langle C\rangle\vartheta^0_{,22}-j\langle Ch_{,2}\rangle\psi_{,2}+\langle D\rangle(\vartheta^0+w_{,2})-jN\vartheta^0_{,22}=0\\
&\langle\rho\rangle\ddot{w}^0-\langle D\rangle(\vartheta^0+w_{,2})_{,2}-Nw^0_{,22}=0\\
&j\langle\rho h^2\rangle\ddot{\psi}+(j\langle Ch^2_{,2}\rangle+\langle Dh^2\rangle)\psi+j\langle Ch_{,2}\rangle\vartheta^0_{,2}+jN\langle h^2_{,2}\rangle\psi=0
\end{aligned}
\tag{5.2}
$$

where $N=const$, $j=d^2/12$.

It can be seen that the coefficients $\langle D\,h^2\rangle=l^2\langle D\rangle/12$, $\langle\rho h^2\rangle=l^2\langle\rho\rangle/12$ depend explicitly on the period length l and describe the length-scale effect.

Assuming the unknown functions in the form

$$
\begin{aligned}
&w^0(x_2,t)=\mathrm{e}^{\mathrm{i}\omega t}\overline{w}(x_2) &\qquad& \vartheta^0(x_2,t)=\mathrm{e}^{\mathrm{i}\omega t}\overline{\vartheta}(x_2)\\
&\psi(x_2,t)=\mathrm{e}^{\mathrm{i}\omega t}\overline{\psi}(x_2)
\end{aligned}
$$

where ω is a vibration frequency, we obtain from $(5.2)_3$

$$
\begin{aligned}
&\overline{\psi}=-\frac{\langle Ch_{,2}\rangle}{R_\omega}\overline{\vartheta}_{,2}\\
&R_\omega=\langle Ch^2_{,2}\rangle+j^{-1}\langle Dh^2\rangle+\langle h^2_{,2}\rangle N-\langle\rho h^2\rangle\omega^2
\end{aligned}
\tag{5.3}
$$

Substituting (5.3) into $(5.2)_1$, taking into account the previous assumption, we can look for a solution to (5.2) in the well known form

$$
\overline{w}(x_2)=w_n\sin k_nx_2 \qquad\qquad \overline{\vartheta}(x_2)=\vartheta_n\cos k_nx_2
$$

where $k_n = n\pi/L$, $n = 1, 2, \ldots$, and w_n, ϑ_n are arbitrary constants, summation convention holds. In this case, we arrive at a system of linear algebraic equations for w_n, ϑ_n

$$\begin{bmatrix} k_n^2 j\Big(\langle C\rangle - \frac{\langle Ch_{,2}\rangle^2}{R_\omega} + N\Big) + \langle D\rangle - \langle\rho\rangle\omega^2 & k_n j\langle D\rangle \\ k_n j\langle D\rangle & k_n^2(\langle D\rangle + N) - \langle\rho\rangle\omega^2 \end{bmatrix} \begin{bmatrix} \vartheta_n \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.4}$$

Equations (5.4) have a nontrivial solution provided that their determinant is equal to zero. In this way, bearing in mind that h, $lh_{,2}$ and d are of an order of the period l and $l \ll L$, we obtain the following approximate formulae for the first three free vibration frequencies

$$\begin{aligned} \omega_1^2 &= \frac{k_n^2 N}{\langle\rho\rangle} + \frac{k_n^4 H}{\langle\rho\rangle} + O(\varepsilon^6) \\ \omega_2^2 &= \frac{\langle D\rangle}{j\langle\rho\rangle} + \frac{1}{\langle\rho\rangle}[k_n^2 H_0 + k_n^2 j(N + \langle D\rangle)] + O(\varepsilon^4) \\ \omega_3^2 &= \frac{\langle Ch_{,2}^2\rangle + N\langle h_{,2}^2\rangle + j^{-1}\langle Dh^2\rangle}{\langle\rho h^2\rangle} + \frac{k_n^2}{\langle\rho\rangle}\frac{\langle Ch_{,2}\rangle^2}{\langle Ch_{,2}^2\rangle + N\langle h_{,2}^2\rangle} + O(\varepsilon^4) \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} H &= j\Big(\langle C\rangle - \frac{\langle Ch_{,2}\rangle^2}{\langle Ch_{,2}^2\rangle + N\langle h_{,2}^2\rangle + j^{-1}\langle Dh^2\rangle}\Big) \\ H_0 &= j\Big(\langle C\rangle - \frac{\langle Ch_{,2}\rangle^2}{\langle Ch_{,2}^2\rangle + N\langle h_{,2}^2\rangle}\Big) \end{aligned}$$

For medium thickness plates, relations (5.5) have a physical meaning provided that $\varepsilon^2 = k_n^2 j \ll 1$.

Now let us discuss an asymptotic approximation of Eqs. (5.2). By formal transition $l \to 0$, Eqs. $(5.2)_3$ lead to an algebraic equation for ψ

$$\langle Ch_{,2}^2\rangle\psi = -\langle Ch_{,2}\rangle\vartheta^0_{,2} - N\langle h_{,2}^2\rangle$$

and to a system of equations for the mid-plane deflection w^0 and rotation ϑ^0

$$\begin{aligned} & j\langle\rho\rangle\ddot{\vartheta}^0 - H_0\vartheta^0_{,22} + \langle D\rangle(\vartheta^0 + w^0_{,2}) - jN\vartheta^0_{,22} = 0 \\ & \langle\rho\rangle\ddot{w} - \langle D\rangle(\vartheta^0 + w^0_{,2})_{,2} - Nw_{,22} = 0 \end{aligned} \tag{5.6}$$

Substituting $w^0 = \mathrm{e}^{\mathrm{i}\omega t} w_n \sin(k_n x_2)$, $\vartheta^0 = \mathrm{e}^{\mathrm{i}\omega t}\vartheta_n \cos(k_n x_2)$ into (5.6), we arrive at the single frequency equation

$$\begin{aligned} & j\langle\rho\rangle\omega^4 - \langle\rho\rangle[(k_n^2 H_0) + \langle D\rangle + k_n^2 j(2N + \langle D\rangle)]\omega^2 + \\ & + k_n^2 N(k_n^2 H_0 + \langle D\rangle) + k_n^2(k_n^2 H_0 + k_n^2 jN) = 0 \end{aligned} \tag{5.7}$$

Taking into account that $k_n^2 j \ll 1$ and using the approximation

$$\sqrt{1 + k_n^2 j} \approx 1 + \frac{1}{2}k_n^2 j$$

we obtain from (5.7) the following formulae for the free vibration frequencies

$$\begin{aligned} \omega_1^2 &= \frac{k_n^2 N}{\langle\rho\rangle} + \frac{k_n^4}{\langle\rho\rangle}\frac{H_0\langle D\rangle}{\langle D\rangle(1 + k_n^2 j) + k_n^2 H_0} \\ \omega_2^2 &= \frac{\langle D\rangle}{j\langle\rho\rangle} + \frac{1}{j\langle\rho\rangle}[k_n^2 H_0 + k_n^2 j(N + \langle D\rangle)] + \frac{k_n^4}{\langle\rho\rangle}\frac{H_0\langle D\rangle}{\langle D\rangle(1 + k_n^2 j) + k_n^2 H_0} \end{aligned} \tag{5.8}$$

One should remember that d is of an order of l, then it is possible to neglect the terms involving j in formulae (5.6) and (5.7). This assumption is equivalent to the neglecting of the rotational inertia in the model described by (5.6). In that case, considering that $k_n^2 H_0/\langle D\rangle \ll 1$, we obtain an asymptotic model equation

$$\omega_0^2 = \frac{k_n^2(N + k_n^2 H_0)}{\langle\rho\rangle} \tag{5.9}$$

In the course of numerical calculations, the analysis of interrelations between the non-dimensional lower free vibration frequency and geometrical parameter $\kappa = l/d$ is carried out. The obtained results are compared with those corresponding to the asymptotic model.

Let the orthotropic constituents of the plate have mass densities ρ', ρ'' and elastic moduli C', C'' and D', D'', Fig. 3. In this case, by denoting $x = l'/l$, $x \in (0,1)$, the averaging operator reduces to the form

$$\langle f\rangle = xf' + (1 - x)f''$$

and

$$\langle fh^2\rangle = \frac{l^2}{12}\langle f\rangle \qquad \langle fh_{,2}\rangle = f' - f'' \qquad \langle fh_{,2}^2\rangle = \frac{f'}{x} + \frac{f''}{1 - x}$$

For simplicity, we assume that $N = 0$, $\rho' = \rho'' = \rho$. Next we introduce parameters $\eta = C''/C'$, $\zeta = D''/D'$ and $\overline{\nu} = D'/C'$. Multiplying both relations

$(5.5)_1$ and (5.9) by $\rho(C'k_n^4 j)^{-1}$ and taking into account the above denotations and assumptions, we arrive at the following formulae for the non-dimensional free vibration frequencies

$$\Omega_0 = \frac{\eta}{(1-x)+x\eta} \qquad \Omega_1 = \frac{R[x-(1-x)\eta]-(1-\eta)^2}{R}$$
$$R = \frac{1}{x} + \frac{1}{1-x} + \kappa^2\overline{\nu}[x+(1-x)\zeta] \tag{5.10}$$

Calculations were performed for three values of the parameters $\eta = 2$; 10; 20 and $\kappa = 0.5$; 1.0; 2.0. We found that $\overline{\nu} = 0.3$ and $\zeta = \eta$. Diagrams representing the interrelation between non-dimensional frequencies Ω and the size of the periodicity cell (given by x and κ) as well the parameter η are shown in Fig. 4.

Commenting the obtained results it should be stated that, with assumptions made regardless of the material η, and geometrical κ parameters, the asymptotic model gives the lowest values of the vibration frequency. The influence of κ on the frequency values rises with the growth of η. For the given η, the highest frequency is obtained when the period l is of an order the plate thickness. The differences in the vibration frequency depending on κ are the highest when the material of greater material parameters fills up the cell periodicity by about 2/3 of its volume.

The calculation assumptions are fulfilled by glued timber plates which are composed of elements cut along and across the fibres, see Fig. 5. Mechanical properties of timber can be treated in different ways. According to PN-B-03150-2000, timber is a quasi-isotropic material with elastic moduli:

$$C' = E_{90} = 430 \text{ MPa} \qquad C'' = E = 13000 \text{ MPa}$$
$$D' = D'' = G = 810 \text{ MPa} \quad \text{(for timber GL-35)}$$

According to Neuhaus (2004), timber can also be treated as an anisotropic material for which, after certain calculations, we obtain $C' = 428\,\text{MPa}$, $C'' = 12290\,\text{MPa}$, $D' = 558\,\text{MPa}$, $D'' = 37\,\text{MPa}$.

In Fig. 6, diagrams of relation between the vibration frequency and the parameter κ for the both mentioned cases are presented. It is clearly seen that no significant differences in the values of vibration frequencies can be observed both for quasi-isotropic and anisotropic timber if the period l is of an order of the plate thickness or lower. The conclusions obtained beforehand have also been confirmed as far as the comparison to the asymptotic model is concerned.

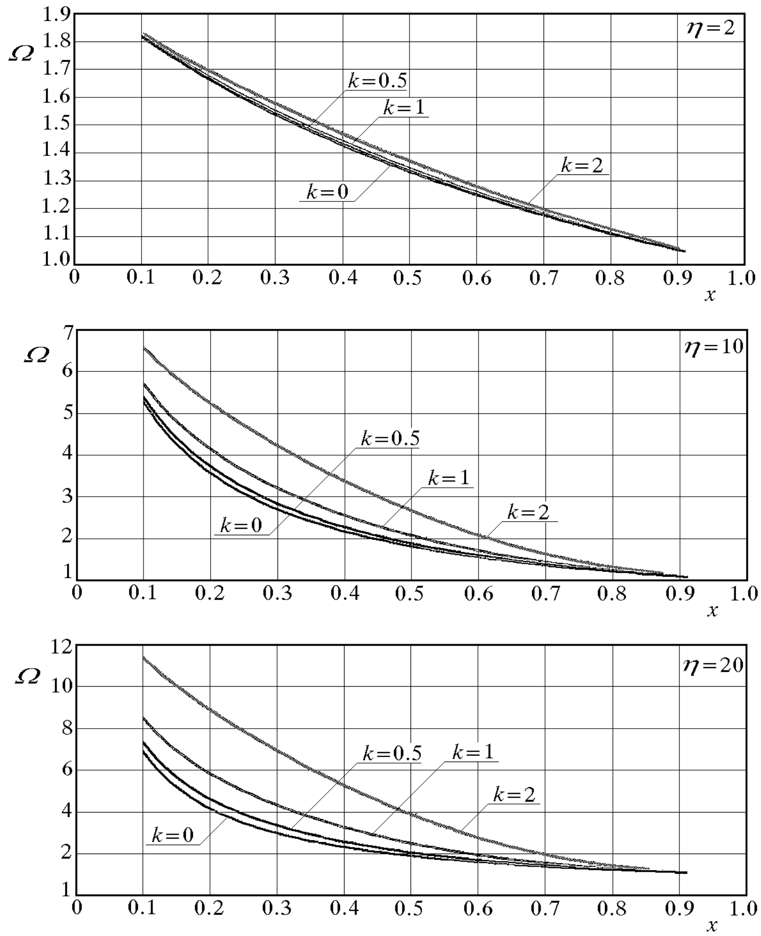

Fig. 4. Diagrams of free vibration frequencies versus different material characteristics

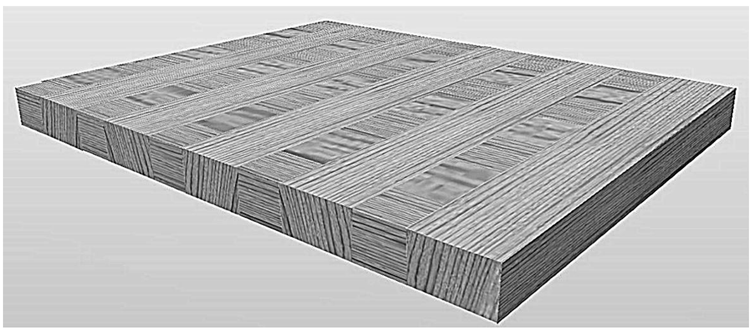

Fig. 5. An example of glued timber plate

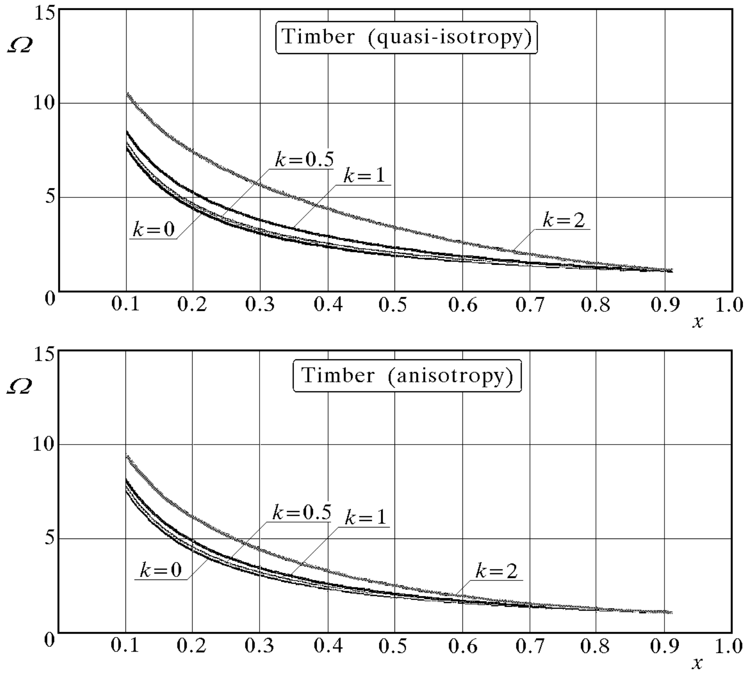

Fig. 6. Interrelation between the frequency and mechanical properties for timber

6. Summary of new results

The following new results and remarks on composite periodic plates have been derived in this paper:

- The obtained 2D-model of periodic Reissner-type plates, makes it possible to investigate dynamic (also and stability) problems, in which the constant plate thickness d is of an order of the period length l.
- In contrast to the homogenized model, the model obtained in this contribution can also be used to determination of higher free vibration frequencies caused by the plate periodic structure.
- The proposed 2D-model is a certain complementation for the model presented by Baron and Woźniak (1995) in which the period lengths were assumed to be much larger than the plate thickness.
- The analysis confirms thesis that if the period lengths are small when compared to the plate thickness then the length-scale effect is reduced; in this case the homogenisation approach is used.

- The analysis of the free vibration problem of a simply supported plate-band leads to the conclusion that the asymptotic model gives the lowest values of the basic free vibration frequency.
- The calculations for glued laminated timber plates prove that no significant differences in the values of vibration frequencies can be observed, both for quasi-isotropic and anisotropic timber, if the period l is equal to the plate thickness or smaller.

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Modelowanie periodycznie niejednorodnych płyt o okresie periodyczności rzędu ich grubości

Streszczenie

W pracy zaproponowano nowy uśredniony model płyt typu Reissnera o strukturze periodycznie niejednorodnej. Jest to model nieasymptotyczny, otrzymany techniką uśredniania tolerancyjnego (tolerance averaging technique, TAT), pozwalający uwzględnić wpływ okresu powtarzalności l na makro-mechaniczne (w sensie mechaniki kompozytów) własności płyty. Wpływ ten nazywamy efektem skali. Dotychczas metodami nieasymptotycznymi modelowano periodyczne płyty średniej grubości spełniające założenie, że okres l jest dużo większy od maksymalnej grubości płyty. TAT stosowano wtedy do uśrednionych na grubości, równań 2Dmodelu płyty.

Elementem oryginalnym jest zastosowanie TAT bezpośrednio do równań trójwymiarowej liniowej teorii sprężystości ośrodka o strukturze periodycznej w kierunkach równoległych do pewnej płaszczyzny środkowej. Uwzględniając w tych równaniach hipotezę kinematyczną Henckey-Bolle'a otrzymano równania 2D-modelu średniej grubości płyt o płaskiej strukturze periodycznej i okresie l rzędu grubości płyty. Jak dotąd modelowano w ten sposób tylko periodyczne płyty spełniające założenia Kichhoffa.

Dla przypadku szczególnego, swobodnie podpartego pasma płytowego wyznaczono częstości drgań własnych w zależności od parametrów geometrycznych oraz materiałowych i porównano je z częstościami uzyskanymi w ramach modelu asymptotycznego.

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